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BOUNDS ON ELASTIC MODULI OF COMPOSITES

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ABSTRACT

General computable bounds are developed for the stored elastic energy of a finite-sized sample of heterogeneous material which consists of a matrix containing disconnected inclusions which may themselves be heterogeneous. These bounds also apply to the limiting cases of elastic solids with disconnected cavities or rigid inclusions. The lower bounds for solids with cavities are nonzero, and the upper bounds for solids with rigid inclusions are finite. As an illustration, closed-form bounds are obtained for the overall elastic parameters of fiber reinforced composites. The fiber packing may be triangular, square, or hexagonal. Though the results are presented for fibers with hexagonal cross sections, the closed-form expressions also apply to arbitrarily shaped cross sections. The general procedure also applies to nonlinear heterogeneous solids which possess convex strain and stress potentials.

1. INTRODUCTION

For heterogeneous materials, the overall mechanical properties are defined through relations among average field quantities taken over a suitably large sample called the *representative volume element* (RVE); see Hill (1963), Hashin (1964, 1983), Kröner (1977), and Willis (1981). These relations in general depend on the boundary conditions to which the RVE is subjected. Hence, in general, the overall properties are not uniquely determined. Recently, Nemat-Nasser and Hori (1993) showed that for any general boundary conditions, the elastic strain energy and the complementary elastic strain energy, stored in any heterogeneous elastic solid, have absolute lower bounds, for given average strain and stress, respectively. Further, for any consistent boundary conditions these energies have an associated upper bound. However, exact computation of these bounds requires solving boundary-value problems involving interface boundary conditions. In this paper, these bounds are extended to avoid any solution involving interface boundary conditions. Our results apply to any heterogeneous solid, with one proviso: the inhomogeneities or clusters of inhomogeneities must be surrounded by the matrix material. Indeed, the general procedure applies to nonlinear heterogeneous solids which possess convex strain and stress potentials (Nemat-Nasser *et al.*, 1994). In Section 2, the averaging methods are outlined. In Section 3, a method is presented to obtain bounds on elastic moduli. In Section 4, it is proved that the resulting expressions are indeed bounds. In Section 5, a method is presented to calculate the bounds for irregularly shaped solids containing irregularly shaped inclusions, based on the known bounds of regularly shaped solids and inclusions. In Section 6, the elastic solid is discretized into subelements, and bounds for the overall composite are obtained, based on the known bounds for each subelement. This is then extended and applied to an RVE with a periodic microstructure. Some examples are given to illustrate the effectiveness of the bounds.

2. AVERAGING METHODS

Consider an elastic solid with volume V bounded by a closed surface ∂V; see Fig. 1. Denote the displacement, strain, and stress fields in the solid by $\mathbf{u}(\mathbf{x})$, $\boldsymbol{\varepsilon}(\mathbf{x})$, and $\boldsymbol{\sigma}(\mathbf{x})$, respectively. The volume average of the field quantities are

$$\bar{\varepsilon} \equiv \frac{1}{V}\int_V \boldsymbol{\varepsilon}(\mathbf{x})\,\mathrm{d}V, \quad \bar{\boldsymbol{\sigma}} \equiv \frac{1}{V}\int_V \boldsymbol{\sigma}(\mathbf{x})\,\mathrm{d}V, \quad \bar{w} \equiv \frac{1}{V}\int_V \tfrac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon}\,\mathrm{d}V. \qquad (2.1\text{a–c})$$

In the absence of body forces, the equations of equilibrium and the strain–displacement relations are

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x}) = 0, \quad \boldsymbol{\varepsilon}(\mathbf{x}) = \tfrac{1}{2}\{\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^{\mathrm{T}}\}. \qquad (2.2\text{a,b})$$

From the Gauss theorem and (2.2), it follows that (Hill, 1963)

$$\bar{\varepsilon} = \frac{1}{V}\int_{\partial V} \tfrac{1}{2}(\mathbf{n} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{n})\,\mathrm{d}S, \quad \bar{\boldsymbol{\sigma}} = \frac{1}{V}\int_{\partial V} \tfrac{1}{2}(\mathbf{t} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{t})\,\mathrm{d}S,$$

$$\bar{w} = \frac{1}{V}\int_{\partial V} \tfrac{1}{2}\mathbf{t} \cdot \mathbf{u}\,\mathrm{d}S, \qquad (2.3\text{a–c})$$

where $\mathbf{t}(\mathbf{x}) = \mathbf{n}(\mathbf{x}) \cdot \boldsymbol{\sigma}(\mathbf{x})$ are the surface tractions acting on ∂V of the unit outward normal $\mathbf{n}(\mathbf{x})$; see Nemat-Nasser and Hori (1993) for details.

Consider an elastic solid consisting of a linearly elastic matrix with uniform elasticity

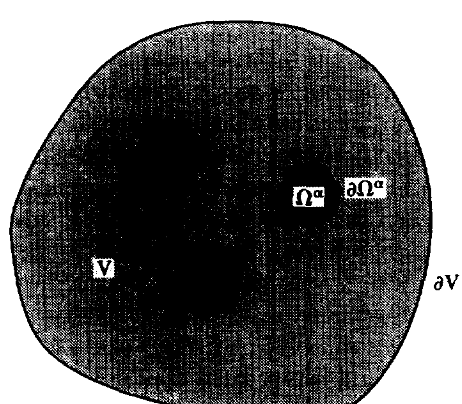

Fig. 1. Finite-sized volume of a composite material.

tensor $\mathbf{C}$ and compliance tensor $\mathbf{D} = \mathbf{C}^{-1}$, containing a set of linearly elastic inclusions Ω^{α}, with elasticities $\mathbf{C}^{\alpha}$ and compliances $\mathbf{D}^{\alpha} = (\mathbf{C}^{\alpha})^{-1}$, $\alpha = 1, 2, \ldots, n$. Denote the displacement, strain, and stress fields in the solid by $\mathbf{u}^{\Sigma}(\mathbf{x})$, $\boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x})$, and $\boldsymbol{\sigma}^{\Sigma}(\mathbf{x})$, respectively, when the boundary data of the solid are prescribed through the overall constant stress $\boldsymbol{\Sigma}$ by

$$\mathbf{t} = \mathbf{n} \cdot \boldsymbol{\Sigma} \quad \text{on } \partial V. \tag{2.4a}$$

Here and throughout this paper, the superscript Σ is used to emphasize that these fields correspond to *uniform* boundary tractions. From (2.3), the average stress and the energy density are given by

$$\bar{\boldsymbol{\sigma}}^{\Sigma} = \boldsymbol{\Sigma}, \quad \bar{w}^{\Sigma} = \tfrac{1}{2}\boldsymbol{\Sigma} : \bar{\boldsymbol{\varepsilon}}. \tag{2.4b}$$

The average strain in this case is

$$\bar{\boldsymbol{\varepsilon}}^{\Sigma} = \mathbf{D} : \boldsymbol{\Sigma} + \sum_{\alpha=1}^{n} f_{\alpha}(\boldsymbol{\varepsilon}^{\alpha} - \mathbf{D} : \boldsymbol{\sigma}^{\alpha}) \equiv \bar{\mathbf{D}}^{\Sigma} : \boldsymbol{\Sigma}, \tag{2.4c}$$

where $\bar{\mathbf{D}}^{\Sigma}$ is the overall compliance tensor associated with the uniform boundary tractions (this is denoted by the superscript Σ); f_{α} is the volume fraction of the αth inclusion; and $\boldsymbol{\varepsilon}^{\alpha}$ and $\boldsymbol{\sigma}^{\alpha}$ are the average strain and stress in the αth inclusion, given, respectively, by

$$\boldsymbol{\varepsilon}^{\alpha} \equiv \frac{1}{\Omega^{\alpha}} \int_{\partial\Omega^{\alpha}} \tfrac{1}{2}(\mathbf{n} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{n})\, \mathrm{d}S, \tag{2.5a}$$

$$\boldsymbol{\sigma}^{\alpha} \equiv \frac{1}{\Omega^{\alpha}} \int_{\partial\Omega^{\alpha}} \tfrac{1}{2}(\mathbf{t} \otimes \mathbf{x} + \mathbf{t} \otimes \mathbf{x})\, \mathrm{d}S, \tag{2.5b}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega^{\alpha}$.

Similarly, denote the displacement, strain, and stress fields in the solid by $\mathbf{u}^{\mathrm{E}}(\mathbf{x})$, $\boldsymbol{\varepsilon}^{\mathrm{E}}(\mathbf{x})$, and $\boldsymbol{\sigma}^{\mathrm{E}}(\mathbf{x})$, respectively, when the boundary data are prescribed through a constant overall strain $\mathbf{E}$ by

$$\mathbf{u} = \mathbf{x} \cdot \mathbf{E} \quad \text{on } \partial V. \tag{2.6a}$$

The average strain and the energy density then become

$$\bar{\boldsymbol{\varepsilon}}^{\mathrm{E}} = \mathbf{E}, \quad \bar{w}^{\mathrm{E}} = \tfrac{1}{2}\bar{\boldsymbol{\sigma}} : \mathbf{E}. \tag{2.6b}$$

In this case, the average stress is

$$\bar{\boldsymbol{\sigma}}^{\mathrm{E}} = \mathbf{C} : \mathbf{E} + \sum_{\alpha=1}^{n} f_{\alpha}(\boldsymbol{\sigma}^{\alpha} - \mathbf{C} : \boldsymbol{\varepsilon}^{\alpha}) \equiv \bar{\mathbf{C}}^{\mathrm{E}} : \mathbf{E}, \tag{2.6c}$$

where $\bar{\mathbf{C}}^{\mathrm{E}}$ is the overall elasticity tensor associated with the linear boundary data (2.6a); note that the superscript E stands for the linear displacement boundary conditions. In general, $\bar{\mathbf{D}}^{\Sigma}$ and $\bar{\mathbf{C}}^{\mathrm{E}}$ are not each other's inverse. Their inverses are denoted by

$$\bar{\mathbf{C}}^{\Sigma} \equiv (\bar{\mathbf{D}}^{\Sigma})^{-1}, \quad \bar{\mathbf{D}}^{\mathrm{E}} \equiv (\bar{\mathbf{C}}^{\mathrm{E}})^{-1}. \tag{2.7a,b}$$

Also, in general, $\bar{\mathbf{C}}^{\Sigma} \neq \bar{\mathbf{C}}^{\mathrm{E}}$. Indeed, as shown by Nemat-Nasser and Hori (1993), in general, $(\bar{\mathbf{C}}^{\mathrm{E}} - \bar{\mathbf{C}}^{\Sigma})$ is positive-definite.

Now consider a set of consistent general (possibly mixed) boundary conditions $\{\mathbf{t}^{\mathrm{G}}, \mathbf{u}^{\mathrm{G}}\}$ which produces in the solid the average stress $\bar{\boldsymbol{\sigma}}^{\mathrm{G}}$, and the average strain $\bar{\boldsymbol{\varepsilon}}^{\mathrm{G}}$. The average energy density is

$$\frac{1}{V}\int_V \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{G}}(\mathbf{x}) : \boldsymbol{\varepsilon}^{\mathrm{G}}(\mathbf{x})\,\mathrm{d}V = \frac{1}{V}\int_{\partial V} \tfrac{1}{2}\mathbf{t}^{\mathrm{G}} \cdot \mathbf{u}^{\mathrm{G}}\,\mathrm{d}S \equiv \tfrac{1}{2}\langle \boldsymbol{\sigma}^{\mathrm{G}} : \boldsymbol{\varepsilon}^{\mathrm{G}} \rangle. \tag{2.8}$$

The overall elasticity and compliance tensors, $\bar{\mathbf{C}}^{\mathrm{G}}$ and $\bar{\mathbf{D}}^{\mathrm{G}}$, for this general boundary data are defined through the corresponding energy density, by

$$\tfrac{1}{2}\bar{\boldsymbol{\varepsilon}}^{\mathrm{G}} : \bar{\mathbf{C}}^{\mathrm{G}} : \bar{\boldsymbol{\varepsilon}}^{\mathrm{G}} \equiv \tfrac{1}{2}\langle \boldsymbol{\sigma}^{\mathrm{G}} : \boldsymbol{\varepsilon}^{\mathrm{G}} \rangle \equiv \tfrac{1}{2}\bar{\boldsymbol{\sigma}}^{\mathrm{G}} : \bar{\mathbf{D}}^{\mathrm{G}} : \bar{\boldsymbol{\sigma}}^{\mathrm{G}}. \tag{2.9a,b}$$

In general, the average energy density is not equal to $\bar{\boldsymbol{\sigma}}^{\mathrm{G}} : \bar{\boldsymbol{\varepsilon}}^{\mathrm{G}}/2$. Therefore, in general,

$$\bar{\boldsymbol{\sigma}}^{\mathrm{G}} \neq \bar{\mathbf{C}}^{\mathrm{G}} : \bar{\boldsymbol{\varepsilon}}^{\mathrm{G}}, \quad \bar{\boldsymbol{\varepsilon}}^{\mathrm{G}} \neq \bar{\mathbf{D}}^{\mathrm{G}} : \bar{\boldsymbol{\sigma}}^{\mathrm{G}}, \quad \bar{\mathbf{C}}^{\mathrm{G}} : \bar{\mathbf{D}}^{\mathrm{G}} \neq \mathbf{I}, \tag{2.10a–c}$$

where $\mathbf{I}$ is the fourth-order symmetric identity tensor.

Consider a special class of boundary data $\{\mathbf{t}^{\mathrm{S}}, \mathbf{u}^{\mathrm{S}}\}$ which produce the average stress $\bar{\boldsymbol{\sigma}}^{\mathrm{S}}$, and the average strain $\bar{\boldsymbol{\varepsilon}}^{\mathrm{S}}$ (superscript S standing for special) such that

$$\langle \boldsymbol{\sigma}^{\mathrm{S}} : \boldsymbol{\varepsilon}^{\mathrm{S}} \rangle = \bar{\boldsymbol{\sigma}}^{\mathrm{S}} : \bar{\boldsymbol{\varepsilon}}^{\mathrm{S}}. \tag{2.11}$$

In this case, the overall moduli defined by (2.9a,b) are such that

$$\bar{\boldsymbol{\sigma}}^{\mathrm{S}} = \bar{\mathbf{C}}^{\mathrm{S}} : \bar{\boldsymbol{\varepsilon}}^{\mathrm{S}}, \quad \bar{\boldsymbol{\varepsilon}}^{\mathrm{S}} = \bar{\mathbf{D}}^{\mathrm{S}} : \bar{\boldsymbol{\sigma}}^{\mathrm{S}}, \quad \bar{\mathbf{C}}^{\mathrm{S}} : \bar{\mathbf{D}}^{\mathrm{S}} = \mathbf{I}. \tag{2.12a–c}$$

Note that, for uniform-traction and linear-displacement boundary data, the energy definition of $\bar{\mathbf{D}}^{\Sigma}$ and $\bar{\mathbf{C}}^{\mathrm{E}}$ coincide, respectively, with definitions (2.4c) and (2.6c).

