SIAM J. APPL. MATH.
Vol. 61, No. 4, pp. 1219–1236


# SINGULAR PERTURBATION SOLUTIONS OF A CLASS OF SYSTEMS OF SINGULAR INTEGRAL EQUATIONS*

LUQUN NI† AND SIA NEMAT-NASSER†

**Abstract.** A new class of systems of strongly singular integrodifferential equations is examined, which emerges in the study of the bridged interface crack growth, e.g., in fiber stitched layered composites. This work generalizes the asymptotic analysis of strongly singular integral equations by Willis and Nemat-Nasser [*Quart. Appl. Math.*, 48 (1990), pp. 741–753]. Singular perturbation solutions up to the first-order in a small paramter $\epsilon$ are presented for a general case which corresponds to the elasticity problem of bridged interface cracks in general anisotropic bimaterials. In particular, detailed solutions are described for a special case which includes the problem of bridged interface cracks in isotropic bimaterials. Methods are illustrated by solving particular examples.

**Key words.** system of singular integral equations, system of integrodifferential equations, asymptotic analysis, singular perturbation

**AMS subject classifications.** 45E10, 45J05

**PII.** S0036139999365742

## 1. Introduction.

The following equation,

$$\epsilon \left\{ \frac{1}{\pi} \mathrm{Re} \left[ \frac{\mathbf{\Lambda}}{(x - i0)^2} \right] * \mathbf{u}(x) + \mathbf{T}(x) \right\} + \mathbf{f}(\mathbf{u}(x); x) = \mathbf{0}, \tag{1.1a}$$

for $\epsilon \ll 1$ and $-1 < x < 1$, with the auxiliary condition

$$\mathbf{u}(\pm 1) = \mathbf{0}, \tag{1.1b}$$

is examined. In (1.1a), $\mathbf{u}(x) = (\mathbf{u_1}(x), \mathbf{u_2}(x))^{\mathbf{T}}$ is a two-dimensional $\mathrm{C}^1$ vector function,

$$\mathbf{\Lambda} = \frac{1}{\alpha_{\mathbf{0}}} \begin{bmatrix} \alpha_1 & \alpha_3 + i\alpha_4 \\ \alpha_3 - i\alpha_4 & \alpha_2 \end{bmatrix} \tag{1.2a}$$

is a complex valued $2 \times 2$ positive-definite matrix with $\alpha_j$ real, for $j = 1, 2, 3, 4$, and $\alpha_0 = \sqrt{\alpha_1 \alpha_2 - \alpha_3}$, Ni and Nemat-Nasser [10], [11], and $*$ denotes the convolution integration, defined as follows:

$$\begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} * \begin{bmatrix} f(x) \\ g(x) \end{bmatrix} = \begin{bmatrix} a * f(x) & b * g(x) \\ c * f(x) & d * g(x) \end{bmatrix}, \tag{1.2b}$$

where, e.g.,

$$a * f(x) = \int_{-1}^{1} a(x - t) f(t) dt. \tag{1.2c}$$

*Received by the editors November 29, 1999; accepted for publication (in revised form) July 7, 2000; published electronically November 8, 2000. This research was supported by the Army Research Office under contract number ARO DAAL 04-95-1-0369 and ARO DAAH 04-96-1-0376 to the University of California, San Diego.

http://www.siam.org/journals/siap/61-4/36574.html

†Center of Excellence for Advanced Materials, Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0416 (lni@ucsd.edu, ceam@ames.ucsd.edu).

Define

$$\mathbf{\Lambda} = \mathbf{\Lambda_1} - \mathbf{i}\mathbf{\Lambda_2}, \tag{1.3a}$$

and

$$\mathbf{\Lambda_1} = \frac{1}{\alpha_{\mathbf{0}}} \begin{bmatrix} \alpha_1 & \alpha_3 \\ \alpha_3 & \alpha_2 \end{bmatrix}, \tag{1.3b}$$

$$\mathbf{\Lambda_2} = \frac{1}{\alpha_{\mathbf{0}}} \begin{bmatrix} 0 & -\alpha_4 \\ \alpha_4 & 0 \end{bmatrix}. \tag{1.3c}$$

Then (1.1a) becomes

$$\epsilon \left\{ \frac{\mathbf{\Lambda_1}}{\pi} \int_{-1}^{1} \frac{\mathbf{u}(\xi)}{(x-\xi)^2} d\xi - \mathbf{\Lambda_2} \frac{d}{dx} \mathbf{u}(x) + \mathbf{T}(x) \right\} + \mathbf{f}(\mathbf{u}; x) = \mathbf{0} \tag{1.4}$$

for $-1 < x < 1$.

In deriving (1.4),

$$\frac{1}{(x - i0)^2} = \frac{1}{x^2} - i\pi\delta^{(1)}(x) \tag{1.5}$$

is used, where $\delta^{(1)}(x)$ is the first-order derivative of the Dirac delta function. In (1.4), the strongly singular integral is understood in the sense of the finite parts of Hadamard [5], or of the distributions, Gel'fand and Shilov [4]. Specifically, use

$$\begin{aligned} \int_{-1}^{1} \frac{dy}{(y-x)^2} \phi(y) = & \int_{-1}^{1} \frac{dy}{(y-x)^2} [\phi(y) - \phi(x) - (y-x)\phi'(x)] \\ & - \frac{2\phi(x)}{1-x^2} + \phi'(x) \ln\left(\frac{1-x}{1+x}\right), \end{aligned} \tag{1.6}$$

provided that $\phi'(x)$ is defined and the integral in the right-hand side exists.

Singular integral equations of this kind emerge in the study of the bridged interface cracks in bimaterials, Ni and Nemat-Nasser [12], and are generalizations of the integral equations considered in Nemat-Nasser and Hori [8], Hori and Nemat-Nasser [6], and Willis and Nemat-Nasser [15]. Indeed, system (1.1a) or (1.4) represents an equilibrium of the total forces acting on the surfaces of a bridged interface crack, where, $\mathbf{u}(x)$, $\mathbf{T}(x)$, and $\mathbf{f}(\mathbf{u}(x); x)/\epsilon$ describe the nondimensional crack-opening-displacement, applied tractions, and the bridging forces, respectively. The convolution term in (1.1a) corresponds to the nondimensional crack resistance, with the matrix $\mathbf{\Lambda}$ determined by the elastic parameters of the bimaterial. For example, as defined in (1.2a), $\alpha_3 = 0$ corresponds to an isotropic bimaterial, and $\alpha_4 = 0$ is associated with cracks in a homogeneous solid.

The parameter $\epsilon$ in the equation is the inverse of a nondimensional crack length $l = K_0 L/\alpha_0$, where $K_0$ is the strength of the bridging force, and $L$ is the physical crack length. Hence, the cases of $\epsilon \ll 1$ and $\epsilon \gg 1$ define the "long" crack and "short" crack problems, respectively.

Unlike the case of the "short" crack, for the "long" crack, i.e., $\epsilon \ll 1$, the routine perturbation method does not work, since in (1.1a) or (1.4), both the integral and derivative terms are multiplied by the small parameter $\epsilon$. Hence, if we set $\epsilon = 0$, then the resulting solution, in general, does not satisfy (1.1b). This indicates that there are boundary layers near $x = -1$ and $x = 1$, respectively, where the integral and derivative terms are both important.

As studied by Angell and Olmstead [1], [2], Lange and Smith [7], and Willis and Nemat-Nasser [15], special techniques are needed for singular perturbation solutions of this class of integral equations. In this paper, the singular perturbation method developed in Willis and Nemat-Nasser [15] is generalized and applied to solve the present problem. Explicit expressions for the asymptotic solutions up to the $\epsilon$ order are presented for (1) the general case of a bridged interface crack problem in an anisotropic bimaterial, where the bimaterial constant matrices $\Lambda_i$, $i = 1, 2$, are unrestricted; and (2) a special case where in the matrix $\Lambda_1$, the component $\alpha_3 = 0$, which is the case of an isotropic bimaterial. Finally, an illustrative example is given where the "inner" terms of the singular perturbation solution are solved by using the Wiener–Hopf technique.

