OVERALL PROPERTIES OF ELASTIC–VISCOPLASTIC PERIODIC COMPOSITES

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Abstract—The assumption of a periodic distribution of inclusions gives accurate estimates of the overall moduli of composites, especially when interactions among inclusions are dominant. In this paper periodic composites with rate-dependent elastic–plastic material response including non-linear power-law hardening are considered. Overall stress–strain relations are obtained and the dependence of these relations on the density of the discretization of the unit cell is studied. Several options, such as the use of mirror image symmetry/antisymmetry, partially analytic summation of Fourier series, and a highly accurate and stable time-integration algorithm help to keep the computational expense low. The formulation is presented in three dimensions as well as for plane-stress and plane-strain problems.

I. INTRODUCTION

Various methods have been developed to evaluate the macroscopic behavior of linearly elastic or viscoelastic composites, e.g. the self-consistent method (Hill [1965b], Budiansky [1965]); the differential scheme (Roscoe [1952, 1973]) or the double inclusion method based on Mori-Tanaka's result (Benveniste [1987], Mori & Tanaka [1973], Nemat-Nasser & Hori [1993]). These methods utilize Eshelby's concept of transformation strain (or eigenstrain) (Eshelby [1957]) in order to calculate the stress disturbances produced by inhomogeneities. In the context of linear material response, an important fact is that a constant eigenstrain within an ellipsoidal domain produces a constant stress therein, when the ellipsoid is embedded in an infinite homogeneous matrix material. In applications where other inhomogeneities are also present, the results are accurate only if there is a weak interaction among the neighboring inhomogeneities. While the simplest theory (method of dilute dispersion) neglects interactions completely, the self-consistent method and the double-inclusion model account for interactions in an averaged sense, as does the differential scheme which, due to its structure, may have a wider range of application.

Another approach for taking inhomogeneity interaction into account, is to assume a certain distribution of inclusions within the matrix and compute their interactions through the solution of a boundary-value problem. A natural choice of the distribution of inclusions is a periodic one, since then only one representative unit cell containing one or more inclusions may be considered. Due to the periodicity, the boundary conditions of the unit cell can be satisfied by expressing all variables in Fourier series. A
detailed survey of several varieties of approximations based on this theory is given by Nemat-Nasser et al. [1982] for elastic material behavior. Other elastic solutions of periodic inhomogeneities can be found in Nemat-Nasser and Taya [1981,1985] for spherical holes, Iwakuma and Nemat-Nasser [1983] for ellipsoidal inclusions, and Nemat-Nasser et al. [1993] for penny-shaped cracks; see Nemat-Nasser and Hori [1993] for further comments. For an alternative approach using the method of cells, see Paley et al. [1992]. It appears that the assumption of periodic composites provides better estimates of inclusion interaction compared to most other alternatives. The resulting boundary-value problem can be solved to within any desired degree of accuracy, and seems to yield excellent results even for randomly distributed inhomogeneities (Nemat-Nasser et al. [1982]). The accuracy of the solution can be improved further by subdiscretization of the unit cell, while other methods do not provide means of refinement. Walker et al. [1991, 1993a] use a subdiscretization consisting of cuboidal (rectangular) subvolumes, to obtain the local stress distributions in periodic composites. Within such a theory, upper and lower bounds based on Hashin and Shtrikman's [1963] formulation may also be calculated to any level of accuracy. These positive features, however, are at the price of an increased amount of numerical computation.

The self-consistent method has also been used when plastic deformations are taken into account, e.g. in the calculation of deformations of polycrystals; Budiansky and Wu [1962], Hill [1965a], Hutchinson [1970], Nemat-Nasser and Iwakuma [1984] and Nemat-Nasser and Obata [1986]. Applications of the self-consistent estimate to elastic-plastic composites have been considered by Nemat-Nasser and Iwakuma [1985], and periodic distributions of ellipsoidal cavities in a visco-elastic matrix have been investigated by Nemat-Nasser et al. [1986]. Accorsi and Nemat-Nasser [1986] calculate bounds of overall elastoplastic moduli of periodic composites with rate-independent linear strain-hardening materials. Walker et al. [1994] calculate the viscoplastic response of two-dimensional rectangular unit cells with triangular subregions.

In the present paper, we treat periodically distributed anisotropic inclusions in an isotropic rate-dependent matrix including power-law strain-hardening. In addition, we include several options to decrease computational expense, which is the only major drawback of this method. The most time-consuming elements in this approach are the summation of multiple Fourier series in obtaining the Green's function of a periodic domain, the space-subdiscretization of the unit cell, and the time discretization due to the nonlinear constitutive behavior. All three of these subjects will be addressed by advanced computational algorithms in order to accelerate the resulting performance. Space discretization will be reduced by a factor of 8 in 3D, and 4 in 2D, by considering mirror-image symmetries. This cuts down the terms of Fourier series to only non-negative coordinates in the Fourier space. A partly analytic summation reduces the order of Fourier sums by one, leaving the remaining sums rapidly convergent; Fotiu [1994]. Finally, time discretization is minimized by applying an accurate and unconditionally stable time-integration algorithm for elastoplastic constitutive equations based on the generalized midpoint rule.

In the case of elastic material behavior, numerical studies (Nemat-Nasser et al. [1993] for penny shaped cracks, Fotiu [1994] for cuboidal inclusions) have shown that a subdiscretization of the unit cell (in this case, only of the inclusion) leads to minor improvement of the estimate of overall properties, as the results are already quite accurate without such refinements. This, however, might not be the case when plastic deformations are considered, mainly because in a nonlinear case, the averaged quantities no
longer obey the same equations as the local quantities. Therefore, we present a detailed study of the micromechanics of the matrix-inclusion composite using different degrees of subdiscretization.

The paper presents a three-dimensional formulation as well as a two-dimensional one for plane-strain and plane-stress. Many of the equations in two dimensions will be analogous to those in three dimensions, and in that case we will present only the three-dimensional ones. Whenever more distinct differences between these cases occur, we will make an explicit reference to the two-dimensional problem.

Latin indices $i, j, k, l, \text{etc.} = 1, \ldots, 3 \ (= 1, 2 \text{ in } 2D)$ obey the conventional rules of tensor index notation including the Einstein summation convention. Greek indices $\alpha, \beta, \gamma, \text{etc.} = 1, \ldots, 3 \ (= 1, 2 \text{ in } 2D)$ do not obey the summation convention, and each symbol stands for a different number, that is, $\alpha \neq \beta \neq \gamma$, etc., always.

II. REPRESENTATIVE UNIT CELL

Consider a parallelepipedic (rectangular in 2D) unit cell with volume $V = V^I + V^M$ (area $A = A^I + A^M$)\(^\dagger\) containing a centered inhomogeneity with volume $V^I$ (area $A^I$) of the same shape (see Fig. 1). The dimensions of the cell are given by $\Lambda_i, i = 1, 2, 3 \ (i = 1, 2)$. This inclusion shape allows an extremely simple numerical treatment when a further discretization into subcells is performed, and hence, the corresponding improvements can be assessed directly. For small to moderate volume fractions of inclusions, the inclusion shape seems to have minor influence on the overall properties of the material, and even for relatively large volume fractions we might still obtain satisfactory approximations by choosing a cuboidal form similar to the real inclusion shape (e.g. substitute a sphere (circle) by a cube (square) of the same volume (area)). In particle-reinforced materials, the shape of the particles is at best given only in rough

\(^\dagger\)Here and throughout the paper superscripts $M$ and $I$ refer to the matrix and the inclusion, respectively.
dimensions, and assuming them as a cuboid (rectangle) would be as good as, e.g. an ellipsoid (ellipse). The latter form is taken mostly because then the homogenizing eigenstrains within the inclusion are uniform, when the ellipsoid is embedded in an infinite solid. For a periodic distribution of inclusions, however, the homogenizing eigenstrains are no longer uniform.

Let the overall strain rate\(^1\) be \(\dot{\varepsilon}^0\), so that \(\dot{\varepsilon} = \langle \dot{\varepsilon}^0 \rangle\), where the total strain rate is

\[
\dot{\varepsilon}^t = \dot{\varepsilon}^0 + \dot{\varepsilon}, \quad \dot{\varepsilon}_{ij} = (\dot{u}_{ij} + \dot{u}_{ji})/2,
\]

with \(\varepsilon\) being the perturbation strain due to the presence of the inclusion and any possible plastic or thermal strains, and \(u\) being the corresponding displacements. The relations between the strain rate and total stress rate \(\dot{\sigma}^t\), in the matrix and in the inclusion are,

\[
\dot{\sigma}^t = C^M : (\dot{\varepsilon}^0 + \dot{\varepsilon} - \dot{\varepsilon}^p), \quad \text{x in } V^M, \quad \dot{\sigma}^t = C^I : (\dot{\varepsilon}^0 + \dot{\varepsilon} - \dot{\varepsilon}^p), \quad \text{x in } V^I.
\]

Here, \(C\) is the elasticity tensor and \(\dot{\varepsilon}^p\) denotes the plastic strain rate.

Introduce transformation strain rate \(\dot{\varepsilon}^*\) to homogenize the unit cell. Thus, set

\[
C^I : (\dot{\varepsilon}^0 + \dot{\varepsilon} - \dot{\varepsilon}^p) = C^M : (\dot{\varepsilon}^0 + \dot{\varepsilon} - \dot{\varepsilon}^p - \dot{\varepsilon}^*), \quad \text{x in } V^I.
\]

Then,

\[
\dot{\varepsilon}^* = A : (\dot{\varepsilon}^0 + \dot{\varepsilon} - \dot{\varepsilon}^p),
\]

with

\[
A = I - (C^M)^{-1} C^I, \quad \text{x in } V^I
\]

\[
A = 0, \quad \text{x in } V^M,
\]

where \(I\) is the identity tensor. The periodicity of the perturbation fields is guaranteed by Fourier series expansions, for example

\[
\dot{\varepsilon}(x) = \sum_{n=-\infty}^{+\infty} \tilde{\varepsilon}(\xi) \exp(i \cdot \xi), \quad p = 1, 2, 3 (= 1, 2 \text{ in } 2D), \quad i = \sqrt{-1},
\]

where

\[
\tilde{\varepsilon}(\xi) = \frac{1}{V} \int_V \dot{\varepsilon}(x) \exp(-i \cdot \xi) dV, \quad \xi_o = \frac{2 \pi n_o}{\Lambda_o}.
\]

The prime on the summation sign indicates that the term with \(n_1 = n_2 = n_3 = 0\) is excluded. This term corresponds to the mean value over the volume \(V\), and is zero since the averages of the disturbance fields vanish identically in a periodic structure.

In Fourier space, the following relation holds between \(\tilde{\varepsilon}^*, \tilde{\varepsilon}^p\), and the resulting strain disturbance \(\tilde{\varepsilon}\):

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\(^1\)The rate formulation is necessary for plasticity.