Nemat-Nasser and Hori (1993) show that,† for any general consistent boundary data, $(\bar{\mathbf{C}}^{\mathrm{G}} - \bar{\mathbf{C}}^{\Sigma})$ and $(\bar{\mathbf{D}}^{\mathrm{G}} - \bar{\mathbf{D}}^{\mathrm{E}})$ are positive semi-definite, i.e.

$$\mathbf{E} : (\bar{\mathbf{C}}^{\mathrm{G}} - \bar{\mathbf{C}}^{\Sigma}) : \mathbf{E} \geqslant 0, \quad \boldsymbol{\Sigma} : (\bar{\mathbf{D}}^{\mathrm{G}} - \bar{\mathbf{D}}^{\mathrm{E}}) : \boldsymbol{\Sigma} \geqslant 0, \tag{2.13a,b}$$

for any constant $\boldsymbol{\Sigma}$ and $\mathbf{E}$. Thus the energy density, and hence the corresponding overall moduli and compliances, $\bar{\mathbf{C}}^{\mathrm{G}}$ and $\bar{\mathbf{D}}^{\mathrm{G}}$, can be bounded if lower bounds for $\bar{\mathbf{C}}^{\Sigma}$ and $\bar{\mathbf{D}}^{\mathrm{E}}$ are obtained. In addition to this, it is shown in the next section that for any special boundary data which results in (2.11), $(\bar{\mathbf{C}}^{\mathrm{E}} - \bar{\mathbf{C}}^{\mathrm{S}})$, $(\bar{\mathbf{C}}^{\mathrm{S}} - \bar{\mathbf{C}}^{\Sigma})$, $(\bar{\mathbf{D}}^{\Sigma} - \bar{\mathbf{D}}^{\mathrm{S}})$, and $(\bar{\mathbf{D}}^{\mathrm{S}} - \bar{\mathbf{D}}^{\mathrm{E}})$ are positive semi-definite, i.e.

$$\mathbf{E} : (\bar{\mathbf{C}}^{\mathrm{E}} - \bar{\mathbf{C}}^{\mathrm{S}}) : \mathbf{E} \geqslant 0, \quad \mathbf{E} : (\bar{\mathbf{C}}^{\mathrm{S}} - \bar{\mathbf{C}}^{\Sigma}) : \mathbf{E} \geqslant 0,$$

$$\boldsymbol{\Sigma} : (\bar{\mathbf{D}}^{\Sigma} - \bar{\mathbf{D}}^{\mathrm{S}}) : \boldsymbol{\Sigma} \geqslant 0, \quad \boldsymbol{\Sigma} : (\bar{\mathbf{D}}^{\mathrm{S}} - \bar{\mathbf{D}}^{\mathrm{E}}) : \boldsymbol{\Sigma} \geqslant 0, \tag{2.14a–d}$$

for any constant $\boldsymbol{\Sigma}$ and $\mathbf{E}$.

Though the boundary conditions (2.4a) and (2.6a) are simple, the evaluation of $\bar{\mathbf{D}}^{\Sigma}$

†It is shown that, among all consistent boundary data which produce the same average overall strain (stress), uniform-traction (linear-displacement) boundary data render the strain energy (the complementary strain energy) an absolute minimum. Note that, for uniform boundary tractions, the overall strain is fixed, and for linear boundary displacements, the overall stress is fixed.

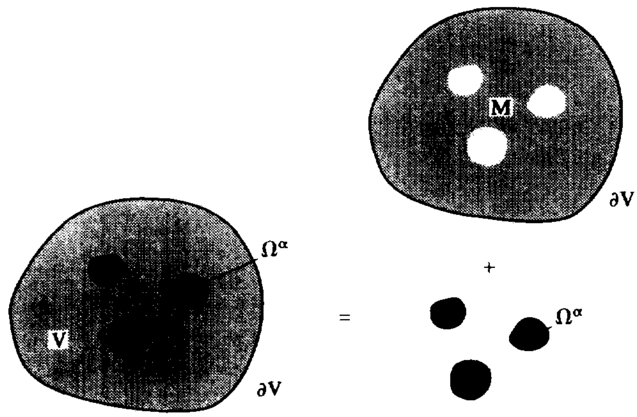

Fig. 2. Modified V: inclusions are removed from the matrix which then contains cavities; the inclusions are isolated.

and $\bar{\mathbf{C}}^{E}$ for an arbitrary geometry is rather complicated, because it requires the calculation of $\boldsymbol{\varepsilon}^{\alpha}$ and $\boldsymbol{\sigma}^{\alpha}$. This involves the solution of integral equations resulting from the boundary conditions on $\partial\Omega^{\alpha}$. Instead, seek to develop computable upper bounds for $\bar{\mathbf{D}}^{\Sigma}$ and $\bar{\mathbf{C}}^{E}$ (lower bounds for $\bar{\mathbf{C}}^{\Sigma}$ and $\bar{\mathbf{D}}^{E}$), as discussed in the sequel, where these computable bounds are denoted, respectively, by $\hat{\mathbf{D}}^{\Sigma}$ and $\hat{\mathbf{C}}^{E}$. First, techniques for calculating $\hat{\mathbf{D}}^{\Sigma}$ and $\hat{\mathbf{C}}^{E}$ are developed in Section 3, and then in Section 4 it is shown that these are indeed the required bounds.

3. ESTIMATES OF OVERALL MODULI AND COMPLIANCES

3.1. *Calculation technique for upper bound on $\bar{\mathbf{D}}^{\Sigma}$*

To estimate the overall compliance tensor, $\hat{\mathbf{D}}^{\Sigma}$, when uniform tractions are prescribed on ∂V, first assume that all inclusions, Ω^{α}, $\alpha = 1, 2, \ldots, n$, are removed, resulting in an elastic matrix M, containing n cavities, Ω^{α}, see Fig. 2. Then apply uniform tractions corresponding to constant stresses $\hat{\boldsymbol{\sigma}}^{\alpha}$, on $\partial\Omega^{\alpha}$ of the resulting cavities and the isolated inclusions. Finally, calculate $\hat{\boldsymbol{\sigma}}^{\alpha}$ such that the average strain $\hat{\boldsymbol{\varepsilon}}^{\alpha}$ for the cavities and the isolated inclusions are the same.

Denote the displacement, strain, and stress fields in the matrix by $\hat{\mathbf{u}}^{\Sigma}(\mathbf{x})$, $\hat{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})$, and $\hat{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})$, when the matrix is subjected to the following boundary conditions:

$$\hat{\mathbf{t}}^{\Sigma} = \begin{cases} \mathbf{n}\cdot\hat{\boldsymbol{\sigma}}^{\Sigma} = \mathbf{n}\cdot\hat{\boldsymbol{\Sigma}}, & \text{on } \partial V \\ -\mathbf{n}^{\alpha}\cdot\hat{\boldsymbol{\sigma}}^{\Sigma} = -\mathbf{n}^{\alpha}\cdot\hat{\boldsymbol{\sigma}}^{\alpha}, & \text{on } \partial\Omega^{\alpha}, \alpha = 1, 2, \ldots, n, \end{cases} \tag{3.1}$$

where $\hat{\boldsymbol{\Sigma}}$ and $\hat{\boldsymbol{\sigma}}^{\alpha}$ are constant† stress tensors, and $\mathbf{n}^{\alpha}$ is the unit outward normal on $\partial\Omega^{\alpha}$. Denote the average strain of the cavity Ω^{α} resulting from the displacement field $\hat{\mathbf{u}}^{\Sigma}$ by $\hat{\boldsymbol{\varepsilon}}^{\alpha M}$.

†In the presence of the inclusions, the interface tractions on $\partial\Omega^{\alpha}$ are not, in general, uniform, even for uniform tractions applied on ∂V. The caret on field quantities emphasizes that these are not the actual fields for the composite.

$$\boldsymbol{\varepsilon}^{\alpha M} \equiv \frac{1}{\Omega^{\alpha}} \int_{\partial\Omega^{\alpha}} \tfrac{1}{2}(\hat{\mathbf{u}}^{\Sigma} \otimes \mathbf{n}^{\alpha} + \mathbf{n}^{\alpha} \otimes \hat{\mathbf{u}}^{\Sigma})\,\mathrm{d}S. \tag{3.2}$$

Since the matrix material is linearly elastic, $\boldsymbol{\varepsilon}^{\alpha M}$ is linearly related to $\hat{\boldsymbol{\Sigma}}$ and $\hat{\boldsymbol{\sigma}}^{\alpha}$, $\alpha = 1, 2, \ldots, n$. This linear relation may be expressed as

$$\boldsymbol{\varepsilon}^{\alpha M} = \mathbf{D} : \hat{\boldsymbol{\Sigma}} - \sum_{\beta=1}^{n} \mathbf{M}^{\alpha\beta} : (\hat{\boldsymbol{\sigma}}^{\beta} - \hat{\boldsymbol{\Sigma}}). \tag{3.3}$$

The fourth order tensor $-\mathbf{M}^{\alpha\beta}$ relates the average strain in Ω^{α} to the constant stress tensor $\hat{\boldsymbol{\sigma}}^{\beta}$ when uniform tractions $\hat{\mathbf{t}} = -\mathbf{n}^{\beta} \cdot \hat{\boldsymbol{\sigma}}^{\beta}$ are applied on Ω^{β} while keeping the boundaries ∂V and all $\partial\Omega^{\alpha}$'s, $\alpha \neq \beta$, traction free. The minus sign is used following Hill (1965).

Now, consider the volume integral of the strain and energy density in the matrix. It follows from (2.3a,c) and (3.2) that

$$\int_{M} \boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \int_{\partial V} \tfrac{1}{2}(\hat{\mathbf{u}}^{\Sigma} \otimes \mathbf{n} + \mathbf{n} \otimes \hat{\mathbf{u}}^{\Sigma})\,\mathrm{d}S - \sum_{\alpha=1}^{n} \Omega^{\alpha} \boldsymbol{\varepsilon}^{\alpha M}, \tag{3.4a}$$

$$\int_{M} \tfrac{1}{2}\boldsymbol{\sigma}^{\Sigma}(\mathbf{x}) : \boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \frac{1}{2}\int_{\partial V} \hat{\mathbf{t}}^{\Sigma} \cdot \hat{\mathbf{u}}^{\Sigma}\,\mathrm{d}A + \frac{1}{2}\sum_{\alpha=1}^{n} \int_{\partial\Omega^{\alpha}} \hat{\mathbf{t}}^{\Sigma} \cdot \hat{\mathbf{u}}^{\Sigma}\,\mathrm{d}A$$

$$= \tfrac{1}{2}\hat{\boldsymbol{\Sigma}} : \int_{\partial V} \tfrac{1}{2}(\hat{\mathbf{u}}^{\Sigma} \otimes \mathbf{n} + \mathbf{n} \otimes \hat{\mathbf{u}}^{\Sigma})\,\mathrm{d}S - \frac{1}{2}\sum_{\alpha=1}^{n} \Omega^{\alpha} \hat{\boldsymbol{\sigma}}^{\alpha} : \boldsymbol{\varepsilon}^{\alpha M}. \tag{3.4b}$$

Alternatively, since the compliance tensor $\mathbf{D}$ is constant in the matrix, the volume integral of strain in the matrix can be expressed as

$$\int_{M} \boldsymbol{\varepsilon}^{\Sigma}(\mathbf{X})\,\mathrm{d}V = \mathbf{D} : \int_{M} \boldsymbol{\sigma}^{\Sigma}(\mathbf{X})\,\mathrm{d}V = V\mathbf{D} : \left\{\hat{\boldsymbol{\Sigma}} - \sum_{\alpha=1}^{n} f_{\alpha} \hat{\boldsymbol{\sigma}}^{\alpha}\right\}. \tag{3.4c}$$

Substitute (3.4c) into (3.4a) to obtain

$$\bar{\boldsymbol{\varepsilon}}^{\Sigma} \equiv \frac{1}{V}\int_{\partial V} \tfrac{1}{2}(\hat{\mathbf{u}}^{\Sigma} \otimes \mathbf{n} + \mathbf{n} \otimes \hat{\mathbf{u}}^{\Sigma})\,\mathrm{d}S$$

$$= \mathbf{D} : \hat{\boldsymbol{\Sigma}} + \sum_{\alpha=1}^{n} f_{\alpha}(\boldsymbol{\varepsilon}^{\alpha M} - \mathbf{D} : \hat{\boldsymbol{\sigma}}^{\alpha}). \tag{3.5}$$

Now assume uniform stress fields in the isolated inclusions,

$$\hat{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x}) = \hat{\boldsymbol{\sigma}}^{\alpha}, \quad \text{in}\,\Omega^{\alpha}, \quad \alpha = 1, 2, \ldots, n. \tag{3.6}$$

Denote the corresponding average strains by $\boldsymbol{\varepsilon}^{\alpha I}$,

$$\boldsymbol{\varepsilon}^{\alpha I} \equiv \bar{\mathbf{D}}^{\alpha} : \hat{\boldsymbol{\sigma}}^{\alpha}, \quad \bar{\mathbf{D}}^{\alpha} \equiv \frac{1}{\Omega^{\alpha}}\int_{\Omega^{\alpha}} \mathbf{D}^{\alpha}(\mathbf{x})\,\mathrm{d}V, \quad \alpha = 1, 2, \ldots, n. \tag{3.7a,b}$$

Note that the inclusions may be heterogeneous. If they are homogeneous then

$\bar{\mathbf{D}}^\alpha = \mathbf{D}^\alpha$. Then, from (3.4a,b) and (3.5), the average strain and energy density in this modified solid† are given by

$$\frac{1}{V}\int_V \hat{\varepsilon}^\Sigma(\mathbf{x})\,dV = \frac{1}{V}\int_M \hat{\varepsilon}^\Sigma(\mathbf{x})\,dV + \sum_{\alpha=1}^{n} f_\alpha \hat{\varepsilon}^{\alpha I}$$

$$= \hat{\varepsilon}^\Sigma + \sum_{\alpha=1}^{n} f_\alpha(\hat{\varepsilon}^{\alpha I} - \hat{\varepsilon}^{\alpha M}).$$

$$\frac{1}{V}\int_V \tfrac{1}{2}\hat{\sigma}^\Sigma(\mathbf{x}):\hat{\varepsilon}^\Sigma(\mathbf{x})\,dV = \frac{1}{V}\int_M \tfrac{1}{2}\hat{\sigma}^\Sigma(\mathbf{x}):\hat{\varepsilon}^\Sigma(\mathbf{x})\,dV + \sum_{\alpha=1}^{n} f_\alpha \tfrac{1}{2} f_\alpha \hat{\sigma}^\alpha : \hat{\varepsilon}^{\alpha I}$$

$$= \tfrac{1}{2}\hat{\Sigma}:\hat{\varepsilon}^\Sigma + \sum_{\alpha=1}^{n} \tfrac{1}{2} f_\alpha \hat{\sigma}^\alpha : (\hat{\varepsilon}^{\alpha I} - \hat{\varepsilon}^{\alpha M}). \qquad (3.8a,b)$$

Finally impose a weak kinematical admissibility (see Nemat-Nasser and Hori, 1993) that the cavity strain $\hat{\varepsilon}^{\alpha M}$ should equal the average inclusion strain $\hat{\varepsilon}^{\alpha I}$, i.e.

$$\hat{\varepsilon}^{\alpha M} = \hat{\varepsilon}^{\alpha I} \equiv \hat{\varepsilon}^\alpha, \quad \alpha = 1, 2, \ldots, n. \qquad (3.9)$$

From (3.3) and (3.7a), then obtain

$$\hat{\sigma}^\alpha = \hat{\mathbf{J}}^\alpha : \hat{\Sigma}, \quad \hat{\mathbf{J}}^\alpha = \sum_{\beta=1}^{n} \mathbf{K}^{\alpha\beta} : (\mathbf{D} + \bar{\mathbf{M}}^\beta), \quad \alpha = 1, 2, \ldots, n, \qquad (3.10a,b)$$

where

$$\bar{\mathbf{M}}^\beta \equiv \sum_{\gamma=1}^{n} \mathbf{M}^{\beta\gamma}, \quad \beta = 1, 2, \ldots, n. \qquad (3.11a)$$

and $\mathbf{K}^{\alpha\beta}$ is the inverse of $(\mathbf{M}^{\alpha\beta} + \bar{\mathbf{D}}^\alpha \delta_{\alpha\beta})$ defined by

$$\sum_{\gamma=1}^{n} \mathbf{K}^{\alpha\gamma} : (\mathbf{M}^{\gamma\beta} + \bar{\mathbf{D}}^\gamma \delta_{\gamma\beta}) = \mathbf{I}\delta_{\alpha\beta}, \qquad (3.11b)$$

where $\mathbf{I}$ is the fourth order symmetric identity tensor, and $\delta_{\alpha\beta}$ is the Krönecker delta.