**2. Singular perturbation solution: The general case.** Consider system (1.4) for general matrices $\Lambda_i$, $i = 1, 2$. To generalize the singular perturbation method in Willis and Nemat-Nasser [15], introduce the following notation: For two-dimensional vector functions $\mathbf{f}(x) = (\mathbf{f_1}(x), \mathbf{f_2}(x))^{\mathbf{T}}$ and $\mathbf{g}(x) = (\mathbf{g_1}(x), \mathbf{g_2}(x))^{\mathbf{T}}$, denote

$$\mathbf{f}(x) = \mathrm{o}(\mathbf{g}(x)) \quad \text{as } x \to x_0 \tag{2.1a}$$

if and only if

$$f_i(x) = \mathrm{o}(\mathrm{g}_i(x)) \quad \text{as } x \to x_0 \tag{2.1b}$$

for $i = 1, 2$.

As pointed out by Angell and Olmstead [1],[2], Lange and Smith [7], and Willis and Nemat-Nasser [15], unlike the techniques used in solving differential equations, singular perturbations for integral equations require that a uniformly valid expansion be used in the integrals. Hence, consider the following composite expansion:

$$\boldsymbol{u}(x;\epsilon) \sim \sum_{k=0}^{n} [\boldsymbol{\phi}_k(x) + \tilde{\boldsymbol{\phi}}_k(x') + \hat{\boldsymbol{\phi}}_k(x'') - \boldsymbol{S}_k^n(x') - \boldsymbol{S}_k^n(x'')] + \mathrm{o}(\epsilon^n), \tag{2.2}$$

where $x' = \frac{1+x}{\epsilon}$ and $x'' = \frac{1-x}{\epsilon}$ are the "inner" variables for the boundary layers near $x = -1$ and $x = 1$, respectively; $\boldsymbol{\phi}_k(x)$ are the "outer" terms; $\tilde{\boldsymbol{\phi}}_k(x')$ and $\hat{\boldsymbol{\phi}}_k(x'')$ are the inner terms in the boundary layers near $x = -1$ and $x = 1$, respectively; and $\boldsymbol{S}_k^n(x')$ and $\boldsymbol{S}_k^n(x'')$ are matching terms which are defined as the $n$-term inner expansions of the $k$th outer term in the inner variables $x'$ and $x''$, respectively. Similarly, define the matching terms $\tilde{\boldsymbol{S}}_k^m(x)$ and $\hat{\boldsymbol{S}}_k^m(x)$ as the $m$-term outer expansions of the $k$th inner term $\tilde{\boldsymbol{\phi}}_k(x')$ and $\hat{\boldsymbol{\phi}}_k(x'')$ in the outer variable $x$, respectively.

Note that the matching terms $\boldsymbol{S}_k^n(x')$, $\boldsymbol{S}_k^n(x'')$, and $\tilde{\boldsymbol{S}}_k^m(x)$, $\hat{\boldsymbol{S}}_k^m(x)$ contain terms up to the orders $\epsilon^n$, and $\epsilon^m$, respectively; i.e.,

$$\boldsymbol{\phi}_k(-1 + \epsilon x') \sim \boldsymbol{S}_k^n(x') + \mathrm{o}(\epsilon^n) \quad \text{as } \epsilon \to 0; \tag{2.3a}$$

$$\tilde{\boldsymbol{\phi}}_k\left(\frac{1+x}{\epsilon}\right) \sim \tilde{\boldsymbol{S}}_k^m(x) + \mathrm{o}(\epsilon^m) \quad \text{as } \epsilon \to 0. \tag{2.3b}$$

Then the Van Dyke matching principle (Van Dyke [14]) states

$$\sum_{k=0}^{m} \boldsymbol{S}_k^n \left(\frac{1+x}{\epsilon}\right) = \sum_{k=0}^{n} \tilde{\boldsymbol{S}}_k^m(x). \tag{2.4}$$

To simplify the discussion and without loss of generality, concentrate on the asymptotic behavior near $x = -1$, and write

$$\boldsymbol{u}(x;\epsilon) \sim \sum_{k=0}^{n} [\boldsymbol{\phi}_k(x) + \tilde{\boldsymbol{\phi}}_k(x') - \boldsymbol{S}_k^n(x')] + \mathrm{o}(\epsilon^n), \tag{2.5}$$

uniformly for $-1 \le x < 1 - \delta$, for some $\delta > 0$. Then the composite expansion near $x = -1$, up to the $\epsilon$ order, i.e., $n = 1$ in (2.5), is

$$\begin{aligned} \boldsymbol{u}(x;\epsilon) &\sim \boldsymbol{\phi}_0(x) + \tilde{\boldsymbol{\phi}}_0(x') - \boldsymbol{S}_0^1(x') \\ &+ \boldsymbol{\phi}_1(x) + \tilde{\boldsymbol{\phi}}_1(x') - \boldsymbol{S}_1^1(x') + \mathrm{o}(\epsilon), \end{aligned} \tag{2.6}$$

uniformly for $-1 \le x < 1 - \delta$, for some $\delta > 0$.

We now proceed to find the outer, inner, and matching terms.

**(1) Lowest-order terms.**

(i) *Lowest-outer term*, $\boldsymbol{\phi}_0(x)$.

Substitute the uniformly valid composite expansion (2.5) into the system (1.4). Note that, in view of the evaluation of the singular integration defined in (1.6), for a fixed $x$ (not equal to $\pm 1$) in the outer region, the integral and derivative terms are bounded. Thus, setting $\epsilon \to 0$, it follows that

$$\boldsymbol{f}(\boldsymbol{\phi}_0(x); x) = \boldsymbol{0} \tag{2.7}$$

which is the equation for determining the lowest term, $\boldsymbol{\phi}_0(x)$; we assume that the prescribed function $\boldsymbol{f}$ is such that (2.7) admits a solution.

(ii) *Matching terms*, $\boldsymbol{S}_0^0(x')$.

By definition, $\boldsymbol{S}_0^0(x')$ is the lowest term of the expansion of $\boldsymbol{\phi}_0(x)$ in the inner variable $x' = (1+x)/\epsilon$, so that

$$\boldsymbol{\phi}_0(-1+\epsilon x') \to \boldsymbol{\phi}_0(-1) = \boldsymbol{S}_0^0(x') \quad \text{as } \epsilon \to 0. \tag{2.8a}$$

And by Van Dyke's matching principle (2.4), it follows that

$$\boldsymbol{S}_0^0(x') = \tilde{\boldsymbol{S}}_0^0(x) = \boldsymbol{\phi}_0(-1). \tag{2.8b}$$

(iii) *Lowest-inner term*, $\tilde{\boldsymbol{\phi}}_0(x')$.

Substitute the composite expansion (2.5) for $n = 0$ into (1.4) and collect the lowest terms of O(1) when $x$ is in the inner region near $x = -1$, and $x' = (1+x)/\epsilon$ is fixed. As in Willis and Nemat-Nasser [15], decompose the integral term in (1.4) into two parts:

$$\begin{aligned} &\epsilon \int_{-1}^{1} \frac{dy}{(y-x)^2} [\boldsymbol{\phi}_0(y) + \tilde{\boldsymbol{\phi}}_0(y') - \boldsymbol{\phi}_0(-1) + \mathrm{o}(\epsilon)] \\ &= \epsilon \int_{-1}^{1} \frac{dy}{(y-x)^2} \tilde{\boldsymbol{\phi}}_0(y') + \epsilon \int_{-1}^{1} \frac{dy}{(y-x)^2} [\boldsymbol{\phi}_0(y) - \boldsymbol{\phi}_0(-1)] + \mathrm{o}(\epsilon). \end{aligned} \tag{2.9}$$

