ON CERTAIN MACROSCOPIC AND MICROSCOPIC ASPECTS OF PLASTIC FLOW OF DUCTILE MATERIALS

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Microscopic aspects of ductile flow of two-phase alloys are reviewed and the influence of microvoids, microscopic slip, and hydrostatic pressure (tension) on macroscopic plastic flow is discussed. Then, in light of experimental results, a phenomenological plasticity theory with nonassociative flow rule that accounts for pressure sensitivity and plastic volume expansion, as well as possible localization of deformation at the microlevel, is presented. Various associated parameters are discussed and given physical interpretation. As an alternative complementary approach, rate equations for rate independent plastic flow are obtained by the consideration of microslides accompanied by microscopic plastic volumetric changes affected by pressure. Various special cases are discussed and compared with published results. Then, localization in biaxial loading is examined, including pressure and compressibility effects. Finally, some experimental results on localization in axial tension and compression of certain maraging steels are reviewed and used to estimate at the critical state the values of several microscopic parameters that enter the microstructurally based theory.

1. Introduction

Ductility refers to the extent to which a material can deform plastically before fracture. It is dominant in certain metals over a limited temperature range. The existence of nonmetallic inclusions in alloys substantially limits their ductility, because of the formation of microvoids at these inclusions, followed by the growth of these voids and their interconnection which may occur either by internal necking of the matrix that separated two adjacent voids (void coalescence), or by the formation of void sheets at smaller precipitates in intense shear bands (localization of plastic distortion) which connect adjacent large voids; for literature review and rather extensive references, see Nemat-Nasser (1977); Goods and Brown (1979). To understand and quantify the phenomenon of ductile flow, the following basic problems must be addressed:

(1) the mechanism of void initiation and void growth;
(2) the mechanism of localization of plastic distortion at the microlevel;
(3) the effect of plastic work accumulation within the matrix, on macroscopic behavior;
(4) the effect of plastic volumetric strain on macroscopic behavior;
(5) macroscopic tension sensitivity.

In addition, and despite the long-held (by the mechanics community) myth of rate-independent plasticity, rate effects are also significant, even at room temperature, as commonly recognized by material scientists, and recently accentuated by Krempl (1979). Here, however, a rate-independent theory is presented in order to bring to focus the essential features enumerated above. Two approaches are considered: (1) a macroscopic phenomenological approach based on a minor but significant generalization of the usual $J_2$ plasticity theory; and (2) a microscopic approach based on the consideration of slip-lines, plastic localization, and void growth at the microscale. Both approaches permit further generalization to include rate effects. The resulting constitutive relations include: (a) material softening due to initiation and growth of voids; (b) material hardening due to plastic distortion of the matrix; (c) material softening due to the localization of plastic distortion at the microlevel; and (d) pressure (tension) sensitivity.

This paper is organized in the following manner. In the remaining part of the present section, the above-mentioned effects are briefly discussed. In Section 2, a plasticity theory is presented, and various associated parameters are discussed and given physical interpretation; the theory represents an extension of the recent work by Nemat-Nasser and Shokooh (1980). In Section 3, rate equations for rate-independent plastic flow are obtained by consideration of microslips accompanied by microscopic plastic volumetric changes affected by pressure (tension). This part represents a generalization of Asaro's (1979) recent contribution which did not include plastic volume expansion and pressure sensitivity; the general theory is also specialized and results are compared with Asaro's. In Section 4, localization in biaxial loading is examined in light of the theory developed in Section 3, and various special cases are discussed. In particular, experimental results by Anand and Spitzig (1980) on the formation of localized shear bands in maraging steels, are reviewed and used to estimate the critical values of the microscopic constitutive parameters for the special case of the rigid-plastic model of Section 3. Anand and Spitzig have shown that the predictions of the usual $J_2$ flow theory and those of a deformation theory do not accord with the experimental facts. In view of this, their experimental data are reviewed and are shown to be consistent, satisfying with good accuracy the usual characteristic equation for localization in incrementally linear materials.