Substitute (3.9) into (3.8) and in view of (3.5) obtain

$$\frac{1}{V}\int_V \hat{\varepsilon}^\Sigma(\mathbf{x}) \equiv \hat{\varepsilon}^\Sigma = \mathbf{D}:\hat{\Sigma} + \sum_{\alpha=1}^{n} f_\alpha(\bar{\mathbf{D}}^\alpha - \mathbf{D}):\hat{\mathbf{J}}^\alpha:\hat{\Sigma} \equiv \hat{\mathbf{D}}^\Sigma:\hat{\Sigma},$$

$$\hat{\bar{\sigma}}^\Sigma \equiv \frac{1}{V}\int_V \hat{\sigma}^\Sigma(\mathbf{x})\,dV = \hat{\Sigma}, \quad \hat{\bar{w}}^\Sigma \equiv \frac{1}{V}\int_V \tfrac{1}{2}\hat{\sigma}^\Sigma(\mathbf{x}):\hat{\varepsilon}^\Sigma(\mathbf{x})\,dV = \tfrac{1}{2}\hat{\Sigma}:\hat{\bar{\varepsilon}}^\Sigma. \qquad (3.12a–c)$$

This $\hat{\mathbf{D}}^\Sigma$ is the required estimate of the overall compliance $\bar{\mathbf{D}}^\Sigma$. In the next section, it is shown that this estimate is actually an upper bound for $\bar{\mathbf{D}}^\Sigma$ (or its inverse $(\hat{\mathbf{D}}^\Sigma)^{-1} = \hat{\mathbf{C}}^\Sigma$ is a lower bound for $\bar{\mathbf{C}}^\Sigma$). Note that, the displacement field $\hat{\mathbf{u}}^\Sigma$ may

†Here, the matrix M contains cavities under uniform tractions and the inclusions are isolated and have uniform boundary tractions (uniform stresses when they are homogeneous).

not be continuous across all interfaces $\partial\Omega^\alpha$, $\alpha = 1, 2, \ldots, n$. However, the tractions $\hat{\mathbf{t}}^\Sigma = \mathbf{n}^\alpha \cdot \hat{\sigma}^\Sigma$ are continuous across all interfaces.

3.2. *Calculation technique for upper bound on $\bar{\mathbf{C}}^E$*

An upper bound for the overall elasticity tensor $\bar{\mathbf{C}}^E$ can be obtained by considering the displacement, strain, and stress fields, $\hat{\mathbf{u}}^E(\mathbf{x})$, $\hat{\boldsymbol{\varepsilon}}^E(\mathbf{x})$, and $\hat{\boldsymbol{\sigma}}^E(\mathbf{x})$, in the matrix when the inclusions are removed and the matrix M, containing n cavities Ω^α, is subjected to the following boundary conditions:

$$\hat{\mathbf{u}}^E = \begin{cases} \mathbf{x} \cdot \hat{\mathbf{E}}, & \text{on } \partial V \\ \mathbf{x} \cdot \hat{\boldsymbol{\varepsilon}}^\alpha, & \textit{on } \partial\Omega^\alpha, \quad \alpha = 1, 2, \ldots, n. \end{cases} \tag{3.13}$$

Here, $\hat{\mathbf{E}}$ and $\hat{\boldsymbol{\varepsilon}}^\alpha$ are constant strain tensors. The idea again is to calculate the constant strain tensors $\hat{\boldsymbol{\varepsilon}}^\alpha$ such that the average stress in the inclusion Ω^α is the same as the average resistance provided by the matrix.

Denote the average stress produced by the matrix on Ω^α, $\alpha = 1, 2, \ldots, n$, resulting from the stress field $\hat{\boldsymbol{\sigma}}^E$, by $\hat{\boldsymbol{\sigma}}^{\alpha M}$. Then it follows that

$$\hat{\boldsymbol{\sigma}}^{\alpha M} \equiv \frac{1}{\Omega^\alpha} \int_{\partial\Omega^\alpha} \tfrac{1}{2}(\hat{\mathbf{t}}^E \otimes \mathbf{x} + \mathbf{x} \otimes \hat{\mathbf{t}}^E)\, \mathrm{d}S, \quad \alpha = 1, 2, \ldots, n,$$

$$\hat{\mathbf{t}}^E(\mathbf{x}) = \mathbf{n}^\alpha \cdot \hat{\boldsymbol{\sigma}}^E(\mathbf{x}), \quad \text{on } \partial\Omega^\alpha. \tag{3.14a,b}$$

Since the matrix material is linearly elastic, $\hat{\boldsymbol{\sigma}}^{\alpha M}$ is linearly related to $\hat{\mathbf{E}}$ and $\hat{\boldsymbol{\varepsilon}}^\alpha$, $\alpha = 1, 2, \ldots, n$. This linear relation may be expressed as

$$\hat{\boldsymbol{\sigma}}^{\alpha M} = \mathbf{C} : \hat{\mathbf{E}} - \sum_{\beta=1}^{n} \mathbf{L}^{\alpha\beta} : (\hat{\boldsymbol{\varepsilon}}^\beta - \hat{\mathbf{E}}). \tag{3.15}$$

The fourth order tensor $-\mathbf{L}^{\alpha\beta}$ relates the average stress in Ω^α to the uniform strain in Ω^β when the boundaries ∂V and all $\partial\Omega^\alpha$s, $\alpha \neq \beta$, are fixed.

Now, consider the volume integral of the stress and energy density in the matrix. It follows from (2.3b,c) and (3.14) that

$$\int_M \hat{\boldsymbol{\sigma}}^E(\mathbf{x})\, \mathrm{d}V = \int_{\partial V} \tfrac{1}{2}(\hat{\mathbf{t}}^E \otimes \mathbf{x} + \mathbf{x} \otimes \hat{\mathbf{t}}^E)\, \mathrm{d}S - \sum_{\alpha=1}^{n} \Omega^\alpha \hat{\boldsymbol{\sigma}}^{\alpha M},$$

$$\int_M \tfrac{1}{2}\hat{\boldsymbol{\sigma}}^E(\mathbf{x}) : \hat{\boldsymbol{\varepsilon}}^E(\mathbf{x})\, \mathrm{d}V = \int_{\partial V} \tfrac{1}{2}\hat{\mathbf{t}}^E \cdot \hat{\mathbf{u}}^E\, \mathrm{d}A + \sum_{\alpha=1}^{n} \int_{\partial\Omega^\alpha} \tfrac{1}{2}\hat{\mathbf{t}}^E \cdot \hat{\mathbf{u}}^E\, \mathrm{d}S$$

$$= \tfrac{1}{2}\hat{\mathbf{E}} : \int_{\partial V} \tfrac{1}{2}(\hat{\mathbf{t}}^E \otimes \mathbf{x} + \mathbf{x} \otimes \hat{\mathbf{t}}^E)\, \mathrm{d}S - \sum_{\alpha=1}^{n} \Omega^\alpha \tfrac{1}{2} \hat{\boldsymbol{\sigma}}^{\alpha M} : \hat{\boldsymbol{\varepsilon}}^\alpha. \tag{3.16a,b}$$

Since the elasticity tensor $\mathbf{C}$ is constant in the matrix, the volume integral of stress in the matrix can be expressed as

$$\int_M \hat{\sigma}^E(\mathbf{x}) = \mathbf{C} : \int_M \hat{\varepsilon}^E(\mathbf{x})\, dV = V\mathbf{C} : \left\{ \mathbf{E} - \sum_{\alpha=1}^{n} f_\alpha \hat{\varepsilon}^\alpha \right\}. \tag{3.16c}$$

Substitute (3.16c) into (3.16a) to obtain

$$\bar{\sigma}^E \equiv \frac{1}{V} \int_{\partial V} \tfrac{1}{2}(\hat{\mathbf{t}}^E \otimes \mathbf{x} + \mathbf{x} \otimes \hat{\mathbf{t}}^E)\, dS = \mathbf{C} : \hat{\mathbf{E}} + \sum_{\alpha=1}^{n} f_\alpha (\hat{\sigma}^{\alpha M} - \mathbf{C} : \hat{\varepsilon}^\alpha). \tag{3.17}$$

Now, assume uniform strain fields in the isolated inclusions,

$$\hat{\varepsilon}^F(\mathbf{x}) = \hat{\varepsilon}^\alpha, \quad \text{in } \Omega^\alpha, \quad \alpha = 1, 2, \ldots, n. \tag{3.18a}$$

Denote the corresponding average stresses by $\hat{\sigma}^{\alpha I}$,

$$\hat{\sigma}^{\alpha I} = \bar{\mathbf{C}}^\alpha : \hat{\varepsilon}^\alpha, \quad \bar{\mathbf{C}}^\alpha \equiv \frac{1}{\Omega^\alpha} \int_{\Omega^\alpha} \mathbf{C}^\alpha(\mathbf{x})\, dV, \quad \alpha = 1, 2, \ldots, n. \tag{3.18b,c}$$

If the inclusions are homogeneous, then $\bar{\mathbf{C}}^\alpha = \mathbf{C}^\alpha$. From (3.16a,b) and (3.17), the average stress and energy density in the modified solid are given by

$$\frac{1}{V} \int_V \hat{\sigma}^E(\mathbf{x})\, dV = \frac{1}{V} \int_M \hat{\sigma}^E(\mathbf{x})\, dV + \sum_{\alpha=1}^{n} f_\alpha \hat{\sigma}^{\alpha I}$$

$$= \bar{\sigma}^E + \sum_{\alpha=1}^{n} (\hat{\sigma}^{\alpha I} - \hat{\sigma}^{\alpha M}),$$

$$\frac{1}{V} \int_V \tfrac{1}{2} \hat{\sigma}^E(\mathbf{x}) : \hat{\varepsilon}^F(\mathbf{x})\, dV = \frac{1}{V} \int_M \tfrac{1}{2} \hat{\sigma}^E(\mathbf{x}) : \hat{\varepsilon}^E(\mathbf{x})\, dV + \sum_{\alpha=1}^{n} f_\alpha \tfrac{1}{2} \hat{\sigma}^{\alpha I} : \hat{\varepsilon}^\alpha$$

$$= \tfrac{1}{2} \bar{\sigma}^E : \mathbf{E} + \sum_{\alpha=1}^{n} f_\alpha \tfrac{1}{2} (\hat{\sigma}^{\alpha I} - \hat{\sigma}^{\alpha M}) : \hat{\varepsilon}^\alpha. \tag{3.19a,b}$$

Finally, impose a weak statical admissibility (see Nemat-Nasser and Hori, 1993) that the stress associated with the average matrix resistance at cavity Ω^α should equal the corresponding average inclusion stress, i.e.

$$\hat{\sigma}^{\alpha M} = \hat{\sigma}^{\alpha I} \equiv \hat{\sigma}^\alpha. \tag{3.20}$$

From (3.15), (3.18b), and (3.20), then obtain

$$\hat{\varepsilon}^\alpha = \hat{\mathbf{H}}^\alpha : \hat{\mathbf{E}}, \quad \hat{\mathbf{H}}^\alpha = \sum_{\beta=1}^{n} \mathbf{N}^{\alpha\beta} : (\mathbf{C} + \bar{\mathbf{L}}^\beta), \quad \alpha = 1, 2, \ldots, n, \tag{3.21a,b}$$

where

$$\bar{\mathbf{L}}^\alpha \equiv \sum_{\beta=1}^{n} \mathbf{L}^{\alpha\beta}, \quad \alpha = 1, 2, \ldots, n, \tag{3.22a}$$

and $\mathbf{N}^{\alpha\beta}$ is the inverse of $(\mathbf{L}^{\alpha\beta} + \bar{\mathbf{C}}^\alpha \delta_{\alpha\beta})$ defined by

$$\sum_{\gamma=1}^{n} \mathbf{N}^{\alpha\gamma} : (\mathbf{L}^{\gamma\beta} + \bar{\mathbf{C}}^{\gamma}\delta_{\gamma\beta}) = \mathbf{I}\delta_{\alpha\beta}. \tag{3.22b}$$

Substitute (3.20) into (3.19a,b) and in view of (3.17) obtain

$$\frac{1}{V}\int_V \hat{\boldsymbol{\sigma}}^E(\mathbf{x})\,\mathrm{d}V \equiv \bar{\hat{\boldsymbol{\sigma}}}^E = \mathbf{C}:\hat{\mathbf{E}} + \sum_{\alpha=1}^{n} f_\alpha(\bar{\mathbf{C}}^\alpha - \mathbf{C}):\mathbf{H}^\alpha:\hat{\mathbf{E}} \equiv \hat{\mathbf{C}}^E:\hat{\mathbf{E}},$$

$$\bar{\hat{\boldsymbol{\varepsilon}}}^E \equiv \frac{1}{V}\int_V \hat{\boldsymbol{\varepsilon}}^E(\mathbf{x})\,\mathrm{d}V = \hat{\mathbf{E}}, \quad \bar{\hat{w}}^E \equiv \frac{1}{V}\int_V \tfrac{1}{2}\hat{\boldsymbol{\sigma}}^E(\mathbf{x}):\hat{\boldsymbol{\varepsilon}}^E(\mathbf{x})\,\mathrm{d}V = \tfrac{1}{2}\hat{\mathbf{E}}:\bar{\hat{\boldsymbol{\sigma}}}, \tag{3.23a–c}$$

where $\hat{\mathbf{C}}^E$ is the estimate of the overall elasticity tensor $\bar{\mathbf{C}}^E$. It is shown in the next section that $\hat{\mathbf{C}}^E$ is actually an upper bound for $\bar{\mathbf{C}}^E$.