To evaluate the singular integral (1.6), set $x = -1 + \epsilon x'$, fix $x'$, and obtain (Willis and Nemat-Nasser [15])

$$\int_{-1}^{1} \frac{\boldsymbol{\phi}(y)dy}{(y+1-\epsilon x')^2} \sim \int_{-1}^{1} \frac{dy}{(y+1)^2}[\boldsymbol{\phi}(y) - \boldsymbol{\phi}(-1) - (y+1)\boldsymbol{\phi}'(-1)]$$
$$-\boldsymbol{\phi}(-1)/2 - \frac{\boldsymbol{\phi}(-1)}{\epsilon x'} - \boldsymbol{\phi}'(-1) + \boldsymbol{\phi}'(-1)\ln\frac{2}{\epsilon x'} + \mathrm{O}(\epsilon). \tag{2.10}$$

Using (2.10) for $\boldsymbol{\phi}(y) = \boldsymbol{\phi}_0(y) - \boldsymbol{\phi}_0(-1)$, it is seen that, in the right-hand side of (2.9), the second term is of the order of $\epsilon$, and the first term, by changing the integration variable to $y' = (1+y)/\epsilon$, becomes

$$\epsilon \int_{-1}^{1} \frac{dy}{(y-x)^2}\tilde{\boldsymbol{\phi}}_0(y') = \int_{0}^{2/\epsilon} \frac{dy'}{(y'-x')^2}\tilde{\boldsymbol{\phi}}_0(y')$$
$$\sim \int_{0}^{\infty} \frac{dy'}{(y'-x')^2}\tilde{\boldsymbol{\phi}}_0(y') + \mathrm{O}(\epsilon), \tag{2.11a}$$

since

$$\tilde{\boldsymbol{\phi}}_0(x') = \tilde{\boldsymbol{S}}_0^0(x) + \mathrm{O}(\epsilon) = \boldsymbol{\phi}_0(-1) + \mathrm{O}(\epsilon) \tag{2.11b}$$

and

$$\int_{2/\epsilon}^{\infty} \frac{dy}{(y-x)^2}\tilde{\boldsymbol{\phi}}_0(y') = \frac{\epsilon}{2}\boldsymbol{\phi}_0(-1) + \mathrm{o}(\epsilon). \tag{2.11c}$$

Therefore the contribution of the integral term is given by

$$\int_{0}^{\infty} \frac{dy'}{(y'-x')^2}\tilde{\boldsymbol{\phi}}_0(y') + \mathrm{O}(\epsilon). \tag{2.12}$$

As for the remaining terms in (1.4), for $x = -1 + \epsilon x'$ in the inner region, obtain

$$\epsilon\frac{d}{dx}[\boldsymbol{\phi}_0(x) + \tilde{\boldsymbol{\phi}}_0(x') - \boldsymbol{S}_0^0(x)] = \epsilon\frac{d}{dx}[\boldsymbol{\phi}_0(x) + \tilde{\boldsymbol{\phi}}_0(x') - \boldsymbol{\phi}_0(-1)]$$
$$\sim \frac{d}{dx'}\tilde{\boldsymbol{\phi}}_0(x'), \tag{2.13a}$$

and

$$\boldsymbol{f}(\boldsymbol{\phi}_0(x) + \tilde{\boldsymbol{\phi}}_0(x') - \boldsymbol{S}_0^0(x); x) \sim \boldsymbol{f}(\tilde{\boldsymbol{\phi}}_0(x'); -1) \tag{2.13b}$$

for $x'$ fixed and as $\epsilon \to 0$.

Therefore the lowest-inner term $\tilde{\boldsymbol{\phi}}_0(x')$ is determined by the following equation:

$$\frac{\boldsymbol{\Lambda}_1}{\pi}\int_{0}^{\infty} \frac{dy'}{(y'-x')^2}\tilde{\boldsymbol{\phi}}_0(y') - \boldsymbol{\Lambda}_2\frac{d}{dx'}\tilde{\boldsymbol{\phi}}_0(x') + \boldsymbol{f}(\tilde{\boldsymbol{\phi}}_0(x'); -1) = \boldsymbol{0}, \tag{2.14a}$$

with the condition that

$$\tilde{\boldsymbol{\phi}}_0(-1) = \boldsymbol{0}. \tag{2.14b}$$

**(2) $\epsilon$-order terms.**

(i) *Matching term,* $\boldsymbol{S}_0^1(x')$.

$\boldsymbol{S}_0^1(x')$ is the inner expansion of the outer term $\boldsymbol{\phi}_0(x)$, up to the $\epsilon$ order in the inner variable $x' = (1+x)/\epsilon$. Hence,

$$\boldsymbol{\phi}_0(x) = \boldsymbol{\phi}_0(-1+\epsilon x') \sim \boldsymbol{\phi}_0(-1) + \epsilon x' \boldsymbol{\phi}'_0(-1) + \mathrm{o}(\epsilon) \tag{2.15a}$$

and

$$\boldsymbol{S}_0^1(x') = \boldsymbol{\phi}_0(-1) + \epsilon x' \boldsymbol{\phi}'_0(-1). \tag{2.15b}$$

The explicit expression for $\boldsymbol{\phi}'_0(-1)$ can be found through a differentiation of (2.7), which leads to

$$\left[\frac{\partial}{\partial x}\boldsymbol{f}(\boldsymbol{\phi}_0(x);x) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}(\boldsymbol{\phi}_0(x);x)\boldsymbol{\phi}'_0(x)\right]|_{x=-1} = \mathbf{0}; \tag{2.16a}$$

here and in what follows, we assume that the prescribed $\boldsymbol{f}$ is such that all indicated derivatives exist and are well behaved; hence, define the matrix

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}} = \left[\frac{\partial f_j}{\partial \phi_k}\right]_{j,k=1,2} \tag{2.16b}$$

for $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$ and $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{\phi}; x)$. Then we have

$$\boldsymbol{\phi}'_0(-1) = -\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}\right]^{-1}(\boldsymbol{\phi}_0(-1);-1)\frac{\partial}{\partial x}\boldsymbol{f}(\boldsymbol{\phi}_0(-1);-1), \tag{2.17}$$

provided that the inverse matrix exists.

(ii) *$\epsilon$-order term,* $\boldsymbol{\phi}_1(x)$.

Now assume that $x$ is in the outer region and substitute the composite expansion (2.6) into (1.4). Note that the integral term in (1.4) contributes the term

$$\epsilon \int_{-1}^{1} \frac{dy}{(y-x)^2}[\boldsymbol{\phi}_0(y) + \mathrm{O}(\epsilon)] \sim \epsilon \int_{-1}^{1} \frac{\mathrm{d}y}{(y-x)^2}\boldsymbol{\phi}_0(y) + \mathrm{o}(\epsilon), \tag{2.18a}$$

while the derivative term and the $\boldsymbol{f}$ term, respectively, reduce to

$$\epsilon \frac{d}{dx}\boldsymbol{\phi}_0(x) + \mathrm{o}(\epsilon) \tag{2.18b}$$

and

$$\boldsymbol{f}(\boldsymbol{\phi}_0(x) + \boldsymbol{\phi}_1(x) + \mathrm{o}(\epsilon); x) \sim \boldsymbol{f}(\boldsymbol{\phi}_0(x);x) + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}(\boldsymbol{\phi}_0(x);x)\boldsymbol{\phi}_1(x). \tag{2.18c}$$

Collecting the terms up to the order $\epsilon$, it results in

$$\begin{aligned} &\epsilon\left[\frac{\mathbf{\Lambda}_1}{\pi}\int_{-1}^{1}\frac{dy}{(y-x)^2}\boldsymbol{\phi}_0(y) - \mathbf{\Lambda}_2\frac{d}{dx}\boldsymbol{\phi}_0(x) + \boldsymbol{T}(x)\right] \\ &\qquad + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}(\boldsymbol{\phi}_0(x);x)\boldsymbol{\phi}_1(x) = \mathbf{0}. \end{aligned} \tag{2.19}$$

Therefore the $\epsilon$-order outer term is given by

$$\boldsymbol{\phi}_1(x) = \epsilon\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}\right]^{-1}(\boldsymbol{\phi}_0(x);x)\mathbf{A}(x), \tag{2.20a}$$

where

$$\mathbf{A}(x) \equiv \left[ \frac{\boldsymbol{\Lambda}_1}{\pi} \int_{-1}^{1} \frac{dy}{(y-x)^2} \boldsymbol{\phi}_0(y) - \boldsymbol{\Lambda}_2 \frac{d}{dx} \boldsymbol{\phi}_0(x) + \boldsymbol{T}(x) \right]. \tag{2.20b}$$