1.1. Void initiation

Recent experiments on various steels have indicated that initially plastic volumetric expansion accompanies plastic distortion in all deformation
modes, i.e., tension, compression, and torsion. This plastic volume increase appears to be independent of the hydrostatic stress or the value of the principal stresses, and seems to depend only on the total effective plastic distortion. The plastic volume expansion is, however, small, but nevertheless it exists. In fact, experiments by Dyson et al. (1976) suggest that the rate of plastic volumetric expansion per unit distortional strain is (initially) constant, its magnitude of course depending on the considered material; see also Spitzig et al. (1975, 1976).

At later stages of plastic distortion, void growth becomes a dominant factor. It is reasonable to expect that hydrostatic tension and stress triaxiality can enhance the void growth process.

1.2. Localization of plastic distortion

Microheterogeneity in material structure, e.g. grain boundaries, second-phase particles, and voids, promotes plastic distortion by localized deformations. The formation of localized deformations at the microlevel has a macroscopic softening effect which tends to become more significant as microscopic localization tends to magnify during the course of deformation. This is a very significant feature of ductile fracture, which has not yet been fully quantified in terms of macroscopic constitutive relations, although corner theories of plasticity are motivated by this fact; see, e.g., Christoffersen and Hutchinson (1979).

At the scale of tens of microns the existence of voids and inclusions promotes plastic localization. This has been shown for a two-dimensional model by Nemat-Nasser and Taya (1978), where a unit cell consisting of a single void has been subjected to its boundary to the deformation history which has been separately calculated for a necked bar.

1.3. Pressure sensitivity

Experiments by Spitzig et al. (1975, 1976) show that the yield function is affected by hydrostatic tension or compression, i.e., yield stress increases with increasing hydrostatic pressure. These authors note that, since the pressure sensitivity of yielding exceeds the corresponding plastic volumetric expansion by an order of magnitude, the usual normality rule used in plasticity models for metals does not apply. A consideration of a nonassociative flow rule, however, permits retention of normality with respect to the flow potential, yielding results compatible with experimental facts, as shown in the next section.
2. Macroscopic approach

Based on the above observations the following yield function and flow potential are introduced:

\[ f = \bar{\sigma} - F(I, \Delta, \bar{\varepsilon}, \Sigma), \quad \text{yield function}; \]
\[ g = \bar{\sigma} + G(I, \Delta, \bar{\varepsilon}, \Sigma), \quad \text{flow potential}, \]

where

\[ \bar{\sigma}^2 = \frac{1}{2} \sigma_{ij} \sigma_{ij}, \quad I = \sigma_{kk}, \]
\[ \Delta = \int_0^\theta \frac{\rho_0}{\rho} D_{kk} \, d\theta, \quad \bar{\varepsilon} = \int_0^\theta (2 D_{ij}^p D_{ij}^p)^{1/2} \, d\theta; \]

here, a fixed rectangular Cartesian coordinate system is used; \( \sigma_{ij} \) are the corresponding stress components; prime denotes the deviatoric part; \( \rho_0 \) and \( \rho \) are the reference and current mass densities; \( D_{ij}^p \) are the components of the plastic part of the deformation rate tensor; \( \theta \) is a monotone increasing load parameter; \( \bar{\sigma} \) and \( \bar{\varepsilon} \) are the effective stress and strain; \( \Delta \) is the total plastic volumetric strain measured per unit reference volume; and \( \Sigma \) is the localization parameter defined on the microscale and characterizes the extent of localization of deformation within the matrix in a typical sample of the material; see Nemat-Nasser and Taya (1978). It is convenient to use the effective plastic strain, \( \bar{\varepsilon} \), as the load parameter. Then all superposed dots denote differentiation with respect to \( \bar{\varepsilon} \). This is followed in the sequel.

From (1) the plastic part of the strain rate is

\[ D_{ij}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma_{ij}} = \dot{\lambda} \left[ \frac{\sigma_{ij}'}{2 \bar{\sigma}} + \frac{\partial G}{\partial I} \delta_{ij} \right], \]

which, in view of the consistency relation \( f = 0 \), becomes

\[ D_{ij}^p = \frac{1}{H} \left[ \frac{\sigma_{ij}'}{2 \bar{\sigma}} + \frac{\partial G}{\partial I} \delta_{ij} \right] \left[ \frac{\sigma_{kl}'}{2 \bar{\sigma}} - \frac{\partial F}{\partial I} \delta_{kl} \right] \sigma_{kl}'; \]

where \( \sigma_{kl}' \) denotes the Jaumann rate of Cauchy stress,

\[ \dot{\sigma}_{kl} = \sigma_{kl} - W_{kl} \sigma_{ij} - W_{ij} \sigma_{kl}; \]

\( W_{ij} \) is the spin tensor.