Note that, when the inclusions are actually cavities ($\bar{\mathbf{C}}^\alpha = 0$), then $\hat{\mathbf{D}}^\Sigma = \bar{\mathbf{D}}^\Sigma$, exactly. Similarly, when the inclusions are rigid ($\bar{\mathbf{D}}^\alpha = 0$) then $\hat{\mathbf{C}}^E = \bar{\mathbf{C}}^E$ exactly. In general, $\hat{\mathbf{D}}^\Sigma$ and $\hat{\mathbf{C}}^E$ are not each other's inverse: they are in fact unrelated, as they correspond to different unrelated boundary data. Their inverses are denoted by

$$\hat{\mathbf{C}}^\Sigma \equiv (\hat{\mathbf{D}}^\Sigma)^{-1}, \quad \hat{\mathbf{D}}^E \equiv (\hat{\mathbf{C}}^E)^{-1}. \tag{3.24a,b}$$

4. BOUNDS ON POTENTIALS

Now, following the procedure of Nemat-Nasser and Hori (1993), seek to show that $(\bar{\mathbf{C}}^\Sigma - \hat{\mathbf{C}}^\Sigma)$ and $(\bar{\mathbf{D}}^E - \hat{\mathbf{D}}^E)$, and hence $(\bar{\mathbf{C}}^G - \hat{\mathbf{C}}^\Sigma)$ and $(\bar{\mathbf{D}}^G - \hat{\mathbf{D}}^E)$, are positive semi-definite tensors.

Since the constituents of the solid are assumed to be linearly elastic, having positive-definite elasticity and compliance tensors, these constituents admit a convex stress potential $\phi = \phi(\boldsymbol{\varepsilon})$ and a convex strain potential $\psi = \psi(\boldsymbol{\sigma})$ such that

$$\boldsymbol{\sigma} = \frac{\partial \phi}{\partial \boldsymbol{\varepsilon}}, \quad \boldsymbol{\varepsilon} = \frac{\partial \psi}{\partial \boldsymbol{\sigma}}. \tag{4.1a,b}$$

It follows from Nemat-Nasser and Hori (1993) that*

$$\phi\{\boldsymbol{\varepsilon}^\Sigma(\mathbf{x})\} - \phi\{\hat{\boldsymbol{\varepsilon}}^\Sigma(\mathbf{x})\} \geq \{\boldsymbol{\varepsilon}^\Sigma(\mathbf{x}) - \hat{\boldsymbol{\varepsilon}}^\Sigma(\mathbf{x})\} : \hat{\boldsymbol{\sigma}}^\Sigma(\mathbf{x}). \tag{4.2}$$

Since $\hat{\boldsymbol{\sigma}}^\Sigma(\mathbf{x})$ and $\boldsymbol{\varepsilon}^\Sigma(\mathbf{x})$ are statically and kinematically admissible fields, respectively, it follows that

$$\frac{1}{V}\int_V \boldsymbol{\varepsilon}^\Sigma(\mathbf{x}):\hat{\boldsymbol{\sigma}}^\Sigma(\mathbf{x})\,\mathrm{d}V = \bar{\boldsymbol{\varepsilon}}^\Sigma:\hat{\Sigma}. \tag{4.3}$$

Sum the volume integrals of (4.2) over the matrix and inclusions, and in view of (3.12c) and (4.3) obtain

$$\Phi^\Sigma(\bar{\boldsymbol{\varepsilon}}^\Sigma) - \hat{\Phi}^\Sigma(\bar{\hat{\boldsymbol{\varepsilon}}}^\Sigma) \geq (\bar{\boldsymbol{\varepsilon}}^\Sigma - \bar{\hat{\boldsymbol{\varepsilon}}}^\Sigma):\hat{\Sigma}, \tag{4.4a}$$

where $\Phi^\Sigma(\bar{\boldsymbol{\varepsilon}}^\Sigma)$ and $\hat{\Phi}^\Sigma(\bar{\hat{\boldsymbol{\varepsilon}}}^\Sigma)$ are the corresponding overall stress potentials, defined by

*Note again that the superscript Σ denotes the field corresponding to a uniform-traction boundary data.

$$\Phi^{\Sigma}(\bar{\varepsilon}^{\Sigma}) \equiv \frac{1}{V}\int_V \phi\{\varepsilon^{\Sigma}(\mathbf{x})\}\,\mathrm{d}V = \tfrac{1}{2}\bar{\varepsilon}^{\Sigma} : \bar{\mathbf{C}}^{\Sigma} : \bar{\varepsilon}^{\Sigma},$$

$$\hat{\Phi}^{\Sigma}(\hat{\varepsilon}^{\Sigma}) \equiv \frac{1}{V}\int_V \phi\{\varepsilon^{\Sigma}(\mathbf{x})\}\,\mathrm{d}V = \tfrac{1}{2}\bar{\bar{\varepsilon}}^{\Sigma} : \hat{\mathbf{C}}^{\Sigma} : \bar{\bar{\varepsilon}}^{\Sigma}. \tag{4.4b,c}$$

Therefore, when the overall average strain is the same, it follows that†

$$\Phi^{\Sigma}(\bar{\varepsilon}^{\Sigma}) \geqslant \hat{\Phi}^{\Sigma}(\bar{\bar{\varepsilon}}^{\Sigma}), \quad \text{for } \bar{\varepsilon}^{\Sigma} = \bar{\bar{\varepsilon}}^{\Sigma}. \tag{4.5a}$$

Note that while $\hat{\sigma}^{\Sigma}(\mathbf{x})$ is statically admissible, $\hat{\varepsilon}^{\Sigma}(\mathbf{x})$ is not kinematically admissible. Hence, from the minimum complementary energy theorem

$$\Psi^{\Sigma}(\Sigma) \equiv \tfrac{1}{2}\Sigma : \bar{\mathbf{D}}^{\Sigma} : \Sigma \leqslant \hat{\Psi}^{\Sigma}(\hat{\Sigma}) \equiv \tfrac{1}{2}\hat{\Sigma} : \hat{\mathbf{D}}^{\Sigma} : \hat{\Sigma} \quad \text{for } \Sigma = \hat{\Sigma}; \tag{4.5b}$$

see also Huet (1990). Hence, it follows that, for any constant $\mathbf{E}$ and Σ,

$$\mathbf{E} : (\bar{\mathbf{C}}^{\Sigma} - \hat{\mathbf{C}}^{\Sigma}) : \mathbf{E} \geqslant 0, \quad \Sigma : (\bar{\mathbf{D}}^{\Sigma} - \hat{\mathbf{D}}^{\Sigma}) : \Sigma \leqslant 0. \tag{4.6a,b}$$

Since $\bar{\mathbf{C}}^{\Sigma}$ and $\hat{\mathbf{C}}^{\Sigma}$ correspond to uniform boundary tractions, they are, respectively, the inverse of $\bar{\mathbf{D}}^{\Sigma}$ and $\hat{\mathbf{D}}^{\Sigma}$. Thus, from the identity (see, e.g. Nemat-Nasser and Hori, 1993, p. 278)

$$(\bar{\mathbf{D}}^{\Sigma} - \hat{\mathbf{D}}^{\Sigma})^{-1} = -\hat{\mathbf{C}}^{\Sigma} : (\bar{\mathbf{C}}^{\Sigma} - \hat{\mathbf{C}}^{\Sigma})^{-1} : \hat{\mathbf{C}}^{\Sigma} - \hat{\mathbf{C}}^{\Sigma},$$

(4.6b) also follows directly from (4.6a). Similarly, (4.6a) results from (4.6b) in view of the identity‡

$$(\bar{\mathbf{D}}^{\Sigma} - \hat{\mathbf{D}}^{\Sigma})^{-1} - \bar{\mathbf{C}}^{\Sigma} = -\bar{\mathbf{C}}^{\Sigma} : (\bar{\mathbf{C}}^{\Sigma} - \hat{\mathbf{C}}^{\Sigma})^{-1} : \bar{\mathbf{C}}^{\Sigma}.$$

In a similar manner, it is shown that, for any constant Σ and $\mathbf{E}$,

$$\Sigma : (\bar{\mathbf{D}}^{E} - \hat{\mathbf{D}}^{E}) : \Sigma \geqslant 0, \quad \mathbf{E} : (\bar{\mathbf{C}}^{E} - \hat{\mathbf{C}}^{E}) : \mathbf{E} \leqslant 0. \tag{4.6c,d}$$

This means that $(\bar{\mathbf{C}}^{\Sigma} - \hat{\mathbf{C}}^{\Sigma})$, $(\hat{\mathbf{D}}^{\Sigma} - \bar{\mathbf{D}}^{\Sigma})$, $(\bar{\mathbf{D}}^{E} - \hat{\mathbf{D}}^{E})$, and $(\hat{\mathbf{C}}^{E} - \bar{\mathbf{C}}^{E})$, are positive semi-definite. Then, it follows from Nemat-Nasser and Hori (1993) that $(\bar{\mathbf{C}}^{G} - \hat{\mathbf{C}}^{\Sigma})$ and $(\bar{\mathbf{D}}^{G} - \hat{\mathbf{D}}^{E})$ are positive semi-definite. For the special class of boundary data $\{\mathbf{t}^{S}, \mathbf{u}^{S}\}$ for which $\bar{\mathbf{C}}^{S} : \bar{\mathbf{D}}^{S} = \mathbf{I}$, it follows that $(\bar{\mathbf{C}}^{S} - \hat{\mathbf{C}}^{\Sigma})$, $(\hat{\mathbf{C}}^{E} - \bar{\mathbf{C}}^{S})$, $(\bar{\mathbf{D}}^{S} - \hat{\mathbf{D}}^{E})$, and $(\hat{\mathbf{D}}^{\Sigma} - \bar{\mathbf{D}}^{S})$, are positive semi-definite, i.e. the overall stress and strain potentials are bounded as follows:

$$\tfrac{1}{2}\bar{\varepsilon}^{S} : \hat{\mathbf{C}}^{E} : \bar{\varepsilon}^{S} \geqslant \Phi^{S}(\bar{\varepsilon}^{S}) \geqslant \tfrac{1}{2}\bar{\varepsilon}^{S} : \hat{\mathbf{C}}^{\Sigma} : \bar{\varepsilon}^{S},$$

$$\tfrac{1}{2}\bar{\sigma}^{S} : \hat{\mathbf{D}}^{\Sigma} : \bar{\sigma}^{S} \geqslant \Psi^{S}(\bar{\sigma}^{S}) \geqslant \tfrac{1}{2}\bar{\sigma}^{S} : \hat{\mathbf{D}}^{E} : \bar{\sigma}^{S}. \tag{4.7a,b}$$

Note that, even for solids with cavities, the lower bound $\hat{\mathbf{C}}^{\Sigma}$, in general, is positive-definite and so is the lower bound $\hat{\mathbf{D}}^{E}$ for solids with rigid inclusions. This is in contrast to, say the Hashin–Shtrikman bounds on the overall elasticity, which vanish for the lower bound when cavities are present, and become unbounded (upper) when rigid

†This result remains valid for nonlinear composites with convex potential $\phi(\varepsilon)$.

‡Note that these identities and the argument based on the complementary energy are relevant to the linear case, where as (4.5a) applies also to the nonlinear cases; see Nemat-Nasser *et al.* (1994).

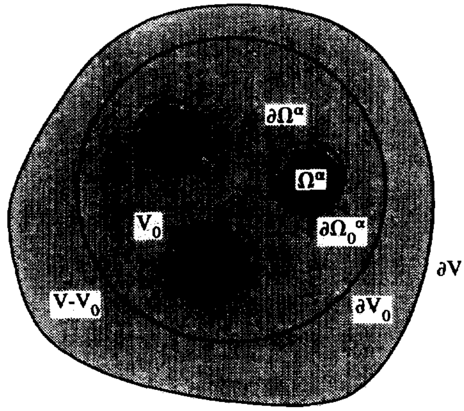

Fig. 3. A solid with fictitious regularly shaped boundaries.

inclusions are present, simply because these bounds are obtained without any assumption on the geometry of the inhomogeneities, whereas here we restrict the cavities or rigid inclusions to be disconnected. Thus, both the lower and the upper bounds developed in the present work for the overall elasticity remain finite even when both cavities and rigid inclusions are present.

5. IRREGULARLY SHAPED SOLID AND INCLUSIONS

It is seen from (3.12a) and (3.19a), that the upper bounds for the compliance and elasticity tensors require the evaluation of the fourth order tensors $\mathbf{M}^{\alpha\beta}$ and $\mathbf{L}^{\alpha\beta}$. This is in general not feasible even if there is only one inclusion in a homogeneous matrix, if the boundaries ∂V and $\partial\Omega^{\gamma}$ are irregularly shaped. The general bounds developed in Section 4 can be used to obtain bounds for irregularly shaped geometries, using a simple modification.

To this end, consider a regularly shaped fictitious boundary ∂V_0 which is inside the solid and bounds all the inclusions in the solid. Denote by V_0 the volume bounded by ∂V_0. For simplicity, consider the case where each inclusion is homogeneous. For each inclusion, consider a regularly shaped boundary $\partial\Omega_0^{\alpha}$ which completely bounds $\partial\Omega^{\alpha}$. Denote by Ω_0^{α} the volume bounded by $\partial\Omega_0^{\alpha}$, $\alpha = 1, 2, \ldots, n$. Denote by M_0 the matrix containing n cavities, Ω_0^{α}, and bounded by ∂V_0; see Fig. 3.