(iii) *Matching term,* $\boldsymbol{S}_1^1(x')$.
Observe the relation

$$\boldsymbol{\phi}_1(x) = \boldsymbol{S}_1^1(x') + 0(\epsilon) \tag{2.21}$$

for $x = -1 + \epsilon x'$ in the inner region near $x = -1$. Using the expression (2.20a), by expanding factors $[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}]^{-1}$ and $\mathbf{A}(x)$, obtain the matching term $\boldsymbol{S}_1^1(x')$, as

$$\boldsymbol{S}_1^1(x') = - \left[ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}} \right]^{-1} (\boldsymbol{\phi}_0(-1); -1) \boldsymbol{\Lambda}_1 \boldsymbol{\phi}_0(-1)/x'$$

$$+\epsilon \left\{ \left[ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}} \right]^{-1} \boldsymbol{b} - \left( \frac{\partial}{\partial \boldsymbol{\phi}} \left[ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}} \right]^{-1} \boldsymbol{\phi'}_0(-1) + \frac{\partial}{\partial x} \left[ \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}} \right]^{-1} \right) \right\} \boldsymbol{\Lambda}_1 \boldsymbol{\phi}_0(-1), \tag{2.22a}$$

where

$$\boldsymbol{b} \equiv \boldsymbol{T}(-1) + \frac{\boldsymbol{\Lambda}_1}{\pi} \boldsymbol{a} - \boldsymbol{\Lambda}_2 \boldsymbol{\phi'}_0(-1) \tag{2.22b}$$

and

$$\boldsymbol{a} \equiv \int_{-1}^{1} \frac{dy}{(y+1)^2} [\boldsymbol{\phi}_0(y) \boldsymbol{\phi}_0(-1) - (y+1) \boldsymbol{\phi'}_0(-1)]$$

$$-\boldsymbol{\phi}_0(-1)/2 - \boldsymbol{\phi'}_0(-1) \left[ 1 + \ln \left( \frac{\epsilon x'}{2} \right) \right]. \tag{2.22c}$$

(iv) $\epsilon$*-order inner term,* $\tilde{\boldsymbol{\phi}}_1(x')$.
Substitute the uniformly valid composite expansion (2.6) into (1.4); assume that $x = -1 + \epsilon x'$ is in the inner region near $x = -1$ and derive a linear equation for $\tilde{\boldsymbol{\phi}}_1(x')$. In doing so, use the expression (2.15b) for the matching term $\boldsymbol{S}_0^1(x')$ in the integral in (1.4) and show that

$$\epsilon \int_{-1}^{1} \frac{\boldsymbol{u}(y)}{(x-y)^2} dy \sim \epsilon \int_{-1}^{1} \frac{dy}{(x-y)^2} [(\boldsymbol{\phi}_0(y) - \boldsymbol{\phi}_0(-1))$$

$$+(\boldsymbol{\phi}_1(y) - \boldsymbol{S}_1^1(y')) + (\tilde{\boldsymbol{\phi}}_1(y') - \epsilon y' \boldsymbol{\phi'}_0(-1)) + \tilde{\boldsymbol{\phi}}_0(y')]. \tag{2.23}$$

Now, for the integral on the right-hand side of (2.23), use (2.10) and consider

$$\epsilon \int_{-1}^{1} \frac{dy}{(x-y)^2}[\boldsymbol{\phi}_0(y) - \boldsymbol{\phi}_0(-1)]$$
$$\sim \epsilon \mathbf{A}_2 + \epsilon \boldsymbol{\phi}'_0(-1)\ln\left(\frac{2}{\epsilon x'}\right) + \frac{\epsilon}{2}\boldsymbol{\phi}_0(-1) + \mathrm{o}(\epsilon), \tag{2.24a}$$

where

$$\mathbf{A}_2 \equiv \int_{-1}^{1} \frac{dy}{(y+1)^2}[\boldsymbol{\phi}_0(y) - \boldsymbol{\phi}_0(-1) - (y+1)\boldsymbol{\phi}'_0(-1)]$$
$$-1/2\boldsymbol{\phi}_0(-1) - \boldsymbol{\phi}'_0(-1). \tag{2.24b}$$

The second part of the integral (2.23) yields a higher-order term, as is seen by considering the integrand in the inner and the outer regions separately: when $y$ is in the inner region, from (2.21), the integrand is of order $\mathrm{o}(\epsilon)$, while when $y$ is in the outer region, both terms, $\boldsymbol{\phi}_1(y)$ and $\boldsymbol{S}_1^1(y')$, are of order $\mathrm{O}(\epsilon)$.

In a similar manner as in deriving (2.11a), the third part integral in (2.23) is reduced to

$$\epsilon \int_{-1}^{1} \frac{dy}{(x-y)^2}[\tilde{\boldsymbol{\phi}}_1(y') - \epsilon y' \boldsymbol{\phi}'_0(-1)]$$
$$\sim \int_0^{\infty} \frac{dy'}{(x'-y')^2}[\tilde{\boldsymbol{\phi}}_1(y') - \epsilon y' \boldsymbol{\phi}'_0(-1)] + \mathrm{o}(\epsilon). \tag{2.25}$$

Therefore, from (2.24a,b) and (2.25) the integral (2.23) is expanded as

$$\epsilon \int_{-1}^{1} \frac{\boldsymbol{u}(y)}{(x-y)^2} dy \sim \int_0^{\infty} \frac{dy'}{(x'-y')^2}[\tilde{\boldsymbol{\phi}}_1(y') - \epsilon y' \boldsymbol{\phi}'_0(-1)] + \frac{\epsilon}{2}\boldsymbol{\phi}_0(-1)$$
$$+ \int_0^{\infty} \frac{dy'}{(x'-y')^2}\tilde{\boldsymbol{\phi}}_0(y') + \epsilon \mathbf{A}_2 + \epsilon \boldsymbol{\phi}'_0(-1)\ln\left(\frac{2}{\epsilon x'}\right) + 0(\epsilon). \tag{2.26}$$

The remaining terms in (1.4) are, respectively, estimated as

$$\epsilon \frac{d}{dx}\boldsymbol{u}(x) \sim \frac{d}{dx'}[\tilde{\boldsymbol{\phi}}_0(x') + \tilde{\boldsymbol{\phi}}_1(x')] + \mathrm{o}(\epsilon), \tag{2.27a}$$

$$\epsilon \boldsymbol{T}(x) \sim \epsilon \boldsymbol{T}(-1) + \mathrm{o}(\epsilon), \tag{2.27b}$$

and

$$\boldsymbol{f}(\boldsymbol{u};x) \sim \boldsymbol{f}(\tilde{\boldsymbol{\phi}}_0(x') + \tilde{\boldsymbol{\phi}}_1(x');x) \sim \boldsymbol{f}(\tilde{\boldsymbol{\phi}}_0(x');-1)$$
$$+\epsilon x' \frac{\partial}{\partial x}\boldsymbol{f}(\tilde{\boldsymbol{\phi}}_0(x');-1) + \left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}\right](\tilde{\boldsymbol{\phi}}_0(x');-1)\tilde{\boldsymbol{\phi}}_1(x') + \mathrm{o}(\epsilon). \tag{2.27c}$$

Combining (2.26) and (2.27a–c), and using (2.14a), obtain the linear equation for the $\epsilon$-order term, $\tilde{\boldsymbol{\phi}}_1(x')$,

$$\frac{\boldsymbol{\Lambda}_1}{\pi} \int_0^{\infty} \frac{dy'}{(y'-x')^2}[\tilde{\boldsymbol{\phi}}_1(y') - \epsilon y' \boldsymbol{\phi}'_0(-1)]$$
$$-\boldsymbol{\Lambda}_2 \frac{d}{dx}\tilde{\boldsymbol{\phi}}_1(x') + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{\phi}}\tilde{\boldsymbol{\phi}}_1(x') + \epsilon \mathbf{A}_3 = \mathbf{0}, \tag{2.28a}$$

where $\mathbf{A}_3$ depending on lower order terms is defined by

$$\text{(2.28b)} \quad \mathbf{A}_3 = \frac{\mathbf{\Lambda}_2}{\pi}\left[\boldsymbol{\phi}'_0(-1)\ln\left(\frac{2}{\epsilon x'}\right) + \mathbf{A}_2\right] + \boldsymbol{T}(-1) + x'\frac{\partial}{\partial x}\boldsymbol{f}(\tilde{\boldsymbol{\phi}}_0(x'); -1).$$