\(^2\)Averages over the unit cell volume are denoted by \(\langle \cdot \rangle\), while averages over subvolumes are marked by an overbar.
\( \mathbf{\ddot{e}} = \mathbf{\ddot{S}} : (\mathbf{\dot{e}}^* + \mathbf{\dot{e}}^p) \),
\( (8) \)

where \( \mathbf{\ddot{S}}(\xi) \) is given by
\[
\mathbf{\ddot{S}}(\xi) = \mathbf{\ddot{F}}(\xi) : \mathbf{C}^M, \quad \mathbf{\ddot{F}} = \text{sym} \left\{ \xi \otimes (\xi \cdot \mathbf{C}^M \cdot \xi)^{-1} \otimes \xi \right\}. \tag{9}
\]

In eqn (9), sym denotes the symmetric part of the fourth-order tensor, i.e. \( \text{sym}\{A_{ijkl}\} = (1/4) (A_{ijkl} + A_{jikl} + A_{ijlk} + A_{jkil}) \) and \( \otimes \) stands for the dyadic tensor product.

Introducing eqns (6) and (8) into (4) obtain the consistency condition,
\[
A : \left\{ \mathbf{\dot{e}}^0(x) - \mathbf{\dot{e}}^p(x) + \frac{1}{V} \sum_{n_\rho = -\infty}^{+\infty} \mathbf{\ddot{S}} : \left[ \int_{V} (\mathbf{\dot{e}}^*(y) + \mathbf{\dot{e}}^p(y)) \exp(-iy \cdot \xi) dV(y) \right] \right. \\
\left. \exp(ix \cdot \xi) \right\} - \mathbf{\ddot{e}}^*(x) = 0. \tag{10}
\]

This is an integral equation for \( \mathbf{\ddot{e}}^*(x) \) when \( \mathbf{\dot{e}}^0(x) \) and \( \mathbf{\dot{e}}^p(x) \) are known; Nemat-Nasser and Taya [1981, 1985] and Nemat-Nasser et al. [1986].

II.1 Averaging over subcells

Of interest here are the average values of strain rates rather than their exact distribution. Therefore, subdivide \( V \) into \( N \) subcells with dimensions \( l_{\alpha}^r, \ r = 1, \ldots, N, \) and volume \( v^r = l_{\alpha}^r l_{\beta}^r l_{\gamma}^r \) (no sum on \( r \))\(^1\) and approximate the integrals\(^2\) over \( V \) in eqn (10) by a sum of integrals over \( v^s \)'s, replacing \( \mathbf{\ddot{e}}^* \) and \( \mathbf{\dot{e}}^p \) in each subcell by their average values \( \mathbf{\ddot{e}}^{s*} \),
\[
A' : (\mathbf{\dot{e}}^0 - \mathbf{\dot{e}}^p) - \mathbf{\ddot{e}}^{s*} + \sum_{s=1}^{N} A' : S^{rs} : (\mathbf{\ddot{e}}^{s*} + \mathbf{\ddot{e}}^{p*}) = 0, \tag{11}
\]

where the tensor \( S^{rs} \) is
\[
S^{rs} = f^s \sum_{n_\rho = -\infty}^{+\infty} g^r(\xi) g^s(-\xi) \mathbf{\ddot{S}}(\xi), \tag{12}
\]

with
\[
g^r(\xi) = \frac{1}{v^r} \int_{v^r} \exp(ix \cdot \xi) dV(x), \quad f^r = \frac{v^r}{V}, \tag{13}
\]

characterizing the geometry of each subcell. For a parallelepiped, it follows that
\[
g^r(\xi) = Q' \exp(i\xi' \cdot \xi), \quad Q' = \frac{\sin a_1^r \sin a_2^r \sin a_3^r}{a_1^r a_2^r a_3^r}, \quad a_\alpha = \frac{\xi_\alpha l_{\alpha}}{2}, \tag{14}
\]

\(^1\)Superscripts \( r \) and \( s \) indicate the subcells and are not subject to the summation convention.
\(^2\)For linearly elastic materials, Nemat-Nasser and Taya [1981] have shown that such an approximation yields excellent results, even when there is no subdiscretization (\( N = 1 \)) and the averaging is performed over the whole inclusion, \( v^i = V^i \). This has been further confirmed by the results reported in Nunan and Keller [1984].
where \( \mathbf{x}' \) is the coordinate of the center of the \( r \)th subcell. The solution for a rectangle in the \( x_1, x_2 \)-plane is simply found by setting \( \alpha' = 0 \) in eqn (14). Since \( \mathbf{S}' \) is real-valued, only the real part of the product \( g^r(\mathbf{x}) g^s(-\mathbf{x}) \) enters eqn (12). This is given by (Accorsi & Nemat-Nasser [1986]),

\[
\text{Re}(g^r(\mathbf{\xi})g^s(-\mathbf{\xi})) = Q^r Q^s \cos(\mathbf{\xi} \cdot (\mathbf{x}' - \mathbf{x}^s)).
\]  

(15)

In the sequel we drop the superscripts \( r \) and \( s \), and use matrix notation, where vectors are written in lower case, e.g. \( \mathbf{\tilde{e}} = [\mathbf{\tilde{e}}_{11}, ..., \mathbf{\tilde{e}}_{12}, ..., \mathbf{\tilde{e}}_{N1}, ..., \mathbf{\tilde{e}}_{N2}]^T \), and tensors, in upper case letters; note that the matrix \( \mathbf{S} \) is fully populated, while \( \mathbf{A} \) has the form \( \mathbf{A} = \text{diag}[\mathbf{A}^1, ..., \mathbf{A}^N] \), with \( \mathbf{A}^r, r = 1, ..., N, \) given by eqn (5). In this notation, the solution of eqn (11) for \( \mathbf{\tilde{e}}' \) is

\[
\mathbf{\tilde{e}}' = (\mathbf{I} - \mathbf{A} \mathbf{S})^{-1} \mathbf{A} (\mathbf{\tilde{e}}^0 - \mathbf{\tilde{e}}^p + \mathbf{S} \mathbf{\tilde{e}}^p) = \mathbf{AB}(\mathbf{\tilde{e}}^0 - (\mathbf{I} - \mathbf{S})\mathbf{\tilde{e}}^p),
\]

(16)

with

\[
\mathbf{B} = (\mathbf{I} - \mathbf{S} \mathbf{A})^{-1}.
\]

(17)

Now the total strain rate in each subvolume is found from eqns (1) and (4), and the stress rate is obtained by introducing these into eqn (2),

\[
\mathbf{\tilde{\epsilon}}' = \mathbf{\tilde{\epsilon}}^0 + \mathbf{\tilde{\epsilon}} = \mathbf{A}^{-1} \mathbf{\tilde{\epsilon}} + \mathbf{\tilde{\epsilon}}^p = \mathbf{B}(\mathbf{\tilde{\epsilon}}^0 - (\mathbf{I} - \mathbf{S})\mathbf{\tilde{\epsilon}}^p) + \mathbf{\tilde{\epsilon}}^p. \tag{18a}
\]

\[
\mathbf{\tilde{\sigma}}' = \mathbf{C}^M(\mathbf{\tilde{\epsilon}}^0 + \mathbf{\tilde{\epsilon}} - \mathbf{\tilde{\epsilon}}^p - \mathbf{\tilde{\epsilon}}^p) = \mathbf{C}^M(\mathbf{I} - \mathbf{A})\mathbf{B}(\mathbf{\tilde{\epsilon}}^0 - (\mathbf{I} - \mathbf{S})\mathbf{\tilde{\epsilon}}^p). \tag{18b}
\]

II.2 Averaging over the whole inclusion

The crudest averaging procedure dispenses with a discretization into subcells and applies averages over the whole inclusion. Because this case differs somewhat from the above formulation, the results are presented here explicitly. Averaging over \( V^I \) yields, instead of eqn (11),

\[
\mathbf{A} : \left( \mathbf{\tilde{\epsilon}}^0 - \mathbf{\tilde{\epsilon}}^p \right) + \mathbf{A} : \mathbf{S}^I : \left( \mathbf{\tilde{\epsilon}}^I + \mathbf{\tilde{\epsilon}}^p - \mathbf{\tilde{\epsilon}}^p \right) - \mathbf{\tilde{\epsilon}}^I = 0, \tag{19}
\]

with

\[
\mathbf{S}^I = f \sum_{\eta_i = -\infty}^{\infty} g^I(\mathbf{\xi})g^I(-\mathbf{\xi})\mathbf{\bar{S}}(\mathbf{\xi}), \tag{20}
\]

\[
g^{I,M}(\mathbf{\xi}) = \frac{1}{V^{I,M}} \int_{V^{I,M}} \exp(i\mathbf{x} \cdot \mathbf{\xi})dV(\mathbf{x}) \tag{21}
\]

\[
g^M = \frac{f}{1-f}g^I, \quad f = \frac{V^I}{V}. \tag{22}
\]

The functions \( g^I(\mathbf{\xi}) = g^I(-\mathbf{\xi}) = Q^I \) are given by eqn (14) with \( I \) substituted for \( r \) and with \( l^I_\alpha \) being the lengths of the inclusion.
In matrix notation, equations similar to eqns (16)–(18) are obtained by averaging over the inclusion and the matrix. This gives, e.g. \( \tilde{\epsilon} = [\tilde{\epsilon}_{[11]}, \ldots, \tilde{\epsilon}_{[12]}, \tilde{\epsilon}_{[11]}, \ldots, \tilde{\epsilon}_{[12]}]^T \). However, in view of eqn (22), only \( S^I \) needs to be established. Matrix \( S \) then has the following form:

\[
S = \begin{bmatrix}
S^I & -S^I \\
-\frac{1}{I-f} S^I & \frac{1}{I-f} S^I
\end{bmatrix}
\]  

(23)

Accordingly, \( B \) is found to be

\[
B = \begin{bmatrix}
B^I & 0 \\
-\frac{1}{I-f} (B^I - I) & I
\end{bmatrix}
\]  

(24)

where

\[
B^I = (I - S^I A)^{-1},
\]

(25)

and \( 0 \) is the zero matrix.