5.1. *Upper bound for* $\hat{\mathbf{D}}^{\Sigma}$

Now, consider a statically admissible stress field $\tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})$ which is uniform in the region bounded by ∂V and ∂V_0 and in Ω_0^{α},

$$\tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x}) = \begin{cases} \tilde{\boldsymbol{\Sigma}} & \text{in } V - V_0 \\ \tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x}) & \text{in } M_0 \\ \tilde{\boldsymbol{\sigma}}^{\alpha} & \text{in } \Omega_0^{\alpha}, \quad \alpha = 1, 2, \ldots, n, \end{cases} \tag{5.1}$$

where $\tilde{\mathbf{\Sigma}}$ and $\tilde{\boldsymbol{\sigma}}^{\alpha}$ are constant stress tensors. The corresponding displacement field $\tilde{\mathbf{u}}^{\Sigma}(\mathbf{x})$ is not continuous across the interfaces $\partial\Omega^{\alpha}$, $\partial\Omega_0^{\alpha}$, $\alpha = 1, 2, \ldots, n$, and ∂V_0. By imposing a weak kinematical admissibility as in (3.9), obtain the average strain, stress, and energy density in the region bounded by ∂V_0,

$$\frac{1}{V_0}\int_{V_0} \tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x}) = \bar{\tilde{\boldsymbol{\varepsilon}}}_0^{\Sigma} = \mathbf{D} : \tilde{\mathbf{\Sigma}} + \sum_{\alpha=1}^{n} f_{\alpha}^{0}(\bar{\mathbf{D}}^{\alpha} - \mathbf{D}) : \tilde{\mathbf{J}}_0^{\alpha} : \tilde{\mathbf{\Sigma}} \equiv \tilde{\mathbf{D}}_0^{\Sigma} : \tilde{\mathbf{\Sigma}},$$

$$\bar{\tilde{\boldsymbol{\sigma}}}_0^{\Sigma} \equiv \frac{1}{V_0}\int_{V_0} \tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \tilde{\mathbf{\Sigma}}, \quad \bar{\tilde{w}}_0^{\Sigma} \equiv \frac{1}{V_0}\int_{V_0} \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x}) : \tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \tfrac{1}{2}\tilde{\mathbf{\Sigma}} : \bar{\tilde{\boldsymbol{\varepsilon}}}_0^{\Sigma}, \tag{5.2a–c}$$

where

$$f_0^{\alpha} = \frac{\Omega_0^{\alpha}}{V_0}, \quad \bar{\mathbf{D}}^{\alpha} = (\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{D}^{\alpha} + (1 - \Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{D}, \quad \alpha = 1, 2, \ldots, n, \tag{5.3a,b}$$

and $\tilde{\mathbf{J}}_0^{\alpha}$ is given by the expressions (3.10b) to (3.11b) when the fourth order tensor $\mathbf{M}^{\alpha\beta}$ is replaced by $\mathbf{M}_0^{\alpha\beta}$. The fourth order tensor $\mathbf{M}_0^{\alpha\beta}$ relates the average strain in Ω_0^{α} to the constant stress tensor $\tilde{\boldsymbol{\sigma}}^{\beta}$ when uniform tractions $\tilde{\mathbf{t}} = -\mathbf{n}^{\beta}\cdot\tilde{\boldsymbol{\sigma}}^{\beta}$ are applied on $\partial\Omega_0^{\beta}$ while keeping the boundaries ∂V_0 and all $\partial\Omega^{\alpha}$'s, $\alpha \neq \beta$, traction free.

Then, the average strain, stress, and energy density in the modified solid are given by

$$\bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma} \equiv \frac{1}{V}\int_{V} \tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = (V_0/V)\tilde{\mathbf{D}}_0^{\Sigma} : \tilde{\mathbf{\Sigma}} + (1 - V_0/V)\mathbf{D} : \tilde{\mathbf{\Sigma}} \equiv \tilde{\mathbf{D}}^{\Sigma} : \tilde{\mathbf{\Sigma}},$$

$$\bar{\tilde{\boldsymbol{\sigma}}}^{\Sigma} \equiv \frac{1}{V}\int_{V} \tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \tilde{\mathbf{\Sigma}}, \quad \bar{\tilde{w}}^{\Sigma} = \frac{1}{V}\int_{V} \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x}) : \tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \tfrac{1}{2}\tilde{\mathbf{\Sigma}} : \bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma}. \tag{5.4a–c}$$

Since, $\tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})$ and $\boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x})$ are statically and kinematically admissible fields, respectively, both corresponding to uniform boundary data, it follows that

$$\frac{1}{V}\int_{V} \boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x}) : \tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \frac{1}{V}\int_{\partial V} \tilde{\mathbf{t}}^{\Sigma}\cdot\mathbf{u}^{\Sigma}\,\mathrm{d}S = \bar{\boldsymbol{\varepsilon}}^{\Sigma} : \tilde{\mathbf{\Sigma}}. \tag{5.5}$$

Therefore, volume integral of the inequality

$$\phi\{\boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x})\} - \phi\{\tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})\} \geqslant \{\boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x}) - \tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})\} : \tilde{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x}) \tag{5.6a}$$

yields

$$\Phi(\bar{\boldsymbol{\varepsilon}}^{\Sigma}) - \tilde{\Phi}(\bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma}) \geqslant (\bar{\boldsymbol{\varepsilon}}^{\Sigma} - \bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma}) : \tilde{\mathbf{\Sigma}}, \tag{5.6b}$$

where

$$\tilde{\Phi}(\bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma}) \equiv \frac{1}{V}\int_{V} \phi\{\tilde{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x})\}\,\mathrm{d}V = \tfrac{1}{2}\bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma} : \tilde{\mathbf{C}}^{\Sigma} : \bar{\tilde{\boldsymbol{\varepsilon}}}^{\Sigma}, \quad \tilde{\mathbf{C}}^{\Sigma} \equiv (\tilde{\mathbf{D}}^{\Sigma})^{-1}. \tag{5.7a,b}$$

This shows that $(\bar{\mathbf{C}}^{\Sigma} - \tilde{\mathbf{C}}^{\Sigma})$ and $(\tilde{\mathbf{D}}^{\Sigma} - \bar{\mathbf{D}}^{\Sigma})$ are positive semi-definite. This also results from the minimum complementary energy theorem.† Indeed, since $\hat{\mathbf{u}}^{\Sigma}(\mathbf{x})$ is weakly kinematically admissible in V, and $\hat{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})$ is statically admissible and constant on the interfaces, it follows that

$$\frac{1}{V}\int_V \hat{\boldsymbol{\varepsilon}}^{\Sigma}(\mathbf{x}):\hat{\boldsymbol{\sigma}}^{\Sigma}(\mathbf{x})\,\mathrm{d}V = \frac{1}{V}\int_{\partial V} \hat{\mathbf{u}}^{\Sigma}\cdot\hat{\mathbf{t}}^{\Sigma}\,\mathrm{d}S = \bar{\hat{\boldsymbol{\varepsilon}}}^{\Sigma}:\boldsymbol{\Sigma}. \tag{5.8}$$

Thus, $(\hat{\mathbf{C}}^{\Sigma}-\tilde{\mathbf{C}}^{\Sigma})$ and $(\tilde{\mathbf{D}}^{\Sigma}-\hat{\mathbf{D}}^{\Sigma})$ are positive semi-definite.

5.2. *Upper bound for* $\tilde{C}^E$

Similarly, an upper bound for $\tilde{\mathbf{C}}^{E}$ is obtained by considering a kinematically admissible displacement field

$$\tilde{\mathbf{u}}^{E}(\mathbf{x}) = \begin{cases} \mathbf{x}\cdot\tilde{\mathbf{E}} & \text{in} \quad V-V_0 \\ \tilde{\mathbf{u}}^{E}(\mathbf{x}) & \text{in} \quad M_0 \\ \mathbf{x}\cdot\tilde{\boldsymbol{\varepsilon}}^{\alpha} & \text{in} \quad \Omega_0^{\alpha}, \quad \alpha = 1,2,\ldots,n, \end{cases} \tag{5.9}$$

where $\tilde{\mathbf{E}}$ and $\tilde{\boldsymbol{\varepsilon}}^{\alpha}$ are constant strain tensors. Then, the corresponding stress field $\tilde{\boldsymbol{\sigma}}(\mathbf{x})$ is not statically admissible. By imposing a weak statical admissibility as in (3.20), obtain the average stress, strain, and energy density in the modified solid,

$$\bar{\tilde{\boldsymbol{\sigma}}}^{E} \equiv \frac{1}{V}\int_V \tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x})\,\mathrm{d}V = (V_0/V)\tilde{\mathbf{C}}_0^{E}:\tilde{\mathbf{E}}+(1-V_0/V)\mathbf{C}:\tilde{\mathbf{E}} \equiv \tilde{\mathbf{C}}^{E}:\tilde{\mathbf{E}},$$

$$\bar{\tilde{\boldsymbol{\varepsilon}}}^{E} \equiv \frac{1}{V}\int_V \tilde{\boldsymbol{\varepsilon}}^{E}(\mathbf{x})\,\mathrm{d}V = \tilde{\mathbf{E}}, \quad \bar{\tilde{w}}^{E} = \frac{1}{V}\int_V \tfrac{1}{2}\tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x}):\tilde{\boldsymbol{\varepsilon}}^{E}(\mathbf{x})\,\mathrm{d}V = \tfrac{1}{2}\tilde{\mathbf{E}}:\bar{\tilde{\boldsymbol{\sigma}}}^{E}, \tag{5.10a–c}$$

where

$$\tilde{\mathbf{C}}_0^{E} = \mathbf{C}+\sum_{\alpha=1}^{n} f_0^{\alpha}(\bar{\mathbf{C}}^{\alpha}-\mathbf{C}):\tilde{\mathbf{H}}_0^{\alpha},$$

$$\bar{\mathbf{C}}^{\alpha} = (\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{C}^{\alpha}+(1-\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{C}, \quad \alpha = 1,2,\ldots,n, \tag{5.11a,b}$$

and $\tilde{\mathbf{H}}_0^{\alpha}$ is given by expressions (3.21b) to (3.22b) when the fourth order tensor $\mathbf{L}^{\alpha\beta}$ is replaced by $\mathbf{L}_0^{\alpha\beta}$. The fourth order tensor $\mathbf{L}_0^{\alpha\beta}$ relates the average stress in Ω_0^{α} to the uniform strain in Ω_0^{β} when the boundaries ∂V_0 and all $\partial\Omega^{\alpha}$s, $\alpha \neq \beta$ are fixed.

Since $\tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x})$ is weakly statically admissible, and $\tilde{\mathbf{u}}(\mathbf{x})$ is kinematically admissible and linear on the interfaces, it follows that

$$\frac{1}{V}\int_V \tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x}):\tilde{\boldsymbol{\varepsilon}}^{E}(\mathbf{x})\,\mathrm{d}V = \frac{1}{V}\int_{\partial V} \tilde{\mathbf{t}}^{E}\cdot\tilde{\mathbf{u}}^{E}\,\mathrm{d}S = \bar{\tilde{\boldsymbol{\sigma}}}^{E}:\tilde{\mathbf{E}}. \tag{5.12}$$

Therefore, volume integral of the inequality

$$\psi\{\tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x})\}-\psi\{\boldsymbol{\sigma}^{E}(\mathbf{x})\} \geq \{\tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x})-\boldsymbol{\sigma}^{E}(\mathbf{x})\}:\boldsymbol{\varepsilon}^{E}(\mathbf{x}) \tag{5.13a}$$

yields

†Note, however, that (5.6b) also applies to nonlinear cases.

$$\hat{\Psi}^{E}(\hat{\bar{\boldsymbol{\sigma}}}^{E}) - \tilde{\Psi}^{E}(\tilde{\bar{\boldsymbol{\sigma}}}^{E}) \geq (\hat{\bar{\boldsymbol{\sigma}}}^{E} - \tilde{\bar{\boldsymbol{\sigma}}}^{E}) : \tilde{\mathbf{E}}, \tag{5.13b}$$

where

$$\hat{\Psi}^{E}(\hat{\bar{\boldsymbol{\sigma}}}^{E}) \equiv \frac{1}{V}\int_{V} \psi\{\hat{\boldsymbol{\sigma}}^{E}(\mathbf{x})\}\, \mathrm{d}V = \tfrac{1}{2}\hat{\bar{\boldsymbol{\sigma}}}^{E} : \hat{\mathbf{D}}^{E} : \hat{\bar{\boldsymbol{\sigma}}}^{E},$$

$$\tilde{\Psi}^{E}(\tilde{\bar{\boldsymbol{\sigma}}}^{E}) \equiv \frac{1}{V}\int_{V} \psi\{\tilde{\boldsymbol{\sigma}}^{E}(\mathbf{x})\}\, \mathrm{d}V = \tfrac{1}{2}\tilde{\bar{\boldsymbol{\sigma}}}^{E} : \tilde{\mathbf{D}}^{E} : \tilde{\bar{\boldsymbol{\sigma}}}^{E}. \tag{5.14a,b}$$

This shows that $(\hat{\mathbf{D}}^{E} - \tilde{\mathbf{D}}^{E})$ and $(\tilde{\mathbf{C}}^{E} - \hat{\mathbf{C}}^{E})$ are positive semi-definite.

6. BOUNDS BASED ON DISCRETIZATION OF SOLID

Now consider the discretization of a solid consisting of matrix material and n inclusions Ω^{α}, $\alpha = 1, 2, \ldots, n$. First divide this solid into n subregions such that each subregion V^{α} bounded by ∂V^{α} consists of the matrix material and an inclusion Ω^{α} bounded by $\partial\Omega^{\alpha}$: see Fig. 4. Obtain upper bounds for the moduli of each subregion, and then combine these to arrive at the required bounds.

6.1. *Solid with single inclusion*

Now, treat each subregion as an RVE with a single inclusion and seek to obtain upper bounds for the compliance and elasticity tensors of the respective region. In this case, from (3.10b) and (3.21b), $\hat{\mathbf{J}}^{\alpha}$ and $\hat{\mathbf{H}}^{\alpha}$ are given by

$$\hat{\mathbf{J}}^{\alpha} = (\bar{\mathbf{D}}^{\alpha} + \mathbf{M}^{\alpha})^{-1} : (\mathbf{D} + \mathbf{M}^{\alpha}),$$

$$\hat{\mathbf{H}}^{\alpha} = (\bar{\mathbf{C}}^{\alpha} + \mathbf{L}^{\alpha})^{-1} : (\mathbf{C} + \mathbf{L}^{\alpha}). \tag{6.1a,b}$$

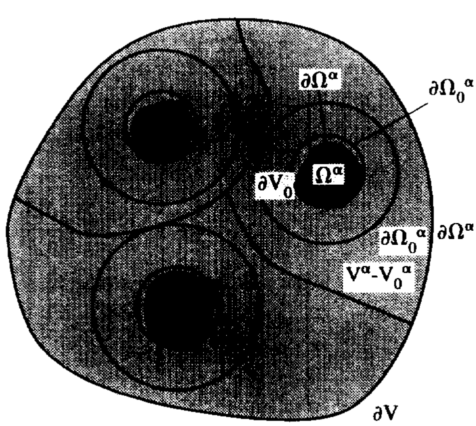

Fig. 4. A solid discretized into subregions such that each subregion contains one inclusion embedded in the material.