In summary, using the method described above, the singular perturbation solution of (1.4) in a composite series expansion can be obtained up to any desired order of $\epsilon$. For the singular perturbation solution up to the $\epsilon$ order, the lowest outer term $\boldsymbol{\phi}_0(x)$ and the lowest inner term $\tilde{\boldsymbol{\phi}}_0(x')$ are determined from (2.7) and (2.14a), respectively (these may be nonlinear equations); the $\epsilon$-order term, $\boldsymbol{\phi}_1(x)$, the matching terms, $\boldsymbol{S}_0^0(x')$, $\boldsymbol{S}_0^1(x')$, and $\boldsymbol{S}_1^1(x')$ are explicitly given by (2.20a), (2.8b), (2.15b), and (2.22a,c), respectively; the $\epsilon$-order inner term, $\tilde{\boldsymbol{\phi}}_1(x')$, is determined by the linear equation (2.28a).

**3. Singular perturbation solution: A special case.** In this section, a special case is examined, for which the material constant matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ in the main equation (1.4) take on the following simple forms:

$$\text{(3.1a)} \qquad \mathbf{\Lambda}_1 = \boldsymbol{I}$$

and

$$\text{(3.1b)} \qquad \mathbf{\Lambda}_2 = \begin{bmatrix} 0 & -\beta_D \\ \beta_D & 0 \end{bmatrix},$$

respectively, where $\boldsymbol{I}$ is the identity matrix and $\beta_D$ is the Dundurs constant (Dundurs [3]), which is a constant characterizing the elastic property of the isotropic bimaterial. The problem of bridged interface cracks in isotropic bimaterials falls into this special case.

Now the singular integral equation (1.4) is simplified to a system of integral equations,

$$\text{(3.2a)} \qquad \epsilon\left[\frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\xi)}{(x-\xi)^2}d\xi + \beta_D\frac{d}{dx}u_2(x) + T_1(x)\right] + f_1(u_1, u_2; x) = 0,$$

$$\text{(3.2b)} \qquad \epsilon\left[\frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\xi)}{(x-\xi)^2}d\xi - \beta_D\frac{d}{dx}u_1(x) + T_2(x)\right] + f_2(u_1, u_2; x) = 0,$$

where

$$\text{(3.2c)} \qquad (u_1(x), u_2(x))^T = \boldsymbol{u}(x),$$

$$\text{(3.2d)} \qquad (T_1(x), T_2(x))^T = \boldsymbol{T}(x),$$

and

$$\text{(3.2e)} \qquad (f_1(u_1, u_2; x), f_2(u_1, u_2; x))^T = \boldsymbol{f}(\boldsymbol{u}(x); x).$$

In components, the composite expansion (2.2) for the solution is rewritten as

$$\text{(3.3a)} \qquad u_1(x;\epsilon) = \sum_{k=0}^{n}[\phi_k(x) + \tilde{\phi}_k(x') - S_k^n(x') + \tilde{\tilde{\phi}}_k(x'') - S_k^n(x'')] + o(\epsilon^n),$$

(3.3b) $$u_2(x;\epsilon) = \sum_{k=0}^{n} [\psi_k(x) + \tilde{\psi}_k(x') - s_k^n(x') + \tilde{\tilde{\psi}}_k(x'') - s_k^n(x'')] + o(\epsilon^n),$$

where $(\phi_k(x), \psi_k(x)) = \boldsymbol{\phi}_k(x)$ are the outer terms; $(\tilde{\phi}_k(x'), \tilde{\psi}_k(x')) = \tilde{\boldsymbol{\phi}}_k(x')$ are the inner terms in the boundary layer near $x = -1$; $(\hat{\phi}_k(x''), \hat{\psi}_k(x'')) = \hat{\boldsymbol{\phi}}_k(x'')$ are the inner terms in the boundary layer near $x = 1$; and $(S_k^n, s_k^n) = \boldsymbol{S}_k^n$ are the matching terms.

Similarly to the discussion in the previous section, focus attention on the solution near $x = -1$, and use the following composite expansion up to the $\epsilon$-order terms, i.e., $n = 1$ in (3.3a,b),

(3.4a) $$u_1(x;\epsilon) \sim \phi_0(x) + \tilde{\phi}_0(x') - S_0^1(x') + \phi_1(x) + \tilde{\phi}_1(x') - S_1^1(x') + \mathrm{o}(\epsilon),$$

(3.4b) $$u_2(x;\epsilon) \sim \psi_0(x) + \tilde{\psi}_0(x') - s_0^1(x') + \psi_1(x) + \tilde{\psi}_1(x') - s_1^1(x') + \mathrm{o}(\epsilon),$$

for $-1 \le x < 1 - \delta$, for some $\delta > 0$.

**(1) Lowest-order terms.**

(i) *Lowest-order outer terms.*

In view of (2.7), the lowest-order outer terms, $\phi_0(x)$ and $\psi_0(x)$, are determined by the following equations:

(3.5a) $$f_1(\phi_0(x), \psi_0(x); x) = 0$$

and

(3.5b) $$f_2(\phi_0(x), \psi_0(x); x) = 0.$$

(ii) *Matching terms*, $S_0^0(x')$, $s_0^0(x')$.

From (2.8b), it follows that

(3.6a) $$S_0^0(x') = \phi_0(-1)$$

and

(3.6b) $$s_0^0(x') = \psi_0(-1).$$

(iii) *Lowest-order inner terms*, $\tilde{\phi}_0(x')$, $\tilde{\psi}_0(x')$.

From (2.14a), a system of equations for determining $(\tilde{\phi}_0(x'), \tilde{\psi}_0(x'))$ is obtained,

(3.7a) $$\int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_0(y') + \beta_D \frac{d}{dx'}\tilde{\psi}_0(x') + f_1(\tilde{\phi}_0(x'), \tilde{\psi}_0(x'); -1) = 0,$$

(3.7b) $$\int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\psi}_0(y') - \beta_D \frac{d}{dx'}\tilde{\phi}_0(x') + f_2(\tilde{\phi}_0(x'), \tilde{\psi}_0(x'); -1) = 0.$$

**(2) $\epsilon$-order terms.**

(i) *Matching terms*, $S_0^1(x')$, $s_0^1(x')$.

From (2.15b), the matching terms, $S_0^1(x')$ and $s_0^1(x')$, are given by

(3.8a) $$S_0^1(x') = \phi_0(-1) + \epsilon x' {\phi'}_0(-1),$$

(3.8b) $$s_0^1(x') = \psi_0(-1) + \epsilon x' {\psi'}_0(-1),$$

where ${\phi'}_0(-1)$ and ${\psi'}_0(-1)$ can be evaluated explicitly,

(3.8c) $${\phi'}_0(-1) = \begin{vmatrix} f_{1,\psi}^{(0)}(-1) & f_{1,x}^{(0)}(-1) \\ f_{2,\psi}^{(0)}(-1) & f_{2,x}^{(0)}(-1) \end{vmatrix} / \Delta(-1),$$

(3.8d) $${\psi'}_0(-1) = -\begin{vmatrix} f_{1,\phi}^{(0)}(-1) & f_{1,x}^{(0)}(-1) \\ f_{2,\phi}^{(0)}(-1) & f_{2,x}^{(0)}(-1) \end{vmatrix} / \Delta(-1);$$

here, a subscript comma followed by an index, e.g., $x$, denotes derivative with respect to the corresponding argument.

(ii) *$\epsilon$-order outer terms*, $\phi_1(x)$, $\psi_1(x)$.