II.3 Isotropic matrix material

Assume an isotropic matrix material with elasticity tensor

\[
C_{ijkl}^M = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj}), \quad i,j,k,l = 1, \ldots, 3 \quad (= 1, 2 \text{ in 2D}),
\]

(26)

where \( \delta_{ij} \) is the Kronecker symbol, and \( \lambda \) and \( \mu \) are the Lamé constants. We may write \( \lambda = 2\mu\zeta \), where \( \zeta \) is a function of Poisson’s ratio \( \nu \),

\[
\zeta = \frac{\nu}{1 - 2\nu} \quad \text{in 3D}; \quad \zeta = \frac{3 - \kappa}{2(\kappa - 1)} \quad \text{in 2D}.
\]

(27)

The parameter \( \kappa \) is given by

\[
\kappa = 3 - 4\nu \quad \text{for plane strain}; \quad \kappa = \frac{3 - \nu}{1 + \nu} \quad \text{for plane stress}.
\]

(28)

With an isotropic \( C^M \), the Fourier representations of \( \tilde{F} (\xi) \) and \( \tilde{S} (\xi) \) take on especially simple forms,

\[
\tilde{F}_{ijkl} (\xi) = \frac{1}{2\mu} \left[ \frac{1}{2} (\xi_i (\delta_{ij} \tilde{\xi}_k + \delta_{jk} \tilde{\xi}_i) + \xi_j (\delta_{il} \tilde{\xi}_k + \delta_{ik} \tilde{\xi}_j)) - \gamma \xi_i \tilde{\xi}_j \tilde{\xi}_k \tilde{\xi}_l \right],
\]

(29)

\[
\tilde{S}_{ijkl} (\xi) = \frac{1}{2} (\xi_i (\delta_{ij} \tilde{\xi}_k + \delta_{jk} \tilde{\xi}_i) + \xi_j (\delta_{il} \tilde{\xi}_k + \delta_{ik} \tilde{\xi}_j)) - \gamma \xi_i \tilde{\xi}_j \tilde{\xi}_k \tilde{\xi}_l + \gamma \tilde{\xi}_i \tilde{\xi}_j \delta_{kl},
\]

(30)

where

\[
\tilde{\xi}_i = \xi_i / \xi, \quad \xi = (\xi_i \xi_j)^{1/2}
\]

(31)
\[
\gamma = \frac{1}{1 - \nu}, \quad \gamma' = \frac{\nu}{1 - \nu} \quad \text{in 3D}; \quad (32)
\]
\[
\gamma = \frac{4}{1 + \kappa}, \quad \gamma' = \frac{3 - \kappa}{1 + \kappa} \quad \text{in 2D.} \quad (33)
\]

The specific components of \( \hat{\mathbf{S}}(\xi) \) can be found in Nemat-Nasser et al. [1982]. For numerical purposes, however, it is better to evaluate \( \mathbf{S} \) by summing the individual terms of \( \hat{\mathbf{S}}(\xi) \) and multiplying by \( \mathbf{C}^{\mu} \) afterwards, since \( \hat{\Gamma}_{ijkl} \) is symmetric in the pairs \( (i,j) \) and \( (k,l) \), while \( \hat{S}_{ijkl} \) is not. The individual components of \( \hat{\Gamma}(\xi) \), as given by eqn (29), are:

\[
\begin{align*}
2\mu \hat{\Gamma}_{\alpha\alpha\alpha\alpha} &= 2\xi^2_\alpha - \gamma \xi^4_\alpha, \\
2\mu \hat{\Gamma}_{\alpha\alpha\beta\beta} &= -\gamma \xi^2_\alpha \xi^2_\beta, \\
2\mu \hat{\Gamma}_{\alpha\alpha\beta\gamma} &= -\gamma \xi^2_\alpha \xi^2_\beta \xi^2_\gamma, \\
2\mu \hat{\Gamma}_{\alpha\beta\beta\gamma} &= \xi^2_\alpha \xi^2_\beta - \gamma \xi^2_\alpha \xi^2_\beta \xi^2_\gamma.
\end{align*} \quad (34)
\]

From the above components, \( \hat{\mathbf{S}} \) is obtained to be

\[
\hat{S}_{ijkl} = 2\mu \left( \hat{\Gamma}_{ijkl} + \zeta \hat{T}_{ijmm} \delta_{kl} \right). \quad (35)
\]

II.4 Anisotropic inclusion material: Cubic symmetry

The averaged Eshelby tensor according to eqn (35) may be used for any type of inclusion material. Isotropy or anisotropy of the inclusion is represented through the tensor \( \mathbf{A} \) only. Let us assume for example that the inclusion material has cubic symmetry with respect to the coordinate axes of the unit cell,

\[
C^1_{ijkl} = \lambda^1 \delta_{ij} \delta_{kl} + \mu^1 (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \mu^1 \delta_{ijkl}. \quad (36)
\]

In eqn (36), \( \delta_{ijkl} \) equal to one if \( i = j = k = l \) and zero otherwise. With eqns (26) and (36) introduced into eqn (5), \( \mathbf{A} \) also has cubic symmetry,

\[
A_{ijkl} = \frac{(1 + \zeta^1) \zeta - \zeta^1}{1 + \rho \zeta} \bar{\mu} \delta_{ij} \delta_{kl} + \frac{1}{2} (1 - \bar{\mu})(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \zeta^1 \bar{\mu} \delta_{ijkl}, \quad (37)
\]

where

\[
\bar{\mu} = \mu^1 / \mu, \quad \zeta^1 = \frac{\lambda^1}{2\mu^1}, \quad \zeta^1 = \frac{\mu^1}{2\mu^1}, \quad \rho = 2 \quad \text{in 2D}; \quad 3 \quad \text{in 3D.} \quad (38)
\]

The isotropic form of \( \mathbf{A} \) may be found simply by setting \( \zeta^1 = 0 \).

II.5 Tensor \( \mathbf{S} \) in plane problems

In obtaining two-dimensional nominal overall moduli in the \( x_1, x_2 \)-plane, it is not necessary to consider the 33-component of the transformation strain rate \( \dot{\varepsilon}^{\text{tr}} \). However,

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1In estimating the effective in-plane Young's modulus, however, the influence of the third dimension must be taken into account; see Nemat-Nasser and Hori [1993] for further comments.
there is always a nonzero 33-component of the plastic strain rate. In plane-strain problems, \( \varepsilon_{33}^p \) contributes also to the in-plane components of the perturbation strain rate. Since \( \varepsilon_{33}^p = -(\varepsilon_{11}^p + \varepsilon_{22}^p) \), \( \mathbf{S} \) in eqn (18) has to be replaced by (Appendix B)

\[
\bar{S}_{ijkl} = \Gamma_{ijkl} \bar{C}_{mnkl}^M, \quad \bar{C}_{mnkl}^M = C_{mnkl}^M - C_{mn33}^M \delta_{kl}, \quad i,j,k,l,m,n = 1,2. \tag{39}
\]

However, matrix \( \mathbf{B} \), eqn (17), which refers to \( \dot{\varepsilon}^* \), still has to be calculated with \( \mathbf{S} = \mathbf{I} : \mathbf{C}^M \). Hence, the plane-strain problem requires different Eshelby tensors for the transformation strain and the plastic strain. Furthermore, we point out that in the two-dimensional formulation presented here, we use two-dimensional tensors from the onset. The overall moduli obtained in this way are, in general, different from those derived by a three-dimensional analysis and a subsequent reduction to two dimensions; for a more detailed discussion of this issue we refer to Nemat-Nasser and Hori [1993].

With \( \mathbf{C}^M \) being isotropic, \( \bar{C}_{ijkl}^M \) is obtained from eqn (39) as

\[
\bar{C}_{ijkl}^M = 2 \mu \delta_{ik} \delta_{jl}. \tag{40}
\]

Contrary to the plane-strain case, the 33-component of the eigenstrain rate in plane stress does not have any influence on the in-plane components of \( \dot{\varepsilon} \) or \( \dot{\sigma} \) (Appendix B). Consequently, the same Eshelby tensor \( \mathbf{S} \) appears both in eqns (17) and (18) in the plane-stress case.

III. MIRROR-IMAGE SYMMETRIES

If the unit cell and the external loading show a certain degree of symmetry, only a part of the unit cell might be actually considered in numerical computations. Any external loading can be suitably decomposed into symmetric and antisymmetric parts. If the material properties are symmetric with respect to all coordinate planes, each mirror image will represent a closed set, that is, they can be evaluated independently. For linearly elastic materials, these mirror images can then be superposed to yield the final response. For elastic–plastic materials superposition must be performed at each instant on the rate quantities. For materials with a yield surface, it will then be necessary to check whether or not the yield conditions are satisfied. For illustration, and in order to retain the advantages of the mirror-image decomposition, we consider here only separate sets of external loading, each of which falls completely into one symmetry class. In addition, we only account for such kinds of anisotropy of the inclusion, which preserves symmetry with respect to all coordinate planes.

An arbitrary tensor field \( T_{ijkl}(\mathbf{x}) \) can be decomposed into eight (four in 2D) distinct mirror images \( T_{ijkl}^{(u)}(\mathbf{x}) \), \( u = \pm 1, ..., \pm 4 \). Following Nemat-Nasser and Hori [1993], we designate the completely symmetric portion of \( \mathbf{T} \) by \( \mathbf{T}^{(4)} \) and the completely antisymmetric part by \( \mathbf{T}^{(-4)} \). With \( \mathbf{T}^{(u)}, u = 1, ..., 3 \), we denote for each \( u \) the image, which is antisymmetric with respect to the plane \( x_u = 0 \) and symmetric with respect to all other planes. Then \( \mathbf{T}^{(-u)} \), for each \( u \) is the image, which is symmetric with respect to \( x_u = 0 \) and antisymmetric with respect to all other planes. In two dimensions, we have only four symmetric/antisymmetric parts, \( T_{ijkl}^{(u)}(\mathbf{x}), u = 1, 2, \pm 4 \).

Consider the loading cases of (i) normal strain rates, \( \varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{33}^0 \) in any combination and (ii) simple shear strain rates \( \varepsilon_{23}^0, \varepsilon_{31}^0 \) or \( \varepsilon_{12}^0 \), all of which are represented by one mirror image (see Table 1).
Table 1. Classes of mirror-image symmetries/antisymmetries for different external loadings

<table>
<thead>
<tr>
<th>External loading</th>
<th>Mirror image index</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_{11}^0$, $\varepsilon_{22}^0$, $\varepsilon_{33}^0$</td>
<td>4</td>
</tr>
<tr>
<td>$\varepsilon_{23}^0$</td>
<td>-1</td>
</tr>
<tr>
<td>$\varepsilon_{\bar{3}1}^0$</td>
<td>-2</td>
</tr>
<tr>
<td>$\varepsilon_{12}^0$</td>
<td>-3</td>
</tr>
<tr>
<td>$\varepsilon_{11}^0$, $\varepsilon_{22}^0$</td>
<td>4</td>
</tr>
<tr>
<td>$\varepsilon_{12}^0$</td>
<td>-4</td>
</tr>
</tbody>
</table>

With the elastic constants $C^M(x), C^I(x)$ being completely symmetric, eqn (8) holds also for each mirror image, i.e.

$$\mathfrak{e}^{(u)} = \widehat{S} : \left( \mathfrak{e}^{(u)} + \mathfrak{e}^{(u)} \right) \quad (41)$$

However, instead of eqn (6), $\mathfrak{e}^{(u)}(x)$ is given by the following series:

$$\mathfrak{e}^{(u)}_{k,l}(x) = \sum_{n_p=0}^{\infty} w(n_p) \mathfrak{e}^{(u)}_{k,l}(\xi) \exp(u')(ix \cdot \xi), \quad (42)$$

where $u'(u; k, l)$ depends on the mirror image and the components of $\mathfrak{e}$. The function $w(n_p)$ in eqn (42) is a weighting factor, $w(n_p) = 2^K$, where $K$ is equal to the number of nonzero components of the vector $n$. The dependencies of $u'$ on $u$, $k$, $l$ are shown explicitly in Table 2.