Then, from (3.12a) and (3.23a), the upper bounds for the compliance and elasticity tensors are given by

$$\hat{\mathbf{D}}^{\alpha\Sigma} = \mathbf{D}+\rho^{\alpha}(\mathbf{D}^{\alpha}-\mathbf{D})-\rho^{\alpha}(\mathbf{D}^{\alpha}-\mathbf{D}):(\mathbf{D}^{\alpha}+\mathbf{M}^{\alpha})^{-1}:(\mathbf{D}^{\alpha}-\mathbf{D}),$$

$$\hat{\mathbf{C}}^{\alpha E} = \mathbf{C}+\rho^{\alpha}(\mathbf{C}^{\alpha}-\mathbf{C})-\rho^{\alpha}(\mathbf{C}^{\alpha}-\mathbf{C}):(\mathbf{C}^{\alpha}+\mathbf{L}^{\alpha})^{-1}:(\mathbf{C}^{\alpha}-\mathbf{C}), \qquad (6.2\text{a,b})$$

where

$$\rho^{\alpha} = \Omega^{\alpha}/V^{\alpha}. \qquad (6.2\text{c})$$

The fourth order tensor $\mathbf{M}^{\alpha}$ relates the average strain in Ω^{α} to the constant stress tensor $\tilde{\boldsymbol{\sigma}}^{\alpha}$ when uniform tractions $\tilde{\mathbf{t}} = -\mathbf{n}^{\alpha}\cdot\tilde{\boldsymbol{\sigma}}^{\alpha}$ are applied on $\partial\Omega^{\alpha}$ while keeping the boundary ∂V^{α} traction free. Similarly, the fourth order tensor $\mathbf{L}^{\alpha}$ relates the average stress in Ω^{α} to constant strain tensor $\tilde{\boldsymbol{\varepsilon}}^{\alpha}$ when linear displacement $\mathbf{u} = \tilde{\boldsymbol{\varepsilon}}^{\alpha}\cdot\mathbf{x}$ is prescribed on $\partial\Omega^{\alpha}$ while ∂V^{α} is fixed.

If the boundaries ∂V^{α} and $\partial\Omega^{\alpha}$ are irregular, then consider two fictitious boundaries $\partial\Omega_0^{\alpha}$ and ∂V_0^{α} with regular geometries, which are completely within the matrix region $V^{\alpha}-\Omega^{\alpha}$. Then, from (5.2), (5.3), (5.4), (5.10), and (5.11), the upper bounds for $\hat{\mathbf{D}}^{\alpha\Sigma}$ and $\hat{\mathbf{C}}^{\alpha E}$ are given by

$$\tilde{\mathbf{D}}^{\alpha\Sigma} = (V_0^{\alpha}/V^{\alpha})\tilde{\mathbf{D}}_0^{\alpha\Sigma}+(1-V_0^{\alpha}/V^{\alpha})\mathbf{D},$$

$$\tilde{\mathbf{C}}^{\alpha E} = (V_0^{\alpha}/V^{\alpha})\tilde{\mathbf{C}}_0^{\alpha E}+(1-V_0^{\alpha}/V^{\alpha})\mathbf{C}, \qquad (6.3\text{a,b})$$

$$\tilde{\mathbf{D}}_0^{\alpha\Sigma} = \mathbf{D}+\rho_0^{\alpha}(\bar{\mathbf{D}}^{\alpha}-\mathbf{D})-\rho_0^{\alpha}(\bar{\mathbf{D}}^{\alpha}-\mathbf{D}):(\bar{\mathbf{D}}^{\alpha}+\mathbf{M}_0^{\alpha})^{-1}:(\bar{\mathbf{D}}^{\alpha}-\mathbf{D}),$$

$$\tilde{\mathbf{C}}_0^{\alpha E} = \mathbf{C}+\rho_0^{\alpha}(\bar{\mathbf{C}}^{\alpha}-\mathbf{C})-\rho_0^{\alpha}(\bar{\mathbf{C}}^{\alpha}-\mathbf{C}):(\bar{\mathbf{C}}^{\alpha}+\mathbf{L}_0^{\alpha})^{-1}:(\bar{\mathbf{C}}^{\alpha}-\mathbf{C}), \qquad (6.4\text{a,b})$$

where

$$\rho_0^{\alpha} = \frac{\Omega_0^{\alpha}}{V_0^{\alpha}}, \quad \bar{\mathbf{D}}^{\alpha} = (\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{D}^{\alpha}+(1-\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{D},$$

$$\bar{\mathbf{C}}^{\alpha} = (\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{C}^{\alpha}+(1-\Omega^{\alpha}/\Omega_0^{\alpha})\mathbf{C}. \qquad (6.5\text{a–c})$$

The fourth order tensor $\mathbf{M}_0^{\alpha}$ relates the average strain in Ω_0^{α} to the constant stress tensor $\tilde{\boldsymbol{\sigma}}^{\alpha}$ when uniform tractions $\tilde{\mathbf{t}} = -\mathbf{n}^{\alpha}\cdot\tilde{\boldsymbol{\sigma}}^{\alpha}$ are applied on $\partial\Omega_0^{\alpha}$ while keeping the boundary ∂V_0 traction free. Similarly, the fourth order tensor $\mathbf{L}_0^{\alpha}$ relates the average stress in Ω_0^{α} to constant strain tensor $\tilde{\boldsymbol{\varepsilon}}^{\alpha}$ when linear displacement $\mathbf{u} = \tilde{\boldsymbol{\varepsilon}}^{\alpha}\cdot\mathbf{x}$ is prescribed on $\partial\Omega_0^{\alpha}$ while ∂V_0 is fixed.

6.2. *Upper bound for* $\bar{\mathbf{D}}^{\Sigma}$

In each subregion consider a statically admissible stress field

$$\overset{\Delta\Sigma}{\boldsymbol{\sigma}}(\mathbf{x}) = \begin{cases} \overset{\Delta}{\Sigma} & \text{in } V^{\alpha}-V_0^{\alpha} \\ \overset{\Delta\Sigma}{\sigma}(\mathbf{x}) & \text{in } M_0^{\alpha} \quad \alpha = 1,2,\ldots,n, \\ \overset{\Delta}{\sigma}{}^{\alpha} & \text{in } \Omega_0^{\alpha}, \end{cases} \qquad (6.6)$$

where $\overset{\Delta}{\boldsymbol{\Sigma}}$ and $\overset{\Delta}{\boldsymbol{\sigma}}{}^{\alpha}$ are constant stress tensors. Note that this stress field is statically admissible everywhere in solid with volume V bounded by ∂V. Then, by imposing a weak kinematical admissibility as in (3.9), obtain the average strain, stress, and energy density in each subregion as follows:

$$\bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\alpha\Sigma} \equiv \frac{1}{V^{\alpha}} \int_{V^{\alpha}} \overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \tilde{\mathbf{D}}^{\alpha\Sigma} : \overset{\Delta}{\boldsymbol{\Sigma}}, \quad \bar{\overset{\Delta}{\boldsymbol{\sigma}}}{}^{\alpha\Sigma} \equiv \frac{1}{V^{\alpha}} \int_{V^{\alpha}} \overset{\Delta}{\boldsymbol{\sigma}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \overset{\Delta}{\boldsymbol{\Sigma}},$$

$$\bar{\overset{\Delta}{w}}{}^{\alpha\Sigma} \equiv \frac{1}{V^{\alpha}} \int_{V^{\alpha}} \tfrac{1}{2} \overset{\Delta}{\boldsymbol{\sigma}}{}^{\Sigma}(\mathbf{x}) : \overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \tfrac{1}{2} \overset{\Delta}{\boldsymbol{\Sigma}} : \bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\alpha\Sigma}, \tag{6.7a–c}$$

where $\tilde{\mathbf{D}}^{\alpha\Sigma}$ is given by (6.3a). Then the average strain, stress, and energy density in the solid are given by

$$\bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma} \equiv \frac{1}{V} \int_{V} \overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \overset{\Delta}{\mathbf{D}}{}^{\Sigma} : \overset{\Delta}{\boldsymbol{\Sigma}}, \quad \bar{\overset{\Delta}{\boldsymbol{\sigma}}}{}^{\Sigma} \equiv \frac{1}{V} \int_{V} \overset{\Delta}{\boldsymbol{\sigma}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \overset{\Delta}{\boldsymbol{\Sigma}},$$

$$\bar{\overset{\Delta}{w}}{}^{\Sigma} \equiv \frac{1}{V} \int_{V} \tfrac{1}{2} \overset{\Delta}{\boldsymbol{\sigma}}{}^{\Sigma}(\mathbf{x}) : \overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \tfrac{1}{2} \overset{\Delta}{\boldsymbol{\Sigma}} : \bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma}, \tag{6.8a–c}$$

where

$$\overset{\Delta}{\mathbf{D}}{}^{\Sigma} = \sum_{\alpha=1}^{n} f V^{\alpha} \tilde{\mathbf{D}}^{\alpha\Sigma}, \quad f V^{\alpha} = V^{\alpha}/V. \tag{6.9a,b}$$

Since, $\overset{\Delta}{\boldsymbol{\sigma}}(\mathbf{x})$ is a statically admissible stress field and $\mathbf{u}^{\Sigma}(\mathbf{x})$ is a kinematically admissible displacement field everywhere in the solid,

$$\frac{1}{V} \int_{V} \boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x}) : \overset{\Delta}{\boldsymbol{\sigma}}{}^{\Sigma}(\mathbf{x})\, \mathrm{d}V = \frac{1}{V} \int_{\partial V} \mathbf{u}^{\Sigma} \cdot \overset{\Delta}{\mathbf{t}}{}^{\Sigma}\, \mathrm{d}S = \bar{\boldsymbol{\varepsilon}}^{\Sigma} : \overset{\Delta}{\boldsymbol{\Sigma}}. \tag{6.10}$$

Therefore, the volume integral over the solid of the inequality

$$\phi\{\boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x})\} - \phi\{\overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\} \geqslant \{\boldsymbol{\varepsilon}^{\Sigma}(\mathbf{x}) - \overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\} : \overset{\Delta}{\boldsymbol{\sigma}}{}^{\Sigma}(\mathbf{x}) \tag{6.11a}$$

yields

$$\Phi^{\Sigma}(\bar{\boldsymbol{\varepsilon}}^{\Sigma}) - \overset{\Delta}{\Phi}{}^{\Sigma}(\bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma}) \geqslant (\bar{\boldsymbol{\varepsilon}}^{\Sigma} - \bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma}) : \overset{\Delta}{\boldsymbol{\Sigma}}, \tag{6.11b}$$

where

$$\overset{\Delta}{\Phi}{}^{\Sigma}(\bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma}) \equiv \frac{1}{V} \int_{V} \phi\{\overset{\Delta}{\boldsymbol{\varepsilon}}{}^{\Sigma}(\mathbf{x})\}\, \mathrm{d}V = \tfrac{1}{2} \bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma} : \overset{\Delta}{\mathbf{C}}{}^{\Sigma} : \bar{\overset{\Delta}{\boldsymbol{\varepsilon}}}{}^{\Sigma}, \quad \overset{\Delta}{\mathbf{C}}{}^{\Sigma} \equiv (\overset{\Delta}{\mathbf{D}}{}^{\Sigma})^{-1}, \tag{6.12a,b}$$

and $\Phi^{\Sigma}(\bar{\boldsymbol{\varepsilon}}^{\Sigma})$ is defined by (4.4a). This shows that $(\bar{\mathbf{C}}^{\Sigma} - \overset{\Delta}{\mathbf{C}}{}^{\Sigma})$ and $(\overset{\Delta}{\mathbf{D}}{}^{\Sigma} - \bar{\mathbf{D}}^{\Sigma})$ are positive semi-definite.

6.3. *Upper bound for* $\bar{\mathbf{C}}^{E}$

In a similar manner, by considering a kinematically admissible displacement field

$$\overset{\wedge}{\mathbf{u}}^{E}(\mathbf{x}) = \begin{cases} \mathbf{x}\cdot\overset{\wedge}{\mathbf{E}} & \text{in } V^{\alpha}-V_0^{\alpha} \\ \overset{\wedge}{\mathbf{u}}^{E}(\mathbf{x}) & \text{in } M_0^{\alpha} \quad \alpha = 1,2,\ldots,n \\ \mathbf{x}\cdot\overset{\wedge}{\boldsymbol{\varepsilon}}^{\alpha} & \text{in } \Omega_0^{\alpha}, \end{cases} \tag{6.13}$$

and by imposing a weak statical admissibility as in (3.20), it is shown that $(\bar{\mathbf{D}}^{E}-\overset{\Delta}{\mathbf{D}}^{E})$ and $(\overset{\wedge}{\mathbf{C}}^{E}-\bar{\mathbf{C}}^{E})$ are positive semi-definite, where

$$\overset{\wedge}{\mathbf{C}}^{E} \equiv \sum_{\alpha=1}^{n} f V^{\alpha}\tilde{\mathbf{C}}^{\alpha E}, \quad \overset{\wedge}{\mathbf{D}}^{E} \equiv (\overset{\Delta}{\mathbf{C}}^{E})^{-1}. \tag{6.14a,b}$$

6.4. *Composites with periodic microstructure*

Consider the case where the inclusions of the same size, shape and material are distributed periodically in an RVE. Then each unit cell has the same upper bounds for the compliance and elasticity tensors. Then, from (6.8a) and (6.13a) the upper bounds for the compliance and elasticity tensors of the RVE are

$$\overset{\wedge}{\mathbf{D}}^{\Sigma} = \tilde{\mathbf{D}}^{\alpha\Sigma}, \quad \overset{\wedge}{\mathbf{C}}^{E} = \tilde{\mathbf{C}}^{\alpha E}. \tag{6.15a,b}$$

6.5. *Improved bounds in special cases*

In certain special cases, the bounds may be improved if exact solutions for subregions, illustrated in Fig. 7, can be constructed for uniform traction and linear displacement-boundary conditions. In the case of concentric spheres in three dimensions and cylinders in two dimensions, such closed-form solutions have been used by Hashin (1962) and Hashin and Rosen (1964) to obtain closed-form bounds for the overall bulk modulus. For spherical or cylindrical cavities (rigid inclusions), the resulting lower (upper) bounds will be the same as that obtained by the present procedure.