The outer terms $\phi_1(x)$ and $\psi_1(x)$ are found to be

(3.9a) $$\phi_1(x) = \epsilon \begin{vmatrix} f_{1,\psi}^{(0)}(x) & A_1(x) \\ f_{2,\psi}^{(0)}(x) & A_2(x) \end{vmatrix} / \Delta(x)$$

and

(3.9b) $$\psi_1(x) = \epsilon \begin{vmatrix} f_{1,\phi}^{(0)}(x) & A_1(x) \\ f_{2,\phi}^{(0)}(x) & A_2(x) \end{vmatrix} / \Delta(x),$$

where

(3.9c) $$f_i^{(0)}(x) = f_i(\phi_0(x), \psi_0(x); x)$$

for $i = 1, 2$;

(3.9d) $$\Delta(x) \equiv \begin{vmatrix} f_{1,\phi}^{(0)}(x) & f_{1,\psi}^{(0)}(x) \\ f_{2,\phi}^{(0)}(x) & f_{2,\psi}^{(0)}(x) \end{vmatrix};$$

and $A_1(x)$ and $A_2(x)$ are defined by

(3.9e) $$A_1(x) = T_1(x) + \int_{-1}^{1} \frac{\phi_0(y)}{(y-x)^2} dy + \beta_D \frac{d}{dx}\psi_0(x),$$

(3.9f) $$A_2(x) = T_2(x) + \int_{-1}^{1} \frac{\psi_0(y)}{(y-x)^2} dy - \beta_D \frac{d}{dx}\phi_0(x).$$

(iii) *Matching terms*, $S_1^1(x')$, $s_1^1(x')$.

The matching terms, $S_1^1(x')$ and $s_1^1(x')$, are obtained as

$$S_1^1(x') = -\frac{1}{x'} \begin{vmatrix} f_{1,\psi}^{(0)}(-1) & \phi_0(-1) \\ f_{2,\psi}^{(0)}(-1) & \psi_0(-1) \end{vmatrix} / \Delta(-1)$$
$$+\epsilon[\begin{vmatrix} f_{1,\psi}^{(0)}(-1) & b_1 \\ f_{2,\psi}^{(0)}(-1) & b_2 \end{vmatrix} + \begin{vmatrix} \phi_0(-1) & Q_{1,\psi} \\ \psi_0(-1) & Q_{2,\psi} \end{vmatrix}]/\Delta(-1)$$
$$+\frac{\epsilon}{x'} \begin{vmatrix} f_{1,\psi}^{(0)}(-1) & \phi_0(-1) \\ f_{2,\psi}^{(0)}(-1) & \psi_0(-1) \end{vmatrix}$$

(3.10a) $$\left[\begin{vmatrix} f_{1,\phi}^{(0)}(-1) & Q_{1,\psi} \\ f_{2,\phi}^{(0)}(-1) & Q_{2,\psi} \end{vmatrix} - \begin{vmatrix} f_{1,\phi}^{(0)}(-1) & Q_{1,\phi} \\ f_{2,\phi}^{(0)}(-1) & Q_{2,\phi} \end{vmatrix}\right] / \Delta^2(-1)$$

and

$$
\begin{aligned}
s_1^1(x') = &-\frac{1}{x'}\begin{vmatrix} f_{1,\phi}^{(0)}(-1) & \phi_0(-1) \\ f_{2,\phi}^{(0)}(-1) & \psi_0(-1) \end{vmatrix} /\Delta(-1) \\
&+\epsilon\left[\begin{vmatrix} f_{1,\phi}^{(0)}(-1) & b_1 \\ f_{2,\phi}^{(0)}(-1) & b_2 \end{vmatrix} + \begin{vmatrix} \phi_0(-1) & Q_{1,\phi} \\ \psi_0(-1) & Q_{2,\phi} \end{vmatrix}\right] /\Delta(-1) \\
&+\frac{\epsilon}{x'}\begin{vmatrix} f_{1,\phi}^{(0)}(-1) & \phi_0(-1) \\ f_{2,\phi}^{(0)}(-1) & \psi_0(-1) \end{vmatrix} \\
&\left[\begin{vmatrix} f_{1,\phi}^{(0)}(-1) & Q_{1,\psi} \\ f_{2,\phi}^{(0)}(-1) & Q_{2,\psi} \end{vmatrix} - \begin{vmatrix} f_{1,\phi}^{(0)}(-1) & Q_{1,\phi} \\ f_{2,\phi}^{(0)}(-1) & Q_{2,\phi} \end{vmatrix}\right] /\Delta^2(-1),
\end{aligned} \tag{3.10b}
$$

where

$$Q_{j,\phi} \equiv [f_{j,x\phi} + \phi_0'(-1)f_{j,\phi\phi} + \psi_0'(-1)P_{j,\psi\phi}](\phi_0(-1), \psi_0(-1); -1), \tag{3.10c}$$

$$Q_{j,\psi} \equiv [f_{j,x\psi} + \phi_0'(-1)f_{j,\phi\psi} + \psi_0'(-1)P_{j,\psi\psi}](\phi_0(-1), \psi_0(-1); -1) \tag{3.10d}$$

for $j = 1, 2$, and

$$b_1 \equiv a_1 + T_1(-1) + \beta_D\psi_0'(-1) + \phi_0'(-1)\ln\left(\frac{2}{\epsilon x'}\right), \tag{3.10e}$$

with

$$
\begin{aligned}
a_1 \equiv &\int_{-1}^{1} \frac{dy}{(y-x)^2}[\phi_0(y) - \phi_0(-1) - (y+1)\phi_0'(-1)] \\
&-\frac{1}{2}\phi_0(-1) - \phi_0'(-1);
\end{aligned} \tag{3.10f}
$$

and similarly,

$$b_2 \equiv a_2 + T_2(-1) - \beta_D\phi_0'(-1) + \psi_0'(-1)\ln\left(\frac{2}{\epsilon x'}\right), \tag{3.10g}$$

with

$$
\begin{aligned}
a_2 \equiv &\int_{-1}^{1} \frac{dy}{(y-x)^2}[\psi_0(y) - \psi_0(-1) - (y+1)\psi_0'(-1)] \\
&-\frac{1}{2}\psi_0(-1) - \psi_0'(-1).
\end{aligned} \tag{3.10h}
$$

(iv) *$\epsilon$-order inner terms*, $\tilde{\phi}_1(x')$, $\tilde{\psi}_1(x')$.

The inner terms, $\tilde{\phi}_1(x')$ and $\tilde{\psi}_1(x')$, are the solutions of the following coupled linear integral equations:

$$
\begin{aligned}
&\int_0^\infty \frac{dy'}{(y'-x')^2}[\tilde{\phi}_1(y') - \epsilon y'\phi_0'(-1)] + \beta_D\frac{d}{dx'}\tilde{\psi}_1(x') \\
&+c_{11}(x')\tilde{\phi}_1(x') + c_{12}(x')\tilde{\psi}_1(x') + \epsilon d_1(x') = 0,
\end{aligned} \tag{3.11a}
$$

$$\int_0^\infty \frac{dy'}{(y'-x')^2}[\tilde{\psi}_1(y') - \epsilon y' \psi_0'(-1)] - \beta_D \frac{d}{dx'}\tilde{\phi}_1(x')$$

$$+c_{21}(x')\tilde{\phi}_1(x') + c_{22}(x')\tilde{\psi}_1(x') + \epsilon d_2(x') = 0, \tag{3.11b}$$

with the conditions

$$\tilde{\phi}_1(0) = \tilde{\psi}_1(0) = 0, \tag{3.11c}$$

where

$$c_{i1}(x') = f_{i,\phi}(\tilde{\phi}_0(x'), \tilde{\psi}_0(x'); -1), \tag{3.11d}$$

$$c_{i2}(x') = f_{i,\psi}(\tilde{\phi}_0(x'), \tilde{\psi}_0(x'); -1) \tag{3.11e}$$

for $i = 1, 2$, and

$$d_1 = a_1 + \phi_0'(-1)\ln\left(\frac{2}{\epsilon x'}\right) + x' \mathrm{f}_{1,x}(\tilde{\phi}_0(x'), \tilde{\psi}_0(x'); -1) + \mathrm{T}_1(-1), \tag{3.11f}$$

$$d_2 = a_2 + \psi_0'(-1)\ln\left(\frac{2}{\epsilon x'}\right) + x' \mathrm{f}_{2,x}(\tilde{\psi}_0(x'), \tilde{\psi}_0(x'); -1) + \mathrm{T}_2(-1). \tag{3.11g}$$