The individual symmetric and antisymmetric parts of the complex kernel $\exp(i \xi \cdot x)$ as they appear in Table 2 are given as follows

$$\exp(i \xi \cdot x) = c_1 c_2 c_3$$

$$\exp(-\alpha)(i \xi \cdot x) = -c_\alpha s_\beta s_\gamma, \quad \alpha, \beta, \gamma = 1, \ldots, 3, \quad (43a)$$

2D:

$$\exp(i \xi \cdot x) = c_1 c_2$$

$$\exp(-i)(i \xi \cdot x) = -s_1 s_2, \quad (43b)$$

where

$$c_\alpha = \cos(\xi_\alpha x_\alpha), \quad s_\alpha = \sin(\xi_\alpha x_\alpha) \quad (44)$$

Table 2. Mirror-image index $u'(u; k, l)$ of complex kernel $\exp(i \xi \cdot x)$

<table>
<thead>
<tr>
<th>$u'$</th>
<th>11</th>
<th>22</th>
<th>33</th>
<th>23</th>
<th>31</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>-1</td>
<td>-2</td>
</tr>
<tr>
<td></td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-3</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>2D</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td>-4</td>
</tr>
</tbody>
</table>
With the above results we are now able to cast eqn (12) into a computationally more attractive form, which only involves a summation over non-negative components of \( n_p \),

\[
S_{klmn}^{rs(u)} = 2^p \rho^r \sum_{n_p=0}^{+\infty} w(n_p) g^{r(u')}(u; k, l; \xi) g^{s(u')}(u; m, n; -\xi) \tilde{S}_{klmn}(\xi),
\]

(45)

where

\[
g^{r(u')}(u; k, l; \xi) = Q^r \exp^{i(u')}(i\xi \cdot x'),
\]

(46)

and \( \rho \) is defined by eqn (38).

Note that due to the dependence of \( u' \) on \( k, l, m, n \) functions \( g^r, g^s \) now differ not only because of their dependence on different space coordinates \( x^r \) and \( x^s \), but also because of association with possibly different components of \( \tilde{S}(\xi) \).

IV. REDUCTION OF THE ORDER OF FOURIER SUMS

To evaluate \( S \), triple (double in 2D) summations need to be performed over trigonometric functions with coefficients which are of the order \( 1/n_p \). The convergence of such sums is rather poor. This necessitates a large number of terms to be included in the summation, which is a numerically costly process, especially in three dimensions and when fine subdiscretizations are used. However, the special structure of \( g^r \) for cuboidal geometries allows us to evaluate one sum analytically and thereby transform the remainder into a much more rapidly converging series. The details of the derivation of these reduced Fourier sums are given in Fotiu [1994].

From eqn (35) we obtain the averaged Eshelby tensor \( S_{ijkl}^{rs(u)} \) as

\[
S_{ijkl}^{rs(u)} = \tilde{\Gamma}_{ijkl}^{rs(u)} + \zeta \chi_{ijkl}^{rs(u)} \delta_{kl},
\]

(47)

where \( \tilde{\Gamma}_{ijkl}^{rs(u)} = 2\mu \chi_{ijkl}^{rs(u)} \), and \( \zeta \) is defined by eqn (27). The reduced series expressions for \( \tilde{\Gamma} \) are presented in Appendix A for both the three-dimensional and the plane problems.

To get an idea of the savings due to the reduced summation, we recall that in 3D the conventional triple sum \( \sum_{n_p=0}^{+\infty} \) has \( (N+1)^3 - 1 \) terms, whereas the reduced sums consist of at most \( N^2 + 2N \) terms (the additional \( 2N \) terms appear only in the \( \alpha\alpha\alpha\alpha \) \((\alpha = 1, ..., 3)\) components in the completely symmetric constellation, see (A.1a), and in the \( \beta\gamma\beta\gamma \) components in the antisymmetric case \( u = -\alpha \), see (A.2n)). Numerical comparisons have shown that each of the remaining reduced sums converges faster than each of the original sums, thus, requiring fewer terms to achieve the same accuracy. A conservative estimate was found to be \( N/2 \). Therefore, the number of summations can be reduced at least by a factor of

\[
s = \frac{N/4 + 1}{N^2 + 3(N + 1)}.
\]

(48)

Taking for example \( N = 50 \), which is reasonable to achieve sufficient accuracy of the triple sum, we would obtain from eqn (48) \( s = 1/197 \). We emphasize that this is only a conservative estimate. Fotiu [1994] has shown an example, where accurate values of some components of \( \tilde{\Gamma} \) can be reached with a savings factor of up to \( s = 1/9000 \).
An alternative method of accelerating the summations of the Fourier series is used by Walker et al. [1993], who employ the Poisson sum formula to the Fourier series expression of $S$. As it has been shown by the same authors earlier (Walker et al. [1990]) the Poisson sum changes the Fourier representation of $S$ into the corresponding Green’s function in real space. The latter has to be summed over all inclusions (which again is a triple series). The convergence of this summation is reported to be much faster than that of the Fourier series; (see Nemat-Nasser and Hori [1993]) for comments and additional references.

V. ELASTIC–PLASTIC CONSTITUTIVE RELATIONS

V.1 Rate formulation

Consider a rate-dependent elastic–plastic model with isotropic power-law strain hardening. The flow rule is assumed to be

$$
\dot{\varepsilon}^p = \frac{3}{2} \dot{\varepsilon}_0 \left( \frac{\sigma^p}{k} \right) \frac{\sigma^p}{k},
$$

or in a scalar form

$$
\dot{\gamma}^p = \left( \frac{\sigma}{k} \right)^\frac{1}{n}, \quad \dot{\gamma} = \sqrt{\frac{2}{3} \dot{\gamma}^p \dot{\gamma}^p}, \quad \sigma = \sqrt{\frac{3}{2} \sigma^p \sigma^p},
$$

where $\dot{\gamma}^p$ is the effective plastic strain rate, $\sigma$ is the effective stress, and $\sigma^p$ denotes the deviatoric part of the Cauchy stress $\sigma$. Furthermore, $\varepsilon_0$ is a reference strain rate and the exponent $n$ is called the rate-sensitivity parameter. Rate-independent plasticity is included in this model as a limiting case, where $n = 0$.

Isotropic hardening is described by the relation

$$
k(\gamma^p) = k_0 + K_0 (\gamma^p)^m, \quad 0 \leq m \leq 1,
$$

where $k$ is the radius of the yield surface, $k_0$ is the initial yield stress, $K_0$ and $m$ are material constants, and the accumulated plastic strain is defined as

$$
\gamma^p = \int_0^t \dot{\gamma}^p dt.
$$

V.2 Incremental formulation in three dimensions

The rate equation (49) is numerically integrated by a version of the generalized midpoint rule for rate-dependent plasticity; see, e.g. Simo and Taylor [1986], for rate-independent plasticity. According to this algorithm, the increment of the equivalent plastic strain $\Delta \gamma^p$, defined by

$$
\Delta \gamma^p = \sqrt{(2/3) \Delta \varepsilon^p : \Delta \varepsilon^p},
$$

is evaluated from a nonlinear scalar equation

$$
\left( \frac{\Delta \gamma^p}{\dot{\varepsilon}_0 \Delta t} \right)^n k_0 + 3\mu \Delta \gamma^p = \sqrt{(3/2)p : p},
$$
where
\[ p = \theta 2 \mu \Delta \varepsilon'' + \sigma''_a, \quad k_\theta = k_0 + K_0 (\gamma''_a + \theta \Delta \gamma^p)^n, \] (55)
and \( \varepsilon'' \) is the deviator of \( \xi' \). In the above equations a subscript \((\cdot)_a\) denotes variables at the beginning of the time step and the parameter \( \theta, 0 \leq \theta \leq 1 \), marks the intermediate point within the interval, where plastic consistency is enforced. Among the choices of \( \theta \), the \textit{backward Euler} method with \( \theta = 1 \) usually gives the best results for larger time steps and, therefore, will be used in the sequel in all numerical calculations.

With \( \Delta \gamma^p \) evaluated from eqn (54), the increments of plastic strain and stress are found as
\[ \Delta e^p = \Gamma p, \quad \Gamma = \frac{(3/2) \Delta \gamma^p}{(\frac{\Delta \gamma^p}{\Delta t})^{\frac{n}{2}} + \theta 3 \mu \Delta \gamma^p}, \] (56a)
\[ \Delta \sigma_{ij} = 2 \mu \left( \Delta \varepsilon_{ij} - \Delta e^p_{ij} + \frac{\nu}{1 - 2\nu} \Delta \varepsilon_{kk} \delta_{ij} \right). \] (56b)

\textbf{V.3 Incremental formulation for plane stress}

While the plane-strain problem essentially can be formulated by the three-dimensional eqns (54)–(56), the plane-stress case requires a slightly different approach. This is because the elastic strain deviators \( \varepsilon'^e \), which appear in Hooke's law
\[ \sigma'_{ij} = 2 \mu \varepsilon'^e_{ij}, \quad i, j = 1, 2, \] (57)
are different in plane strain and in plane stress. In plane stress the deviator \( \varepsilon'^e \) can be expressed by
\[ \varepsilon'^e_{ij} = P_{ijkl} \varepsilon^e_{kl}, \] (58)
where
\[ P_{ijkl} = \delta_{ik} \delta_{jl} - \frac{1 - 2\nu}{3(1 - \nu)} \delta_{ij} \delta_{kl}, \quad i, j, k, l = 1, 2, \] (59)
\[ \varepsilon'^e = \varepsilon' - e^p. \] (60)

But since \( \varepsilon'^e_{ij} \neq P_{ijkl} \varepsilon^e_{kl} \), each of the components \( \sigma'_{11} \) and \( \sigma'_{22} \) depends on both components \( \varepsilon'^e_{11} \) and \( \varepsilon'^e_{22} \). In order to obtain an uncoupled equation for plastic strain increments like (56a), we perform a transformation of \( P \) to principal axes. This finally gives
\[ \Delta \tilde{e}'^p = \theta 2 \mu \Delta \tilde{e}'^l + \Lambda_2 \tilde{\sigma}'_a. \] (61)

The components of the variables with a tilde are obtained from the original ones by the transformation
\[ \begin{pmatrix} \tilde{q}_{11} \\ \tilde{q}_{22} \\ \tilde{q}_{12} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} q_{11} + q_{22} \\ q_{11} - q_{22} \\ \sqrt{2}q_{12} \end{pmatrix}, \] (62)
and the diagonal matrices $\mathbf{A}_1, \mathbf{A}_2$ have the following simple form:

$$\mathbf{A}_1 = \text{diag}[\rho_1 z_1, \rho_2 z_2, \rho_3 z_3], \quad \mathbf{A}_2 = \text{diag}[z_1, z_2, z_3], \quad (63)$$

with

$$z_\alpha = \frac{(3/2)\Delta \gamma^p}{(\Delta \gamma^p/\varepsilon_0 \Delta \Gamma)^{n_k_{th}} + \theta_3 \mu_\alpha \Delta \gamma^p}, \quad (64)$$

and $\rho_\alpha$ being the eigenvalues of $\mathbf{P}$: $\rho_1 = \frac{1+\nu}{3(1-\nu)}, \quad \rho_2 = \rho_3 = 1$, and hence, $z_2 = z_3 = \Gamma$. In arriving at a scalar equation for $\Delta \gamma^p$ like eqn (54), note that

$$\Delta \gamma^p = \sqrt{\frac{2}{3} \left(3(\Delta \varepsilon^p_{11})^2 + (\Delta \varepsilon^p_{22})^2 + 2(\Delta \varepsilon^p_{12})^2 \right)}. \quad (65)$$