7. EXAMPLES

In order to illustrate the application of the bounds developed in this paper, first consider the plane stress/strain elasticity problem of an annulus bounded by two concentric circles with radius $R_1(\partial\Omega_0)$ and $R_2(\partial V_0)$. When this annulus is subjected to the boundary conditions

$$\mathbf{t} = \begin{cases} \mathbf{n}\cdot\boldsymbol{\sigma} = 0 & \text{on } \partial V_0 \\ -\mathbf{n}^{\alpha}\cdot\boldsymbol{\sigma} = -\mathbf{n}^{\alpha}\cdot\boldsymbol{\sigma}^{0} & \text{on } \partial\Omega_0 \end{cases}, \tag{7.1}$$

the average strain in Ω_0^{α} defined by (2.5a) is given by

$$\varepsilon^{\alpha}_{kk} = -\frac{p}{2B}\sigma^{0}_{kk}, \quad \varepsilon'^{\alpha}_{ij} = -\frac{q}{2\mu}\sigma^{0\prime}_{ij}, \tag{7.2a,b}$$

$$p = \frac{2}{(\kappa-1)}\left[1+\frac{(\kappa+1)\rho_0}{2(1-\rho_0)}\right], \tag{7.2c–e}$$

$$q = (\kappa+1)\frac{1-\rho_0^3}{(1-\rho_0)^4}-1, \quad \rho_0 = R_1^2/R_2^2,$$

where

$$\varepsilon'_{ij} = \varepsilon_{ij}-\delta_{ij}\varepsilon_{kk}/2, \quad \sigma^{0\prime}_{ij} = \sigma^{0}_{ij}-\delta_{ij}\sigma^{0}_{kk}/2, \quad i,j,k = 1,2. \tag{7.3a,b}$$

Similarly, when the annulus is subjected to the boundary conditions

$$\mathbf{u} = \begin{cases} 0 & \text{on } \partial V_0 \\ \mathbf{x}\cdot\boldsymbol{\varepsilon}^0 & \text{on } \partial\Omega_0 \end{cases}, \tag{7.4}$$

the average stress in Ω^{α}_0 defined by (2.5b) is given by

$$\sigma^{\alpha}_{kk} = -2rB\varepsilon^{0}_{kk}, \quad \sigma'^{\alpha}_{ij} = -2s\mu\varepsilon^{0\prime}_{ij}, \tag{7.5a,b}$$

$$r = \frac{1}{2}\left[\kappa-1+(\kappa+1)\frac{\rho_0}{1-\rho_0}\right], \tag{7.5c,d}$$

$$s = \frac{\kappa(\kappa+1)(1-\rho_0^3)}{(1-\rho_0)^2\{\kappa^2(1+\rho_0+\rho_0^2)-3\rho_0\}}-1.$$

In (7.2) and (7.5) μ is the in-plane shear modulus, B is the in-plane bulk modulus defined by

$$B \equiv \frac{\sigma_{ii}}{2\varepsilon_{kk}} = \frac{2\mu}{\kappa-1}, \quad i,k = 1,2, \tag{7.6a}$$

and κ is related to the Poisson ratio by

$$\kappa = \begin{cases} (3-4\nu) & \text{for plane strain} \\ (3-\nu)/(1+\nu) & \text{for plane stress.} \end{cases} \tag{7.6b}$$

As the first example consider the composite cylindrical assemblages where the composite material is made up of circular composite cylinders of varying sizes. In all composite cylinders, the fibers have the same volume fraction, say, ρ_0 and may be heterogeneous. In this case, the RVE is the composite cylinder. When the fibers are homogeneous, the response of the RVE is transversely isotropic. However, in general, for the heterogeneous fibers, the response of the composite cylinders is orthotropic. For in-plane problems, the overall moduli of the RVE are defined by

$$\Phi(\bar{\varepsilon}) = \tfrac{1}{2}\bar{B}(\bar{\varepsilon}_{kk})^2+2\bar{\mu}(\bar{\varepsilon}_{12})^2+2\bar{\mu}'(\bar{\varepsilon}'_{11})^2,$$

$$\Psi(\bar{\sigma}) = \frac{1}{8\bar{B}}(\bar{\sigma}_{kk})^2+\frac{1}{2\bar{\mu}}(\bar{\sigma}_{12})^2+\frac{1}{2\bar{\mu}'}(\bar{\sigma}'_{11})^2,$$

It follows from (7.2) and (7.5) that $\mathbf{M}^0$ and $\mathbf{L}^0$ for the composite cylinders are transversely isotropic. Therefore, the bounds $\tilde{\mathbf{D}}^\Sigma$ and $\tilde{\mathbf{C}}^E$ resulting from (6.4) are also transversely isotropic. The bounds for the in-plane shear and bulk moduli implied by (6.4) are

$$\frac{\mu}{\tilde{\mu}_0^\Sigma} = 1+\rho_0(\mu/\bar{\mu}^{\alpha D}-1)(1+q)/(\mu/\bar{\mu}^{\alpha D}+q),$$

$$\frac{B}{\tilde{B}_0^\Sigma} = 1+\rho_0(B/\bar{B}^{\alpha D}-1)(1+p)/(B/\bar{B}^{\alpha D}+p),$$

$$\frac{\tilde{\mu}_0^E}{\mu} = 1+\rho_0(\bar{\mu}^{\alpha C}/\mu-1)(1+s)/(\bar{\mu}^{\alpha C}/\mu+s),$$

$$\frac{\tilde{B}_0^E}{B} = 1+\rho_0(\bar{B}^{\alpha C}/B-1)(1+r)/(\bar{B}^{\alpha C}/B+r), \tag{7.7a–d}$$

where

$$\bar{\mu}^{\alpha C} = \frac{1}{\Omega_0}\int_{\Omega_0} \mu^\alpha \,\mathrm{d}V, \quad \bar{B}^{\alpha C} = \frac{1}{\Omega_0}\int_{\Omega_0} B^\alpha \,\mathrm{d}V,$$

$$\frac{1}{\bar{\mu}^{\alpha D}} = \frac{1}{\Omega_0}\int_{\Omega_0}\frac{1}{\mu^\alpha}\,\mathrm{d}V, \quad \frac{1}{\bar{B}^{\alpha D}} = \frac{1}{\Omega_0}\int_{\Omega_0}\frac{1}{B^\alpha}\,\mathrm{d}V. \tag{7.8a–d}$$

Note that, $\tilde{\mu}^\Sigma$ and $\tilde{\mu}^E$ bound both $\bar{\mu}$ and $\bar{\mu}'$.

In the case of homogeneous fibers, i.e. $\bar{B}^{\alpha D} = \bar{B}^{\alpha C} = B^\alpha$, $\bar{\mu}^{\alpha D} = \bar{\mu}^{\alpha C} = \bar{\mu}^\alpha$, in view of (7.2c) and (7.5c), (7.7b,d) reduce to

$$\frac{\tilde{B}_0^\Sigma}{B} = \frac{\tilde{B}_0^E}{B} = 1+\rho_0\left[\frac{2(1-\rho_0)}{\kappa+1}+\frac{1}{B^\alpha/B-1}\right]^{-1}. \tag{7.9}$$

These results coincide with those obtained by Hashin and Rosen (1964). It should be noted that Hashin and Rosen (1964) obtain the bounds on the elastic moduli of composite cylinders from $\bar{\mathbf{D}}_0^\Sigma$ and $\bar{\mathbf{C}}_0^E$. However, there is no closed form expression for the shear part of $\bar{\mathbf{D}}_0^\Sigma$ and $\bar{\mathbf{C}}_0^E$. Evaluation of these bounds involves solving of six linear equations for six unknowns. When the composite cylinders are hollow cylinders, these bounds are

$$\frac{\bar{\mu}_0^E}{\mu} = 1-\rho_0\frac{(\kappa+1)(\kappa+\rho_0^3)}{(\kappa+\rho_0^3)(1+\kappa\rho_0)+\rho_0^2(\rho_0^2-3)(1-\rho_0)},$$

$$\frac{\mu}{\bar{\mu}_0^\Sigma} = 1+\rho_0(\kappa+1)(1+\rho_0+\rho_0^2)/(1-\rho_0)^3. \tag{7.10a,b}$$

For this case, substitute $\bar{\mu}^{\alpha D} = \bar{\mu}^{\alpha C} = 0$ into (7.7a,c) to obtain

$$\frac{\tilde{\mu}_0^E}{\mu} = 1-\rho_0\frac{\kappa(\kappa+1)(1+\rho_0+\rho_0^2)}{\kappa(1+\kappa\rho_0)(1+\rho_0+\rho_0^2)+3\rho_0(1-\rho_0)},$$

$$\frac{\mu}{\tilde{\mu}_0^{\Sigma}} = 1+\rho_0(\kappa+1)(1+\rho_0+\rho_0^2)/(1-\rho_0)^3. \tag{7.10c,d}$$

For the composite cylinders with rigid fibers, the bounds by Hashin and Rosen (1964) are

$$\begin{aligned}\frac{\tilde{\mu}_0^{\mathrm{E}}}{\mu} &= 1+\rho_0\frac{\kappa(\kappa+1)(1+\rho_0+\rho_0^2)}{(1-\rho_0)\{\kappa^2(1+\rho_0+\rho_0^2)-3\rho_0\}},\\ \frac{\mu}{\tilde{\mu}_0^{\Sigma}} &= 1-\rho_0\frac{(\kappa+1)(1+\kappa\rho_0^3)}{(1+\kappa\rho_0^3)(\kappa+\rho_0)+3\rho_0(1-\rho_0)^2}.\end{aligned} \tag{7.11a,b}$$

For this case, the bounds from the present approach are obtained by substituting $1/\bar{\mu}^{\alpha D} = 1/\bar{\mu}^{\alpha C} = 0$ into (7.7a,c), arriving at

$$\begin{aligned}\frac{\tilde{\mu}_0^{\mathrm{E}}}{\mu} &= 1+\rho_0\frac{\kappa(\kappa+1)(1+\rho_0+\rho_0^2)}{(1-\rho_0)\{\kappa^2(1+\rho_0+\rho_0^2)-3\rho_0\}},\\ \frac{\mu}{\tilde{\mu}_0^{\Sigma}} &= 1-\rho_0\frac{(\kappa+1)(1+\rho_0+\rho_0^2)}{(\kappa+1)(1+\rho_0+\rho_0^2)-(1-\rho_0)^3}.\end{aligned} \tag{7.11c,d}$$

It is seen that when the inclusions of the composite cylinders are tubular cavities (rigid fibers), the lower (upper) bound from the present method is identical to that obtained by Hashin and Rosen (1964); see Section 6.5 for comments. However, the upper (lower) bound of the present method is greater (smaller) than that of Hashin and Rosen (1964); see Fig. 5(a,b).

As the second example, consider a fibrous composite where fibers are periodically distributed on the plane normal to their axis. The fibers may be weaker or stronger than the matrix, including the limiting cases where fibers are cavities or rigid inclusions. The fiber packing may be either triangular, square, or hexagonal, see Fig. 6. The cross sectional shape may be arbitrary and the fibers may be heterogeneous. In this case, the fictitious boundaries ∂V_0^{α}, for the unit cells are the largest circles drawn within the unit cells while $\partial\Omega_0^{\alpha}$ are the smallest circles drawn outside the fibers, see Fig. 7.

Then the bounds for the in-plane shear and bulk moduli implied by (6.3) are

$$\begin{aligned}\frac{1}{\tilde{\mu}^{\Sigma}} &= f^0\frac{1}{\tilde{\mu}_0^{\Sigma}}+(1-f^0)\frac{1}{\mu},\\ \frac{1}{\tilde{B}^{\Sigma}} &= f^0\frac{1}{\tilde{B}_0^{\Sigma}}+(1-f^0)\frac{1}{B},\\ \tilde{\mu}^{\mathrm{E}} &= f^0\tilde{\mu}_0^{\mathrm{E}}+(1-f^0)\mu,\\ \tilde{B}^{\mathrm{E}} &= f^0\tilde{B}_0^{\mathrm{E}}+(1-f^0)B,\end{aligned} \tag{7.12a–d}$$

where

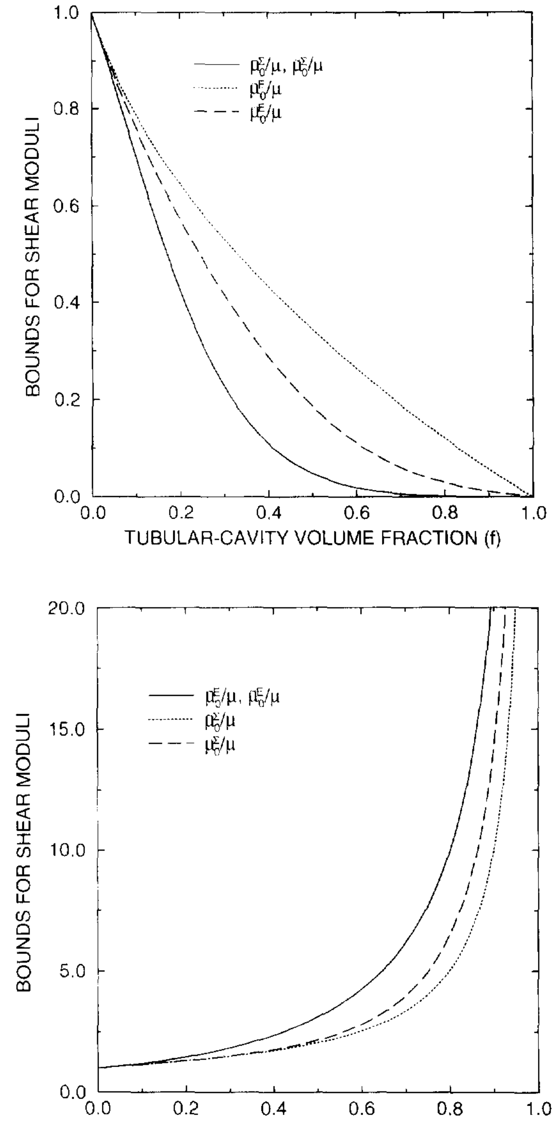

Fig. 5. Bounds for the overall shear moduli of (a) hollow cylindrical assemblages and (b) composite cylindrical assemblages of rigid fibers.

$$f^0 \equiv V_0/V = \begin{cases} \pi/3\sqrt{3} & \text{for triangular} \\ \pi/4 & \text{for square} \\ \pi/2\sqrt{3} & \text{for hexagonal fiber packing.} \end{cases} \tag{7.13a}$$

Then the volume fraction f of fibers is related to f^0 by

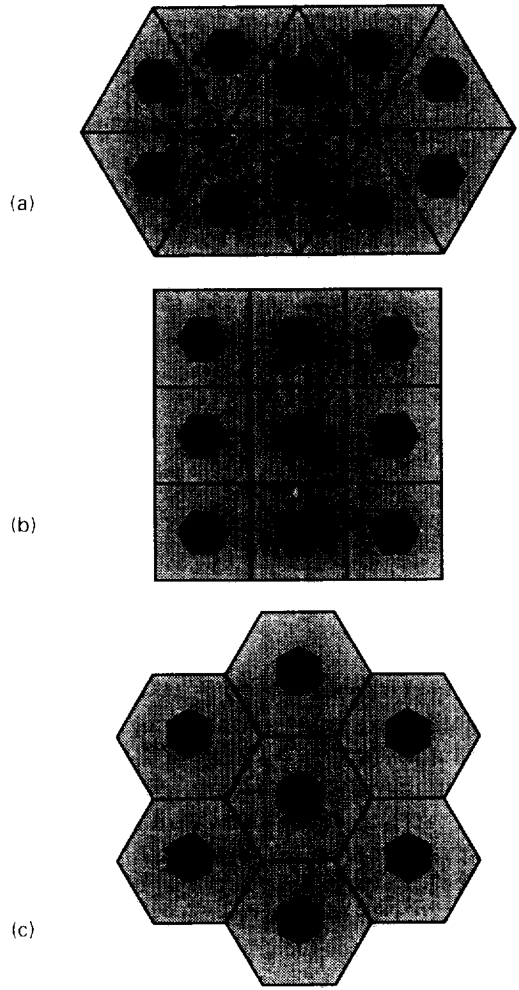

Fig. 6. Periodic distribution of hexagonal fibers. (a) Triangular fiber packing; (b) square fiber packing; and (c) hexagonal fiber packing.