**4. Example.** As an illustration, consider the following example of the special case where the bimaterial constant matrices are given by (3.1a,b), and assume that the term corresponding to the bridging force is defined by

$$\boldsymbol{f}(\mathbf{u}(x);x) = \begin{bmatrix} f_1(\mathbf{u}(x);x) \\ f_2(\mathbf{u}(x);x) \end{bmatrix} = \begin{bmatrix} -u_1(x)+1 \\ -u_2(x)+1 \end{bmatrix}. \tag{4.1}$$

Then the basic system of equations (1.4) reduces to

$$\epsilon\left[\frac{1}{\pi}\int_{-1}^{1}\frac{u_1(\xi)}{(x-\xi)^2}d\xi + \beta_D\frac{d}{dx}u_2(x) + T_1(x)\right] + 1 - u_1(x) = 0, \tag{4.2a}$$

$$\epsilon\left[\frac{1}{\pi}\int_{-1}^{1}\frac{u_2(\xi)}{(x-\xi)^2}d\xi - \beta_D\frac{d}{dx}u_1(x) + T_2(x)\right] + 1 - u_2(x) = 0. \tag{4.2b}$$

We seek a composite expansion of the singular perturbation solution up to the $\epsilon$ order, in the form of (3.4a,b). Now, using the results in section 3, the outer, inner, and matching terms are obtained as follows.

**(1) Lowest-order terms.**

(i) *Lowest-order outer terms.*

From (3.9a,b),

$$\phi_0(x) = \psi_0(x) = 1. \tag{4.3}$$

(ii) *Matching terms*, $S_k^n(x')$, $s_k^n(x')$.

In view of (3.6a) and (4.1),

$$S_0^0(x') = s_0^0(x') = 1. \tag{4.4}$$

(iii) *Inner terms*, $\tilde{\phi}_0(x')$, $\tilde{\psi}_0(x')$.

The lowest-order inner terms, $\tilde{\phi}_0(x')$ and $\tilde{\psi}_0(x')$, are given by the system

$$\int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\phi}_0(y') + \beta_D \frac{d}{dx'}\tilde{\psi}_0(x') + 1 - \tilde{\phi}_0(x') = 0, \tag{4.5a}$$

$$\int_0^\infty \frac{dy'}{(y'-x')^2}\tilde{\psi}_0(y') - \beta_D \frac{d}{dx'}\tilde{\phi}_0(x') + 1 - \tilde{\psi}_0(x') = 0, \tag{4.5b}$$

with the conditions

$$\tilde{\phi}_0(0) = \tilde{\psi}_0(0) = 0. \tag{4.5c}$$

This system of equations is solved below using the Wiener–Hopf technique.

**(2) $\epsilon$-order terms.**

The expressions defined in section 3 and (4.1) are now evaluated to be

$$\Delta(x) = 1, \tag{4.6a}$$

$$f^{(0)}_{1,\psi} = f^{(0)}_{2,\phi} = 0, \qquad f^{(0)}_{1,\phi} = f^{(0)}_{2,\psi} = -1, \tag{4.6b}$$

$$A_1(x) = T_1(x) - \frac{2}{1-x^2}, \qquad A_2(x) = T_2(x) - \frac{2}{1-x^2}, \tag{4.6c}$$

and

$$\phi'{}_0 = \psi'{}_0 = 0. \tag{4.6d}$$

The desired $\epsilon$-order terms are then obtained.

(i) *Matching terms*,

$$S^1_0(x') = s^1_0(x') = 1, \tag{4.7a}$$

$$S^1_1(x') = -\frac{1}{x'} + \epsilon(T_1(-1) - 1/2), \tag{4.7b}$$

$$s^1_1(x') = \frac{1}{x'} + \epsilon(T_2(-1) - 1/2). \tag{4.7c}$$

(ii) *Outer terms*,

$$\phi_1(x) = \epsilon\left(T_1(x) - \frac{2}{1-x^2}\right), \tag{4.8a}$$

$$\psi_1(x) = -\epsilon\left(T_2(x) - \frac{2}{1-x^2}\right). \tag{4.8b}$$

(iii) *Inner terms.*

Using (3.11a,g), with the following values for the parameters,

(4.9a) $$c_{11} = c_{22} = -1, \quad c_{12} = c_{21} = 0,$$

and

(4.9b) $$d_i = T_i(-1) - \frac{1}{2}$$

for $i = 1, 2$; the inner terms, $\tilde{\phi}_1(x')$ and $\tilde{\psi}_1(x')$, satisfy the following system:

(4.10a) $$\int_0^\infty \frac{dy'}{(y'-x')^2}[\tilde{\phi}_1(y')] + \beta_D \frac{d}{dx'}\tilde{\psi}_1(x') \\ -\tilde{\phi}_1(x') + \epsilon(T_1(-1) - 1/2) = 0,$$

(4.10b) $$\int_0^\infty \frac{dy'}{(y'-x')^2}[\tilde{\psi}_1(y')] - \beta_D \frac{d}{dx'}\tilde{\phi}_1(x') \\ -\tilde{\psi}_1(x') + \epsilon(T_1(-1) - 1/2) = 0,$$

and the conditions

(4.10c) $$\tilde{\phi}_1(0) = \tilde{\psi}_1(0) = 0.$$

The systems (4.5a,b) and (4.10a,b) can be treated together, since the only difference between them is due to the constant terms. Systems of this kind can be decoupled by a simple linear transformation. Therefore it suffices to solve the following system of equations:

(4.11a) $$\int_0^\infty \frac{dy'}{(y'-x')^2} g_1(y') - g_1(x') + i\beta_D \frac{d}{dx'} g_1(x') = 1,$$

(4.11b) $$\int_0^\infty \frac{dy'}{(y'-x')^2} g_2(y') - g_2(x') - i\beta_D \frac{d}{dx'} g_2(x') = 1,$$

with the conditions

(4.11c) $$g_1(0) = g_2(0) = 0.$$

It is seen that the two equations of the system (4.11a) are conjugate to each other, so that

(4.12) $$g_2(x') = \bar{g}_1(x').$$

Consequently, the problem reduces to solving a single equation

(4.13a) $$\int_0^\infty \frac{dy'}{(y'-x')^2} g(y') - g(x') + i\beta_D \frac{d}{dx'} g(x') = w(x'),$$

with the condition

(4.13b) $$g(x') = 0.$$

If $g(x')$ is known, then $\tilde{\phi}_0(x')$, $\tilde{\psi}_0(x')$ $\tilde{\phi}_1(x')$, and $\tilde{\psi}_1(x')$ are recovered through

$$\begin{bmatrix} \tilde{\phi}_0(x') \\ \tilde{\psi}_0(x') \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} g+\bar{g} & i(\bar{g}-g) \\ i(g-\bar{g}) & g+\bar{g} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.14a}$$

and

$$\begin{bmatrix} \tilde{\phi}_1(x') \\ \tilde{\psi}_1(x') \end{bmatrix} = -\frac{\epsilon}{2} \begin{bmatrix} g+\bar{g} & i(\bar{g}-g) \\ i(g-\bar{g}) & g+\bar{g} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}. \tag{4.14b}$$

Below, (4.13a) is solved by using the Wiener–Hopf technique, in a similar manner as discussed in Olmstead and Gautesen [13] and Willis and Nemat-Nasser [15]. To employ the Wiener–Hopf technique, apply the Fourier transform to (4.13a) and obtain

$$-(1+|\lambda|-\beta_D\lambda)G_+(\lambda) = W_+(\lambda) + Q_-(\lambda), \tag{4.15a}$$

where

$$G_+(\lambda) = \int_0^\infty g(x')e^{ix'\lambda}dx', \tag{4.15b}$$

$W(x')$ is defined by a similar expression, and $Q_-$ is the Fourier transform of the unknown extension of (4.13a) for $x' < 0$.