Finally, the stress increments are found from the following expression:

$$\Delta \sigma_{ij} = 2\mu \left(\Delta \varepsilon_{ij} - \Delta \varepsilon^p_{ij} + \frac{\nu}{1-\nu} (\Delta \varepsilon_{11} + \Delta \varepsilon_{22} - \Delta \varepsilon^p_{11} - \Delta \varepsilon^p_{22}) \delta_{ij} \right), \quad i,j = 1,2 \quad (66)$$

**VI. COMPUTATION OF SUBCELL AVERAGES**

Equation (56a) or (61) represent local relations between the increments of plastic strain and the total strain increment. We assume now that these equations also hold approximately for the averages in each subcell. This approximation is expected to yield more accurate results for smaller subcells. With the incremental form of eqn (18), we have the following two equations which must be solved simultaneously:

$$\Delta \varepsilon^\prime = \mathbf{B} \Delta \varepsilon^0 + (\mathbf{I} - \mathbf{B}(\mathbf{I} - \mathbf{S})) \Delta \varepsilon^p, \quad (67)$$

$$\Delta \varepsilon^\prime_{ij} = \Gamma \left(\theta \mu \Delta \varepsilon^\prime_{ij} + \sigma^\prime_{ij} \right). \quad (68)$$

The global relation (67) includes two matrices to be calculated, namely $\mathbf{B}$ and $\mathbf{H} = \mathbf{I} - \mathbf{B}(\mathbf{I} - \mathbf{S})$, where $\mathbf{B}$ requires a matrix inversion. It is, however, not necessary to perform this inversion for a complete $(6N \times 6N)$ $(3N \times 3N$ in 2D) matrix, but only for the submatrix corresponding to the subdiscretization of the inclusion. Since in most cases the inclusion covers considerably less volume than the matrix, the dimensions of the inclusion submatrices (denoted by superscript ( )II in the following) will be smaller than those of the total matrices. We conveniently subdivide $\mathbf{S}$ into

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{\text{II}} & \mathbf{S}_{\text{IM}} \\ \mathbf{S}_{\text{MI}} & \mathbf{S}_{\text{MM}} \end{bmatrix}, \quad (69)$$

where the first (second) superscript indicates if $\mathbf{x}$ ( $\mathbf{x}'$) is located inside the inclusion (I) or in the matrix (M). Since $\mathbf{A}$ has nonzero components only in the inclusion domain, we obtain the following expressions for $\mathbf{B}$ and $\mathbf{H}$:

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{\text{II}} & 0 \\ \mathbf{S}_{\text{MI}} \mathbf{A} \mathbf{B}_{\text{II}} & \mathbf{I} \end{bmatrix}, \quad \mathbf{B}_{\text{II}} = (\mathbf{I} - \mathbf{S}_{\text{II}} \mathbf{A})^{-1}, \quad (70a)$$
Elastic–viscoplastic periodic composites

\[ H = \begin{bmatrix} I - B^H (I - S^H) & B^H S^M \\ S^M - S^M A B^H (I - S^H) & S^M A B^H S^M + S^M \end{bmatrix}. \]  

(70b)

This representation involves only an inversion of the submatrix \( I - S^H A \).

For arbitrary subdiscretizations and nonlinear hardening, eqns (67) and (68) have to be solved iteratively. The iteration is usually started with the first guess

\[ \Delta \varepsilon^p_{(1)} = B \Delta \varepsilon^\theta, \]  

(71)

which, when introduced in eqn (68), gives \( \Delta \varepsilon^p_{(1)} \). If this is directly introduced into eqn (67) again, the whole iteration performs as a modified Newton–Raphson algorithm. The iteration will be stopped, if a certain tolerance norm is satisfied, i.e.

\[ \left\| 1 - \Delta \varepsilon^p_{(n)} / \Delta \varepsilon^p_{(n-1)} \right\| < \text{tol}. \]  

(72)

The solution of (67) and (68) by direct substitution of the outcome of one equation into the other may be viewed as an initial stress version of the modified Newton–Raphson method (Argyris & Scharpf [1972]). This algorithm reproduces the exact result within one iteration only, if the solid is supported in a kinematically determined way, which means the introduction of an arbitrary distribution of eigenstrains results in no change of the total strain field. Although this algorithm is absolutely convergent, the rate of convergence might be rather slow, even for a strain-controlled deformation of the unit cell, and especially if larger time/load steps are applied. However, convergence of the global equilibrium iteration can be accelerated without much additional expense by using modified Quasi-Newton updates, which are called Secant-Newton methods by Crisfield [1982] who also gives a detailed description and derivation of these numerical algorithms.

We may combine (67) and (68) into a nonlinear system of equations

\[ r(x) = x - F(x) = 0, \]  

(73)

where \( x = \Delta \varepsilon^p \) is the unknown. Next, we introduce the quantities

\[ \delta_n = x_{n+1} - x_n, \quad \delta_{n}^* = -G_0^{-1} r_n, \quad \rho_n = r_n - r_{n-1}, \]  

(74)

where a subscript \( n \) denotes the \( n \)th iterative estimate, \( r_n = x_n - F_n \), and \( G_0 \) is an approximation of the gradient matrix

\[ G_{n-1} = \left( \frac{\partial r}{\partial x} \right)_{n-1}. \]  

(75)

As in the modified Newton method, \( G_0 \) is kept constant and the increment \( \delta_n \) is found in the following form

\[ \delta_n = A \delta_n^* + B \delta_{n-1} + C \delta_{n-1}^*. \]  

(76)

The modified Newton method simply has \( A = 1, B = C = 0 \), while the Secant-Newton algorithms satisfy the following secant relation

\[ \delta_{n-1} = G_{n-1}^{-1} \rho_n, \]  

(77)
and we obtain eqn (76) from assuming \( \delta_n = -G_n^{-1}r_n \). We consider a Secant formula, which is derived from the BFGS (Broyden–Fletcher–Goldfarb–Shanno) Quasi-Newton method (Broyden [1970], Fletcher [1970]),

\[
C = \frac{\delta^T_{n-1}r_n}{\delta^T_{n-1}\rho_n}, \quad A = 1 - C, \quad B = C\left(\frac{(\delta^*_{n} - \delta^*_{n-1})^T\rho_n}{\delta^T_{n-1}\rho_n} - \frac{\delta^T_{n}\rho_n}{\delta^T_{n-1}\rho_n} - 1\right). \tag{78}
\]

Noting eqn (73) we may simplify the analysis further by setting \( G_0 = I \). This yields \( \delta^*_{n} = -r_n \) and avoids the costly process of establishing a gradient matrix, any updates of this gradient matrix and (in case of the classical Newton scheme) its inversion. Such a version of the Secant formula has been used by Fotiu [1990, 1993] in the numerical computation of vibrations of elastic–plastic porous plates. In the numerical calculations the BFGS updating method needed only about 1/3–1/4 the number of iterations of the modified Newton–Raphson scheme.

VII. NUMERICAL STUDY

VII.1 Plane-strain shear deformation of a quadratic cell-inclusion pattern

We first consider the plane-strain problem of a quadratic inclusion in a quadratic unit cell loaded by a pure shear strain rate \( \epsilon_{12}^0 = 500 \epsilon_0 \). The material properties of the inclusion and the matrix are given in Table 3.

Poisson’s ratio for both matrix and inclusion is \( \nu = 0.3 \). Figure 2 shows the overall shear stress \( <\sigma_{12}>/k_0^M \) over the overall strain \( \epsilon_{12}^0 \) for several volume fractions \( f \). The solid lines correspond to a subdiscretization of the unit cell into 16 and 400 elements, which were computed using 150 time steps and can be viewed as an essentially exact solution within the limits of the graphs. Due to the mirror-image decomposition described in Section 3, only one quarter of the unit cell is considered and the subdivision into 16 cells has been chosen according to Fig. 3. Although this type of discretization is still very rough, there is no visible difference in the overall stress compared to the results using the highly dense mesh of 400 elements. There is, however, a considerable deviation if the simplest approximation of only two elements (the entire inclusion and the entire matrix) is taken (dashed lines). The major differences occur shortly after initial yielding and decrease afterwards. At a strain of about \( \epsilon_{12}^0 = 0.04 \), all levels of approximations yield the same result. This behavior could be expected, because the yield limit will be exceeded first locally at points of stress concentration near the inclusion. At this state the plastic strains are far from uniform, as assumed in the 2-element approximation. The assumption of a uniform plastic strain field therefore introduces additional residual stresses, which elevate the overall stress above its real value. It is interesting that the rather gross mesh of 16 elements, which, naturally, also leads to such residual stresses, already reproduces the exact result to within minimal

<table>
<thead>
<tr>
<th>Table 3. Material properties of inclusion and matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>2(\mu/k_0^M)</td>
</tr>
<tr>
<td>----------------</td>
</tr>
<tr>
<td>Inclusion</td>
</tr>
<tr>
<td>Matrix</td>
</tr>
</tbody>
</table>
150 steps:  
- 2 elements
- 16, 400 elements

3 steps:
- $f = 0$
- $f = 0.1$
- $f = 0.25$
- $f = 0.5$
- $f = 1$

Fig. 2. Overall shear stress vs shear strain in plane-strain deformation of elastic-plastic composite. Comparison of different discretization densities and step sizes.

Fig. 3. Simplest subdiscretization of the unit cell: plane case: 16 elements; three-dimensional case: 64 elements.

errors. Apparently, at this level of refinement, the contributions of these constraint stresses already start to cancel each other during the averaging process.

Figure 2 also shows results for large time steps. The symbols in the graph indicate the solutions using a 50 times larger time step, that is, the whole strain range is covered
within only three steps. Due to the implementation of the highly accurate time integration algorithm shown in Section 5, the exact answer can be found using virtually any time-step size.