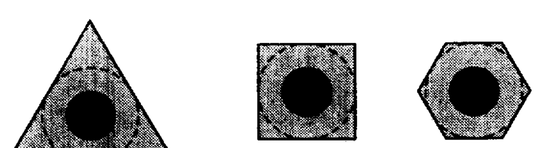

Fig. 7. Typical unit cells with fictitious boundaries.

$$f \equiv \frac{\Omega}{V} = \frac{\Omega}{\Omega_0}\,\frac{\Omega_0}{V_0}\,\frac{V_0}{V} = \rho\rho_0 f^0, \quad \rho \equiv \frac{\Omega}{\Omega_0}. \tag{7.13b,c}$$

Substitute (7.7) into (7.12), and in view of (7.13b,c) obtain

$$\frac{\mu}{\bar{\mu}^{\Sigma}} = 1+(f/\rho)(\mu/\bar{\mu}^{\alpha D}-1)(1+q)/(\mu/\bar{\mu}^{\alpha D}+q),$$

$$\frac{B}{\bar{B}^{\Sigma}} = 1+(f/\rho)(B/\bar{B}^{\alpha D}-1)(1+p)/(B/\bar{B}^{\alpha D}+p),$$

$$\frac{\bar{\mu}^{E}}{\mu} = 1+(f/\rho)(\bar{\mu}^{\alpha C}/\mu-1)(1+s)/(\bar{\mu}^{\alpha C}/\mu+s),$$

$$\frac{\bar{B}^{E}}{B} = 1+(f/\rho)(\bar{B}^{\alpha C}/B-1)(1+r)/(\bar{B}^{\alpha C}/B+r). \tag{7.14a–d}$$

For homogeneous fibers,

$$\bar{\mu}^{\alpha C} = \rho\mu^{\alpha}+(1-\rho)\mu, \quad \bar{B}^{\alpha C} = \rho B^{\alpha}+(1-\rho)B,$$

$$\frac{1}{\bar{\mu}^{\alpha D}} = \rho\frac{1}{\mu^{\alpha}}+(1-\rho)\frac{1}{\mu},$$

$$\frac{1}{\bar{B}^{\alpha D}} = \rho\frac{1}{B^{\alpha}}+(1-\rho)\frac{1}{B}. \tag{7.15a–d}$$

As an example consider homogeneous fibers with hexagonal cross section. In this case $\rho = 3\sqrt{3}/(2\pi)$. The bounds on the bulk and shear moduli for this case are shown in Figs 8 and 9 for the modulus ratios ($\mu^{\alpha}/\mu = B^{\alpha}/B$) of 0.1 and 25. Note that, volume fraction of 1/2, $3\sqrt{3}/8$, and 3/4, in triangular, square, and hexagonal packing, denotes the respective closed packing of hexagonal fibers with arbitrary orientation. These figures show that the bounds are reasonably close when the volume fraction of the fibers is less than about 80% of the closed packing. The bounds for elastic solids with tubular cavities are shown in Fig. 10. It is seen in these figures that the lower bounds are nonzero. They become zero only for the closed packing, which is the exact value. Also shown in these figures are the exact results for the square packing of circular-tubular cavities using periodic method, see Nemat-Nasser *et al.* (1982). As expected, these exact results fall within the bounds. The bounds for elastic solids with rigid fibers are shown in Fig. 11. It is seen that upper bounds are always finite. These become unbounded only for the closed packing, which is exact.

Note that, in Fig. 10, the upper bounds are not zero for the closed packing of tubular cavities. Similarly, the lower bounds in Fig. 11 are finite for the closed packing of rigid fibers. If the bounds are calculated from (6.2) instead of (6.3) and (6.4), the upper and lower bounds would coincide for the two extreme cases, i.e. tubular cavities and rigid fibers. However, this is feasible only if the tensors **M** and **L** are known for an annulus bounded externally by a polygon (triangle, square, and hexagon) and internally by a circle. There is no closed form solution available in the literature for elasticity problems when these annuli are subjected to either uniform tractions or linear displacements on both surfaces. If such solutions are available, then the bounds can be calculated from (6.2). This would bring both upper and lower bounds to the same exact value for the closed packing. Hence, from Figs 10 and 11, it is seen that

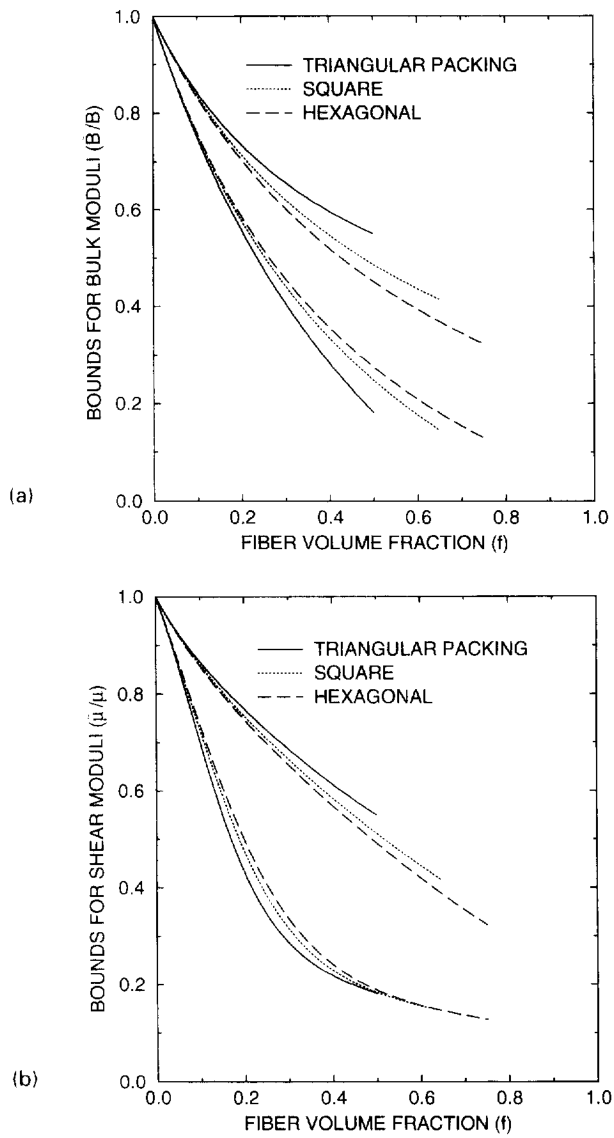

Fig. 8. (a) Bounds for overall bulk moduli: $\mu^{\alpha}/\mu = B^{\alpha}/B = 0.1$. (b) Bounds for overall shear moduli: $\mu^{\alpha}/\mu = B^{\alpha}/B = 0.1$.

(a)

(b)

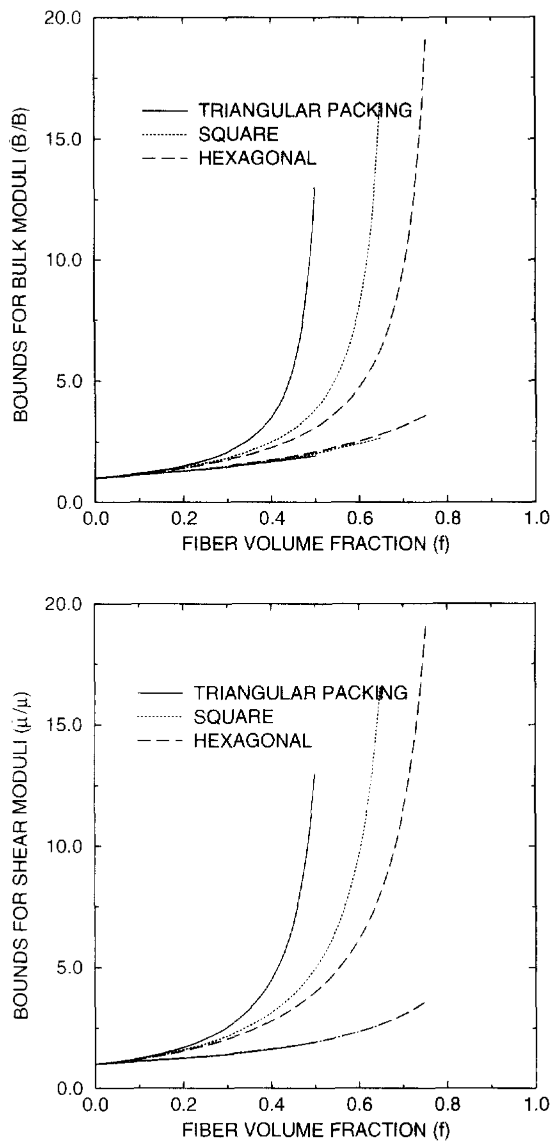

Fig. 9. (a) Bounds for the overall bulk moduli: $\mu'/\mu = B'/B = 25$. (b) Bounds for the overall shear moduli: $\mu'/\mu = B'/B = 25$.

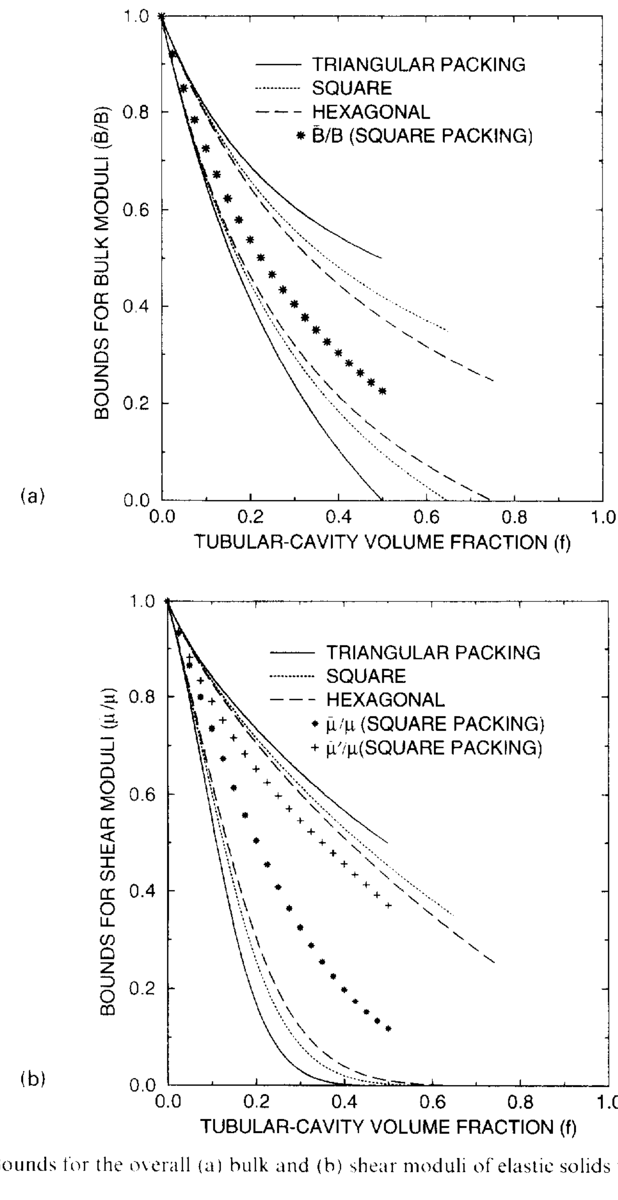

Fig. 10. Bounds for the overall (a) bulk and (b) shear moduli of elastic solids with tubular cavities.

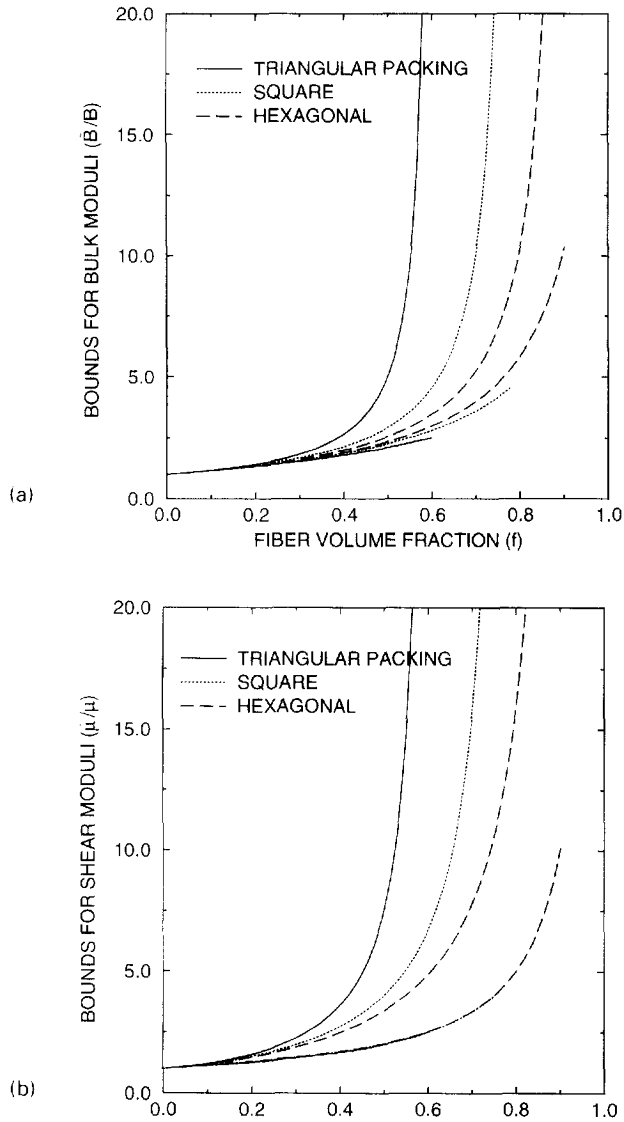

Fig. 11. Bounds for the overall (a) bulk and (b) shear moduli of elastic solids with rigid fibers.

for all volume fractions, the upper and lower bounds would then be closer to each other. Therefore, it is worth computing the tensors **M** and **L** numerically.

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