A crucial step in the Wiener–Hopf technique is the factorization of $1+|\lambda|-\beta_D\lambda$. It is found that

$$1+|\lambda|-\beta_D\lambda = K_+(\lambda)/K_-(\lambda), \tag{4.16a}$$

where

$$K_\pm(\lambda) = \exp\left\{\frac{1}{2\pi \mathrm{i}}\int_0^\infty \frac{\mathrm{s}[\ln(\mathrm{s}\mp\mathrm{i}(1+\beta_\mathrm{D})\lambda) - \ln(\mathrm{s}\mp\mathrm{i}(1-\beta_\mathrm{D})\lambda)]}{1+\mathrm{s}^2}\mathrm{ds}\right.$$
$$\left.\pm\frac{1}{2\pi}\int_0^\infty \frac{\ln(\mathrm{s}\mp\mathrm{i}(1+\beta_\mathrm{D})\lambda) + \ln(\mathrm{s}\mp\mathrm{i}(1-\beta_\mathrm{D})\lambda)}{1+s^2}ds\right\}. \tag{4.16b}$$

Equation (4.15a) now becomes a Hilbert problem,

$$F_+(\lambda) - F_-(\lambda) = -W_+(\lambda)K_-(\lambda), \tag{4.17a}$$

where

$$F_+(\lambda) = K_+(\lambda)G_+(\lambda), \qquad \mathrm{Im}\lambda > 0, \tag{4.17b}$$

$$F_-(\lambda) = -K_-(\lambda)Q_-(\lambda), \qquad \mathrm{Im}\lambda < 0. \tag{4.17c}$$

A solution for (4.17a) is

$$F(\lambda) = -\frac{1}{2\pi i}\int_{-\infty}^\infty \frac{W_+(\lambda)K_-(\lambda)}{z-\lambda}dz. \tag{4.18}$$

The general solution involves an entire function which is zero in the present case.

When $w(x') = 1$,

$$w_+(\lambda) = i(\lambda + 0i)^{-1}. \tag{4.19}$$

It then follows that

$$G_+(\lambda) = \frac{-i}{\lambda K_+(\lambda)}, \tag{4.20}$$

since $K_-(0) = 1$. Therefore $g(x')$ can be obtained from the Fourier inverse transform,

$$g(x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_+(\lambda)\exp(-ix'\lambda)\mathrm{d}\lambda. \tag{4.21}$$

The inner terms are consequently derived.

It is useful to find the asymptotic form of the solution when $x \to -1$ or equivalently $x' \to 0$. Observe that, as $\lambda \to \infty$,

$$K_+(\lambda) \sim e^{-\frac{3\pi}{4}i}\left(\frac{1-\beta_D}{1+\beta_D}\right)^{1/4}(1-\beta_D^2)^{1/4-i\delta/2}\lambda^{1/2-i\delta}, \tag{4.22a}$$

where

$$\delta = \frac{1}{2\pi}\ln\left(\frac{1+\beta_{\mathrm{D}}}{1-\beta_{\mathrm{D}}}\right). \tag{4.22b}$$

Combining the last equation with (4.20), it follows that

$$G_+(\lambda) \sim e^{\frac{3\pi}{4}i}\left(\frac{1+\beta_D}{1-\beta_D}\right)^{1/4}(1-\beta_D^2)^{-1/4+i\delta/2}\lambda^{-3/2+i\delta}. \tag{4.22c}$$

Therefore, as $x' \to 0$,

$$g(x') \sim \frac{(1-\beta_D^2)^{-1/4+i\delta}}{\Gamma(3/2-i\delta)}(x')^{1/2-i\delta}. \tag{4.23}$$

Consequently, as $x' \to 0$, the asymptotic form of the solution is

$$\boldsymbol{u}(x') \sim \begin{bmatrix} (A+B)+\epsilon(Ad_1+Bd_2) \\ (A-B)+\epsilon(Ad_2-Bd_1) \end{bmatrix}(x')^{\frac{1}{2}}, \tag{4.24a}$$

where real numbers $A$ and $B$ are defined by

$$A+Bi = \frac{(1-\beta_D^2)^{-1/4+i\delta}}{\Gamma(3/2-i\delta)}(x')^{-i\delta}, \tag{4.24b}$$

which confirms the singularity and oscillation of the solution near $x = -1$, and reduces to the results in Willis and Nemat-Nasser [15] when $\delta = 0$.

In the problem of bridged interface cracks, (4.24a) gives the asymptotic solution for the crack-opening-displacement near the crack tip, $x = -1$. The dislocation density and interfacial stress can be derived consequently, which are oscillatory and singular at $x = -1$. Although the elastic field variation of various physical quantities for interface cracks is oscillatory, the energy-release-rate is nonoscillatory. From the asymptotic solution (4.24a) for the crack-opening-displacement, the energy-release-rate, $E$, near the crack tip, $x = -1$, is obtained,

$$E = \frac{1}{2\epsilon}[(1+\epsilon d_1)^2 + (1+\epsilon d_2)^2], \tag{4.25}$$

where $d_i$, $i = 1, 2$, is defined by (4.9b). This energy is not oscillatory, confirming the earlier results of Nemat-Nasser and Ni [9].

REFERENCES


[1] J. S. Angell, and W. E. Olmstead, *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math., 47 (1987), pp. 1–14.

[2] J. Angell and W. E. Olmstead, *Singularly perturbed Volterra integral equations* II, SIAM J. Appl. Math., 47 (1987), pp. 1150–1162.

[3] J. Dundurs, *Discussion of a paper by D. B. Bogy*, Trans. ASME J. Appl. Mech., 36 (1969), pp. 650–652.

[4] I. M. Gel'fand and G. E. Shilov, *Generalized Functions, Volume* 1. *Properties and Operations*, Academic Press, New York, 1964.

[5] J. Hadamard, *Lectures on Cauchy's Problem in Linear Differential Equations*, Dover, New York, 1952.

[6] M. Hori and S. Nemat-Nasser, *Asymptotic solution of a class of strongly singular integral equations*, SIAM J. Appl. Math., 50 (1990), pp. 716–725.

[7] C. G. Lange and D. R. Smith, *Singular perturbation analysis of integral equations*, Stud. Appl. Math., 79 (1988), pp. 1–63.

[8] S. Nemat-Nasser and M. Hori, *Toughening by partial or full bridging of cracks in ceramics and fiber reinforced composites*, Mechanics of Materials, 6 (1987), pp. 245–269.

[9] S. Nemat-Nasser and L. Ni, *Mechanics of interface fracture of anisotropic bimaterials*, in Ultrasonic Characterization and Mechanics of Interfaces, S. I. Rokhlin, S. K. Datta, and Y. D. S. Rajapakse, eds., vol. AMD-177, ASME, New York, 1993, pp. 65–77.

[10] L. Ni and S. Nemat-Nasser, *Interface cracks in anisotropic dissimilar materials: An analytic solution*, J. Mech. Phys. Solids, 39 (1991), pp. 113–144.

[11] L. Ni and S. Nemat-Nasser, *Interface cracks in anisotropic dissimilar materials: General case*, Quart. Appl. Math., 50 (1992), pp. 305–322.

[12] L. Ni and S. Nemat-Nasser, *Bridged interface cracks in anisotropic bimaterials*, Philosophical Magazine A, 2000, to appear.

[13] W. E. Olmstead and A. K. Gautesen, *Asymptotic solution of some singularly perturbed Fredholm equations*, Z. Angew. Math. Phys., 40 (1989), pp. 230–244.

[14] M. D. Van Dyke, *Perturbation Methods in Fluid Mechanics*, annotated ed., The Parabolic Press, Stanford, CA, 1975.

[15] J. R. Willis and S. Nemat-Nasser, *Singular perturbation solution of a class of singular integral equations*, Quart. Appl. Math., 48 (1990), pp. 741–753.