Furthermore, we recognize from Fig. 2 that the initial (elastic) slope is represented more or less exactly by each type of approximation. This means that the overall elastic moduli may be found already by the simplest approximation; Fotiu [1994]. In the elastic case and if only linear hardening is considered, the pure-shear solution of the simplest approximation can be given in closed form,

$$<\sigma_{12}^i> = 2\mu \left\{ (1-f-s)\left[ f\bar{\mu}(1-h^1) + (1-f)(1-h^M) \right] + s\bar{\mu}(1-h^1)(1-h^M) \right\} \varepsilon_{12}^0, \quad (79)$$

where

$$\bar{\mu} = \mu^I / \mu^M,$$

$$h^1 = \begin{cases} \frac{2\mu^I}{(2/3)K_0^I + 2\mu^I} & \text{if } \frac{\bar{\mu}}{1 - (1 - \bar{\mu})s} 2\mu |\varepsilon_{12}^0| \geq k_0^I \text{ and } \bar{\sigma}_{12}^I \varepsilon_{12}^0 > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (80a)$$

$$h^M = \begin{cases} \frac{2\mu^M}{(2/3)K_0^M + 2\mu^M} & \text{if } \left( \frac{1-f}{1-(1-\bar{\mu})s} \right) 2\mu |\varepsilon_{12}^0| \geq k_0^M \text{ and } \bar{\sigma}_{12}^M \varepsilon_{12}^0 > 0, \\ 0 & \text{otherwise,} \end{cases} \quad (80b)$$

$$s = 2S_{1212}. \quad (80c)$$

VII.2 Study of yielding in porous cubic cells

Now we focus on the three-dimensional problem of cubic voids in a cubic elastoviscoelastic unit cell. The following material constants were assumed: $2\mu/k_0 = 400$, $\nu = 0.3$, $K_0/k_0 = 1.5$, $n = 0.08$, $m = 0.2$, $b = 1$ (since all these parameters refer to the matrix, the superscript ( )$^M$ has been removed). At first uniaxial tensile loading is simulated by applying the strain rates $\dot{e}_{11}^0 = -2\dot{e}_{22}^0 = -2\dot{e}_{33}^0 = 1000\dot{e}_0$. Figure 4 shows the response of the perfectly dense solid ($f = 0$), as well as for porosities $f = 0.1$ and $f = 0.3$. The solid lines were computed using 100 time steps and the symbols indicate the solutions when only three steps were taken. Again the deviations between large and small step sizes is minimal. Even more interesting, however, is the fact that all these curves can already be obtained by the simplest approximation with two elements. Finer discretizations showed no visible differences in the results. The same was true when the unit cell was loaded by simple shear, $\dot{e}_{12}^0 = 800\dot{e}_0$. The corresponding curves are given in Fig. 5. However, such a high accuracy for even the simplest formulation depends strongly on the type of overall loading.

Both of the above loadings are purely deviatoric, i.e. the overall bulk strain $\epsilon_{kk}^0$ is identically zero. We shall see that an increasing portion of hydrostatic stress components will lead to more significant differences between different mesh densities. In the
Fig. 4. Overall normal stress vs normal strain in cubic unit cell with cubic void. Loading: $\varepsilon_{11}^0 = -2\varepsilon_{22}^0 = -2\varepsilon_{33}^0 = 1000 \varepsilon_0$. Comparison of different step sizes; responses are the same for each degree of subdiscretization.

Fig. 5. Overall shear stress vs shear strain in cubic unit cell with cubic void. Loading: $\varepsilon_{12}^0 = 800 \varepsilon_0$. Comparison of different step sizes; responses are the same for each degree of subdiscretization.

simplest approximation a pure bulk strain introduces only bulk components in the inclusion and in the matrix. Under such conditions the 2-element approximation will never produce yielding, and for a high triaxiality factor $\beta = \sigma_{ii}/3\sigma$ the initial yield stress will be grossly overestimated.
Let us, therefore, consider a case with a dominant bulk component, namely biaxial plane-strain compression $\varepsilon_{11}^0 = \varepsilon_{22}^0 = -\varepsilon^0$ with a triaxiality factor $\beta = \frac{2(1+\nu)}{3(1-2\nu)}$. For $\nu = 0.3$ we get $\beta = 2.17$. In Fig. 6, the responses of several mesh densities are depicted for a rate-independent elastic–perfectly-plastic material with a void fraction of $f = 0.25$ and $2\mu/k_0 = 400$. Within the plotted stress range the simple 2-element approximation gives a purely elastic response, while the 16-element result already accounts for plastic deformation but considerably underestimates its contribution. More realistic answers are found with higher mesh densities (256, 400, 784 elements) but still the differences between these calculations show that even finer meshes are required for a correct result. From this calculation we conclude that the overall elastoplastic behavior in the presence of a dominant hydrostatic stresses cannot be reproduced accurately by a rough subdiscretization of the unit cell.

VII.3 Anisotropic elastic inclusion in ductile matrix

This example includes effects of local anisotropy of the inclusion material as well as overall anisotropy caused by the shape of the inclusion and the unit cell. For a cubic unit cell the elastic overall moduli exhibit cubic symmetry and, thus, are anisotropic even when the inclusion shape is spherical or cubic. However, as shown, for example, in Nemat-Nasser et al. [1982], the degree of anisotropy $<\mu> = <C_{1111}> - <C_{1122}> - 2 <C_{1212}>$ remains small for small $f < 0.1$. This is no longer the case if the inclusion has one or two dominant dimensions.

Let us now consider a cell with dimensions $\Lambda_2/\Lambda_1 = \Lambda_3/\Lambda_1 = 0.5$, $l_2/l_1 = l_3/l_1 = 0.5$ and a cubic symmetric inclusion material with: $2\mu^M/k_0^M = 600$, $\zeta^1 = 0.75$, $\zeta^2 = 0.25$. The matrix is described by the following material parameters: $2\mu^M/k_0^M = 200$, $\mu^M = 0.3$, $n = 0.1$, $m = 0.3$, $K_0/k_0^M = 1$. Loading consists of two different strain rates $\varepsilon_{11}^0 = \ldots$
\[-2\dot{\varepsilon}^0_{22} = -2\dot{\varepsilon}^0_{33} = 600\dot{\varepsilon}_0, 0 \leq \dot{\varepsilon}_0 t \leq 5 \cdot 10^{-5}, \varepsilon^0_{11} = 200\dot{\varepsilon}_0, \varepsilon^0_{22} = \varepsilon^0_{33} = -120\dot{\varepsilon}_0, 5 \cdot 10^{-5} \leq \dot{\varepsilon}_0 t \leq 1.5 \cdot 10^{-4} .\]

In Figs 7 and 8 the overall stresses \(<\sigma_{11}>\) and \(<\sigma_{22}>\) are shown for the pure matrix material (\(f = 0\)) and for a volume fraction of inclusions \(f = 0.05\). We

**Fig. 7.** Overall 11-stress component vs \(\varepsilon^0_{11}\). Elastic–plastic matrix with anisotropic elastic cuboidal inclusion. Loading: \(\dot{\varepsilon}^0_{11} = -2\dot{\varepsilon}^0_{22} = -2\dot{\varepsilon}^0_{33} = 600\dot{\varepsilon}_0, 0 \leq \dot{\varepsilon}_0 t \leq 5 \cdot 10^{-5}, \varepsilon^0_{11} = 200\dot{\varepsilon}_0, \varepsilon^0_{22} = \varepsilon^0_{33} = -120\dot{\varepsilon}_0, 5 \cdot 10^{-5} \leq \dot{\varepsilon}_0 t \leq 1.5 \cdot 10^{-4}\). Solid line: 120 time steps. ■, ○: 4 time steps.

**Fig. 8.** Overall 22-stress component vs \(\varepsilon^0_{22}\). Elastic-plastic matrix with anisotropic elastic cuboidal inclusion. Loading: \(\dot{\varepsilon}^0_{11} = -2\dot{\varepsilon}^0_{22} = -2\dot{\varepsilon}^0_{33} = 600\dot{\varepsilon}_0, 0 \leq \dot{\varepsilon}_0 t \leq 5 \cdot 10^{-5}, \varepsilon^0_{11} = 200\dot{\varepsilon}_0, \varepsilon^0_{22} = \varepsilon^0_{33} = -120\dot{\varepsilon}_0, 5 \cdot 10^{-5} \leq \dot{\varepsilon}_0 t \leq 1.5 \cdot 10^{-4}\). Solid line: 120 time steps. ■, ○: 4 time steps.
observe that the effect of the inclusions on the strengthening of the composite is considerable. Several mesh densities were used to obtain the 5% inclusion result. The simplest approximation did not prove to be accurate enough in this case, but with 64 elements a satisfactory accuracy could be achieved, in the sense that a finer discretization into 216 elements did not show any visible improvement.

The solid curves in Figs 7 and 8 were obtained with 60 time steps in each of the two loading intervals. Again we also performed calculations with very large time increments, covering each loading interval in only two steps. A small difference occurred at the first large step, but the subsequent steps reproduced the result with the same accuracy as the computation with small time steps.

VIII. CONCLUSIONS

We presented a method to obtain the overall response of rate-dependent elastic-plastic composites with periodic microstructure. It is shown that the numerical expense can be reduced considerably by employing mirror image symmetries/antisymmetries, partial analytic summation of Fourier series and, if desired, arbitrarily large time steps. For design purposes often the highest stresses at or near the end of the loading range are of importance. In this case the stress–strain curve can be evaluated using only a small number of time steps and obtaining the response curve by interpolation.

It was found that the overall elastic–plastic stresses can be obtained to a high degree of accuracy employing only a very rough subdiscretization of the cell, as long as the loading is dominantly deviatoric. In case of large triaxiality factors finer meshes are necessary, which increases the numerical effort considerably, especially in the three-dimensional case. In principle, however, any desired level of accuracy could be obtained. Besides, it must be pointed out that other micromechanical averaging theories like, e.g. the self-consistent method, actually fail to produce any results at high hydrostatic loading and, often, no improvement is possible even at the cost of a higher computational effort.

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REFERENCES

APPENDIX A

Here we present the reduced Fourier sums for the components of $\mathbf{\tilde{G}}^{(u)} = 2\mu \mathbf{\tilde{G}}^{(u)}$ for an isotropic matrix material with periodic inclusions of cuboidal (rectangular in 2D)
shape. Note the symmetry relations $\tilde{\Gamma}^{r\gamma}_{a\beta\gamma} = \tilde{\Gamma}^{r\gamma}_{a\beta\gamma} = \tilde{\Gamma}^{r\gamma}_{a\beta\gamma} = \tilde{\Gamma}^{r\gamma}_{a\beta\gamma}$, however, $\tilde{\Gamma}^{r\gamma}_{a\beta\gamma} \neq \tilde{\Gamma}^{r\gamma}_{a\beta\gamma}$. In the subsequent listings the superscripts $r,s,u$ will be abandoned, $\gamma$ is defined by eqns (32) or (33), and $f$ stands for $f^\gamma$.

**AI. THREE-DIMENSIONAL CASE:**

$\alpha, \beta, \gamma = 1, \ldots, 3$

**AI.1 Mirror image $u = 4$:**

$$\tilde{\Gamma}^{a\alpha a\alpha} = f_{\lambda_0} \left\{ \sum_{n_\beta, n_\gamma = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\beta_\gamma} F_{1\alpha}^{\alpha}(\zeta_\beta) + \frac{1}{2} \left[ \sum_{n_\delta = 1}^{\infty} \left[ \mathcal{G}^{\gamma^+}_{\delta_\gamma} F_{1\alpha}^{\alpha}(\zeta_\delta) \right] + \sum_{n_\gamma = 1}^{\infty} \left[ \mathcal{G}^{\gamma^+}_{\gamma_\gamma} F_{1\alpha}^{\alpha}(\zeta_\gamma) \right] + \frac{2 - \gamma}{2} K^{\gamma^+}_a \right] \right\}, \quad (A.1a)$$

$$\tilde{\Gamma}^{a\alpha \beta\beta} = f_{\lambda_0} \sum_{n_\alpha, n_\beta = 1}^{\infty} \mathcal{G}^{\alpha^+}_{\alpha_\alpha} z_{\alpha\beta} z_{\alpha\beta} \gamma F_{2\gamma}^{\beta}(\zeta_{\alpha\beta}), \quad (A.1b)$$

$$\tilde{\Gamma}^{a\alpha \beta\gamma} = f_{\lambda_0} \sum_{n_\alpha, n_\gamma = 1}^{\infty} \mathcal{G}^{\beta^+}_{\beta_\gamma} z_{\beta\gamma} z_{\beta\gamma} \gamma F_{3\alpha}^{\gamma}(\zeta_{\beta\gamma}), \quad (A.1c)$$

$$\tilde{\Gamma}^{a\alpha a\beta} = f_{\lambda_0} \sum_{n_\alpha, n_\beta = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\gamma_\beta} z_{\alpha\beta} z_{\alpha\beta} \left[ F_{4\gamma}^{\alpha}(\zeta_{\alpha\beta}) - \gamma z_{\alpha\beta}^2 F_{2\gamma}^{\alpha}(\zeta_{\alpha\beta}) \right], \quad (A.1d)$$

$$\tilde{\Gamma}^{a\beta a\beta} = -f_{\lambda_0} \sum_{n_\beta, n_\gamma = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\beta_\gamma} \left[ \frac{1}{2} F_{4\gamma}^{\beta}(\zeta_{\alpha\beta}) - \gamma z_{\alpha\beta}^2 F_{2\gamma}^{\alpha}(\zeta_{\alpha\beta}) \right], \quad (A.1e)$$

$$\tilde{\Gamma}^{a\beta \alpha\gamma} = f_{\lambda_0} \sum_{n_\beta, n_\gamma = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\gamma_\beta} z_{\beta\gamma} z_{\beta\gamma} \left[ \frac{1}{2} F_{4\gamma}^{\beta}(\zeta_{\alpha\beta}) + \gamma F_{2\gamma}^{\alpha}(\zeta_{\alpha\beta}) \right], \quad (A.1f)$$

**AI.2 Mirror image $u = -\alpha$:**

$$\tilde{\Gamma}^{a\alpha a\alpha} = f_{\lambda_0} \sum_{n_\beta, n_\gamma = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\gamma_\gamma} F_{1\alpha}^{\alpha}(\zeta_{\beta\gamma}), \quad (A.2a)$$

$$\tilde{\Gamma}^{a\beta \beta\beta} = f_{\lambda_0} \left\{ \sum_{n_\alpha, n_\beta = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\alpha_\gamma} F_{1\alpha}^{\alpha}(\zeta_{\beta\gamma}) + \frac{1}{2} \sum_{n_\gamma = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\gamma_\gamma} F_{1\alpha}^{\alpha}(\zeta_{\beta\gamma}) \right\}, \quad (A.2b)$$

$$\tilde{\Gamma}^{a\alpha \beta\beta} = f_{\lambda_0} \sum_{n_\alpha, n_\beta = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\beta_\gamma} z_{\alpha\beta} z_{\alpha\beta} \gamma F_{2\gamma}^{\alpha}(\zeta_{\alpha\beta}), \quad (A.2c)$$

$$\tilde{\Gamma}^{a\beta \beta\gamma} = f_{\lambda_0} \sum_{n_\beta, n_\gamma = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\beta_\gamma} z_{\beta\gamma} z_{\beta\gamma} \gamma F_{2\alpha}^{\gamma}(\zeta_{\beta\gamma}), \quad (A.2d)$$

$$\tilde{\Gamma}^{a\alpha a\beta} = f_{\lambda_0} \sum_{n_\alpha, n_\beta = 1}^{\infty} \mathcal{G}^{\gamma^+}_{\gamma_\beta} z_{\alpha\beta} z_{\alpha\beta} \left[ F_{4\gamma}^{\alpha}(\zeta_{\alpha\beta}) - \gamma z_{\alpha\beta}^2 F_{2\gamma}^{\alpha}(\zeta_{\alpha\beta}) \right], \quad (A.2e)$$
\[
\tilde{\Gamma}_{\beta\beta\gamma} = f \lambda_{\alpha} \sum_{n_{\beta}, n_{\gamma}=1}^{\infty} \mathcal{G}_{\beta\gamma}^{+\pm} z_{\beta\gamma} z_{\gamma\beta} \left[ F_{4\alpha}^{+} (\zeta_{\beta\gamma}) - \gamma^{2} z_{\beta\gamma} F_{2\alpha}^{+} (\zeta_{\beta\gamma}) \right], \quad (A.2f)
\]

\[
\tilde{\Gamma}_{\beta\alpha\beta} = f \lambda_{\gamma} \sum_{n_{\beta}, n_{\alpha}=1}^{\infty} \mathcal{G}_{\alpha\beta}^{+\pm} z_{\alpha\beta} z_{\beta\alpha} \left[ F_{4\alpha}^{+} (\zeta_{\alpha\beta}) - \gamma^{2} z_{\alpha\beta} F_{2\alpha}^{+} (\zeta_{\alpha\beta}) \right], \quad (A.2g)
\]

\[
\tilde{\Gamma}_{\alpha\alpha\gamma} = f \lambda_{\beta} \sum_{n_{\alpha}, n_{\gamma}=1}^{\infty} \mathcal{G}_{\alpha\gamma}^{+\pm} z_{\alpha\gamma} z_{\gamma\alpha} F_{3\alpha}^{+} (\zeta_{\alpha\gamma}), \quad (A.2h)
\]

\[
\tilde{\Gamma}_{\beta\alpha\gamma} = f \lambda_{\beta} \sum_{n_{\alpha}, n_{\gamma}=1}^{\infty} \mathcal{G}_{\alpha\gamma}^{+\pm} z_{\alpha\gamma} F_{3\beta}^{+} (\zeta_{\alpha\gamma}), \quad (A.2i)
\]

\[
\tilde{\Gamma}_{\alpha\beta\alpha} = -f \lambda_{\gamma} \left\{ \sum_{n_{\alpha}, n_{\beta}=1}^{\infty} \mathcal{G}_{\alpha\beta}^{-\pm} \left[ \frac{1}{2} F_{4\alpha}^{-} (\zeta_{\alpha\beta}) - \gamma^{2} z_{\alpha\beta} F_{2\alpha}^{-} (\zeta_{\alpha\beta}) \right] + \frac{1}{4} \sum_{n_{\alpha}=1}^{\infty} \mathcal{G}_{\alpha}^{-} F_{4\alpha}^{-} (\zeta_{\alpha}) \right\}, \quad (A.2j)
\]

\[
\tilde{\Gamma}_{\alpha\beta\gamma} = f \lambda_{\beta} \sum_{n_{\alpha}, n_{\gamma}=1}^{\infty} \mathcal{G}_{\alpha\gamma}^{+\pm} z_{\alpha\gamma} z_{\gamma\alpha} \left[ \frac{1}{2} F_{4\alpha}^{+} (\zeta_{\alpha\gamma}) + \gamma F_{3\beta}^{+} (\zeta_{\alpha\gamma}) \right], \quad (A.2k)
\]

\[
\tilde{\Gamma}_{\alpha\beta\gamma} = f \lambda_{\alpha} \sum_{n_{\beta}, n_{\gamma}=1}^{\infty} \mathcal{G}_{\beta\gamma}^{+\pm} z_{\beta\gamma} z_{\gamma\beta} \left[ \frac{1}{2} F_{4\alpha}^{+} (\zeta_{\beta\gamma}) + \gamma F_{3\alpha}^{+} (\zeta_{\beta\gamma}) \right], \quad (A.2m)
\]

\[
\tilde{\Gamma}_{\beta\gamma\beta} = -f \lambda_{\beta} \left\{ \sum_{n_{\beta}, n_{\gamma}=1}^{\infty} \mathcal{G}_{\beta\gamma}^{+\pm} \left[ \frac{1}{2} F_{4\beta}^{+} (\zeta_{\beta\gamma}) - \gamma^{2} z_{\beta\gamma} F_{2\beta}^{+} (\zeta_{\beta\gamma}) \right] + \frac{1}{4} \sum_{n_{\beta}=1}^{\infty} \mathcal{G}_{\beta}^{+} F_{4\beta}^{+} (\zeta_{\beta}) + \sum_{n_{\gamma}=1}^{\infty} \mathcal{G}_{\beta}^{+} F_{4\beta}^{+} (\zeta_{\gamma}) \right\}, \quad (A.2n)
\]

The remaining components in (A.1) and (A.2) can be found by interchanging the first and second pairs of indices and simultaneously changing the signs in the superscripts of $\mathcal{G}$. For example, $\tilde{\Gamma}_{\alpha\beta\alpha\alpha}$ is given by the series (A.1)(u = 4) and (A.2e) (u = -\alpha), but with the functions $\mathcal{G}_{\beta\alpha}^{\pm}$ and $\mathcal{G}_{\alpha\beta}^{\pm}$ substituted for $\mathcal{G}_{\alpha\beta}^{\pm}$ and $\mathcal{G}_{\beta\alpha}^{\pm}$, respectively. In the above equations the following abbreviations were used:

\[
\lambda_{\alpha} = \frac{A_{\alpha}^{2}}{f_{a}^{\alpha}}, \quad \zeta_{\beta\gamma} = \Lambda_{\alpha} \sqrt{\zeta_{\beta}^2 + \zeta_{\gamma}^2}, \quad \zeta_{\beta} = \Lambda_{\alpha} \zeta_{\beta}, \quad z_{\alpha\beta} = \frac{\zeta_{\alpha}}{\sqrt{\zeta_{\alpha}^2 + \zeta_{\beta}^2}}, \quad (A.3)
\]

\[
F_{1\alpha}^{\pm} (y) = 2G_{\alpha}^{\pm} (y) + \gamma (H_{\alpha}^{G\pm} (y) + H_{\alpha}^{S\pm} (y)), \quad (A.4a)
\]

\[
F_{2\alpha}^{\pm} (y) = H_{\alpha}^{G\pm} (y) - 3H_{\alpha}^{S\pm} (y) - K_{\alpha}^{\pm} - 4\delta^{\pm} / \lambda_{\alpha}, \quad (A.4b)
\]

\[
F_{3\alpha}^{\pm} (y) = H_{\alpha}^{G\pm} (y) - H_{\alpha}^{S\pm} (y), \quad F_{4\alpha}^{\pm} (y) = G_{\alpha}^{\pm} (y) - K_{\alpha}^{\pm} - 4\delta^{\pm} / \lambda_{\alpha}, \quad (A.4b)
\]

\[
\mathcal{G}_{\alpha\beta}^{\pm} = Q_{\alpha}^{+} Q_{\beta}^{+} \frac{(c_{\alpha}^{+} \pm c_{\beta}^{+}) (c_{\alpha}^{-} \pm c_{\beta}^{+})}{4}, \quad \mathcal{G}_{\alpha\beta}^{\pm} = Q_{\alpha}^{-} Q_{\beta}^{-} \frac{(s_{\alpha}^{+} \pm s_{\beta}^{+}) (s_{\alpha}^{-} \pm s_{\beta}^{+})}{4}. \quad (A.5)
\]
where

\[ Q'^{\alpha}_{\alpha} = \frac{\sin \alpha_{\alpha} \sin \alpha_{\alpha}^d}{\alpha_{\alpha} \alpha_{\alpha}^d}, \quad \epsilon_{\alpha}^d = \cos(\xi_{\alpha}(x_{\alpha}^d \pm x_{\alpha}^d)), \quad s_{\alpha}^d = \sin(\xi_{\alpha}(x_{\alpha}^d \pm x_{\alpha}^d)), \quad (A.6a) \]

\[ \delta^+ = 1, \quad \delta^- = 0. \quad (A.6b) \]

The family of functions \( G, H, \) and \( K \) are defined as follows:

\[ G^{C\pm}_{\alpha}(y) = \frac{1}{y \sinh(y/2)} \sum_{k=1}^{4} (-1)^{k+1} \left[ \cosh((b_{\alpha k}^- - 0.5)y) \pm \cosh((b_{\alpha k}^+ - 0.5)y) \right], \quad (A.7) \]

\[ H^{C\pm}_{\alpha}(y) = \frac{1}{4 \sinh^2(y/2)} \sum_{k=1}^{4} (-1)^{k+1} \left[ b_{\alpha k}^- \cosh((b_{\alpha k}^- - 1)y) - (b_{\alpha k}^- - 1) \cosh(b_{\alpha k}^- y) \pm \{ b_{\alpha k}^+ \cosh((b_{\alpha k}^+ - 1)y) - (b_{\alpha k}^+ - 1) \cosh(b_{\alpha k}^+ y) \} \right], \quad (A.8) \]

\[ H^{S\pm}_{\alpha}(y) = \frac{1}{4 \sinh^2(y/2)} \sum_{k=1}^{4} (-1)^{k+1} \left[ \sinh((b_{\alpha k}^- - 1)y) - \sinh(b_{\alpha k}^- y) \pm \{ \sinh((b_{\alpha k}^+ - 1)y) - \sinh(b_{\alpha k}^+ y) \} \right], \quad (A.9) \]

\[ K^{\pm}_{\alpha} = \sum_{k=1}^{4} (-1)^{k+1} \left[ b_{\alpha k}^- (b_{\alpha k}^- - 1) \pm b_{\alpha k}^+ (b_{\alpha k}^+ - 1) \right], \quad (A.10) \]

where

\[ b_{\alpha k}^\pm = \text{Mod} \left[ \text{Mod} \left[ \tilde{b}_{\alpha k}^\pm, 1 \right] + 1, 1 \right], \quad 0 \leq b_{\alpha k}^\pm < 1, \quad (A.11) \]

\[ \tilde{b}_{\alpha 1}^\pm = \left[ (x_{\alpha}^d \pm x_{\alpha}) - (l_{\alpha}^d - l_{\alpha}^d)/2 \right] / \Lambda_{\alpha}, \quad \tilde{b}_{\alpha 2}^\pm = \left[ (x_{\alpha}^d \pm x_{\alpha}) - (l_{\alpha}^d + l_{\alpha}^d)/2 \right] / \Lambda_{\alpha}, \quad (A.12a) \]

\[ \tilde{b}_{\alpha 3}^\pm = \left[ (x_{\alpha}^d \pm x_{\alpha}) + (l_{\alpha}^d - l_{\alpha}^d)/2 \right] / \Lambda_{\alpha}, \quad \tilde{b}_{\alpha 4}^\pm = \left[ (x_{\alpha}^d \pm x_{\alpha}) + (l_{\alpha}^d + l_{\alpha}^d)/2 \right] / \Lambda_{\alpha}. \quad (A.12b) \]

The definition of \( b_{\alpha k}^\pm \) in (A.11) simply means that \( \tilde{b}_{\alpha k}^\pm \) is repeatedly increased or diminished by one until the result lies in the right sided open interval \([0,1]\).

**AII. TWO-DIMENSIONAL CASE:**

\( \alpha, \beta = 1, 2 \)

**AII.1 Mirror image \( u=4 \):**

\[ \bar{\Gamma}_{\alpha \alpha \alpha \alpha \alpha} = f \frac{\lambda_{\alpha}}{2} \left\{ \sum_{\alpha_{\beta}}^{\alpha_{\beta}} \left[ \mathcal{G}^\pm F^\pm_{1\alpha}(\zeta_{\beta}) \right] + \frac{2 - \gamma}{2} \delta^+ K^\alpha_\alpha \right\}, \quad (A.13a) \]
\[ \tilde{\Gamma}_{\alpha\beta\gamma} = -f \frac{\lambda_\alpha}{2} \sum_{n_\beta=1}^{\infty} \mathcal{G}^+_{\beta\gamma} F_{3\alpha}(\zeta_\beta), \]  
(A.13b)

\[ \tilde{\Gamma}_{\alpha\alpha\gamma} = -f \frac{\lambda_\alpha}{2} \sum_{n_\beta=1}^{\infty} \mathcal{G}^-_{\beta\gamma} \left[ G^+_{\alpha\alpha} - \gamma J^+_{\alpha}(\zeta_\beta) \right], \]  
(A.13c)

\[ \tilde{\Gamma}_{\alpha\beta\alpha} = -f \frac{\lambda_\alpha}{2} \sum_{n_\beta=1}^{\infty} \mathcal{G}^+_{\beta\alpha} \left[ G^+_{\alpha\alpha} - \gamma J^+_{\alpha}(\zeta_\beta) \right], \]  
(A.13d)

\[ \tilde{\Gamma}_{\alpha\beta\alpha} = f \frac{\lambda_\alpha}{2} \left\{ \sum_{n_\beta=1}^{\infty} \mathcal{G}^-_{\beta\alpha} \left[ K^-_{\alpha} + 2 \zeta^-_{\alpha} - \gamma F_{3\alpha}(\zeta_\beta) \right] + \delta^- / 4 \right\}. \]  
(A.13e)

The solutions for \( u = -4 \) are found from eqns (A.13) by changing all superscripts + to – and all – to +. In addition the following definitions were used:

\[ \phi^\pm_{\alpha} = Q^\pm_{\alpha} \frac{\tilde{\epsilon}^\pm_{\alpha} \pm \epsilon^+_{\alpha}}{2}, \quad \sigma^\pm_{\alpha} = Q^\pm_{\alpha} \frac{\tilde{\sigma}^\pm_{\alpha} \pm \sigma^+_{\alpha}}{2}, \quad \zeta_{\alpha} = \Lambda_{\beta} \zeta_{\beta}, \]  
(A.14)

\[ G^\pm_{\alpha}(y) = \frac{1}{y \sinh(y/2)} \sum_{k=1}^{4} (-1)^{k+1} \left[ \sinh((b^-_{ak} - 0.5)y) \pm \sinh((b^+_{ak} - 0.5)y) \right] \]  
\[ - \frac{2}{y} \sum_{k=1}^{4} (-1)^{k+1} [b^-_{ak} \pm b^+_{ak}], \]  
(A.15)

\[ J^\pm_{\alpha}(y) = \frac{1}{4y \sinh^2(y/2)} \sum_{k=1}^{4} (-1)^{k+1} \left[ (b^-_{ak} - 1) \sinh(b^-_{ak}y) - b^-_{ak} \sinh((b^-_{ak} - 1)y) \right] \]  
\[ \pm \{(b^+_{ak} - 1) \sinh(b^+_{ak}y) - b^+_{ak} \sinh((b^+_{ak} - 1)y)\}. \]  
(A.16)

**APPENDIX B**

The Eshelby tensor \( S \) is derived from the corresponding Green’s function \( S(x, x') \), which relates the perturbation strain \( \epsilon_{ij}(x) \) \( (i, j = 1, 2) \) to the eigenstrain \( \epsilon^e_{pq}(x') \) or \( \epsilon^p_{pq}(x') \) \( (p, q = 1, ..., 3) \), i.e.,

\[ \epsilon_{ij}(x) = \int_{A} S_{ijpq}(x, x') \epsilon^e_{pq}(x') dA(x'), \]  
(B.1)

where \( A \) lies in the \( x_1-x_2 \) plane. The function \( S \) can also be written as

\[ S_{ijpq}(x, x') = \frac{1}{2} (\sigma_{pqij} + \sigma_{pquil})(x', x) = \frac{1}{2} C_{pqr}(\epsilon_{ijr} + \epsilon_{isr})(x, x') = C_{pqr} \Gamma_{ijrs}(x, x'), \]  
(B.2)

where \( C \) is the three-dimensional elastic tensor and \( \epsilon_{ij}(x, x') \) \( (\sigma_{pq}(x', x)) \) denotes the strain \( \epsilon_{ij}(x) \) (stress \( \sigma_{pq}(x', x) \)) produced by a unit load \( 1 \cdot e_p \) applied at \( x' \) \( (1 \cdot e_i \) in \( x) \); \( e_p \) is the unit vector along the \( x_p \)-axis.
BI.1. Plane strain

In plane strain $\epsilon_{33p} = 0$ and $\epsilon_{33p,q} = 0$. Due to the symmetry properties of $\Gamma$, there is also $\Gamma_{33pq} = \Gamma_{pq33} = 0$. Hence, the strains due to the 33-component of the plastic strain are

$$\dot{\epsilon}_{ij}(x) = \int_A S_{ij33}(x, x') \dot{\epsilon}_{33}^p(x') \, dA(x')$$

with

$$S_{ij33} = C_{33pq} \epsilon_{ijp,q} = C_{33pq} \Gamma_{ijpq}, \quad i,j,p,q = 1,2.$$  \hspace{1cm} (B.3)

Since $\dot{\epsilon}_{33}^p = -(\dot{\epsilon}_{11}^p + \dot{\epsilon}_{22}^p)$, we can rewrite (B.1) with a tensor $\tilde{S}_{ijpq}$, $i,j,p,q = 1,2$, with the components

$$\tilde{S}_{ijpq} = S_{ijpq} - S_{ij33} \delta_{pq} = \Gamma_{ijrs} (C_{r3pq} - C_{33rs} \delta_{pq}),$$

which is the result (2.39).

BI.2. Plane stress

The plain stress condition requires $\sigma_{33p} = 0$, and with (B.2) this directly gives $S_{ij33} = 0$, or $S_{ijpq} = S_{ijpq}, \quad i,j,p,q = 1,2$.

This result can be immediately understood by considering the plane stress state in a thin sheet containing an inclusion. Extending the inclusion only in the direction normal to the plane will produce anti-plane shear stresses in the sheet, with zero resultant over the sheet thickness. According to St. Venant's principle these stresses are localized within a range of the order of the sheet thickness and are neglected in a plane stress theory.