In (4), \( H \) is the work-hardening parameter,

\[ H = 3 \frac{\rho_0}{\rho} \frac{\partial G}{\partial I} \frac{\partial F}{\partial \Delta} + \frac{\partial F}{\partial \bar{\varepsilon}} + \frac{\partial F}{\partial \Sigma} \frac{\partial \Sigma}{\partial \bar{\varepsilon}}, \]

and \( 3 \partial G / \partial I \) is the dilatancy factor.

\[ \frac{\partial G}{\partial I} = \frac{\bar{\sigma}}{\sigma_{ij}'} D_{ij}^p = \frac{D_{kk}^p}{\bar{\varepsilon}}, \]

being the rate of plastic volumetric change per unit rate of plastic distortion.
The first term on the right-hand side of (6) is material hardening (softening) due to plastic volumetric changes (void growth). Since \( \partial F / \partial \Delta \) is always nonpositive, the sign of this term depends on the sign of \( \Phi K \), which, as pointed out before, at least in the initial stages of deformation, is positive. Thus, the density-hardening
\[
h_1 = 3(1 + \Delta) \frac{\partial G}{\partial F} \frac{\partial F}{\partial \Delta}
\]
is negative during the early stages of plastic flow. Moreover, it continues to remain negative during the process of stable void growth. In fact, Eq. (7) shows that \( \partial G / \partial \delta \) has the same sign as \( \Phi K \), which, for void growth, is positive. Hence, \( h_1 \) represents material softening due to the geometric effects.

The quantity \( \partial F / \partial \delta \) in Eq. (6) represents material hardening due to average plastic distortion of the matrix. It is always nonnegative and can be measured at least approximately, on carefully grown very thin specimens which do not include large inclusions.

The last term in the right-hand side of Eq. (6) represents macroscopic material softening due to formation of microscopic intense localized deformations. In the microscale these localized deformations behave in a similar manner as “plastic hinges” in structural frames. Their distribution and geometric pattern are highly affected by the microstructure of the material. Keeping this in mind, it may be assumed that \( \Sigma \) is essentially a function of the distortional plastic strain, i.e.
\[
\Sigma = \Sigma(\xi).
\]
In fact, calculation by Nemat-Nasser and Taya (1978) suggests that
\[
\Sigma = \Sigma_0 (1 - e^{-k^2})
\]
may be a reasonable approximation, where \( k \) and \( \Sigma_0 \) depend on the microstructure, i.e. the material.

It is helpful to pause at this point and review the experimental support for a nonassociative flow rule. In a series of compression-tension tests of several steels, Spitzig et al. (1975, 1976) report a pronounced pressure effect on the yield stress and a definite plastic volumetric expansion. If the effective stress, \( \delta \), is redefined as
\[
\delta = \sqrt{(\sigma_i' \sigma_j')},
\]
then it equals the magnitude of axial stress, \( \sigma \), imposed on an overall hydrostatic compression (or tension), in a tension-compression test. In this case, \( -\partial F / \partial \delta \) equals the parameter, \( a \), used by Spitzig et al. They report that \( a \) is essentially independent of strain and therefore, the strength differential \( (S-D) \) equals twice this parameter. For an associative flow rule,
$G = -F$ and therefore, $\frac{\partial G}{\partial I} = -\frac{\partial F}{\partial I}$. Based on this assumption, Spitzig et al. calculate the plastic volumetric expansion which turns out to be approximately 15 times larger than the observed values. These authors then conclude that normality (with associative flow rule) may not hold, in general, for materials of this kind.

According to our theory, $\frac{\partial G}{\partial I} = -\frac{\partial F}{\partial I}$, in general, because $\frac{\partial G}{\partial I}$ is a kinematical quantity having the same sign as the instantaneous rate of plastic volumetric change, whereas $-\frac{\partial F}{\partial I}$ pertains to the strength of the material and is always positive. Because of the nature of plastic flow, it so happens that for metals both quantities possess the same sign, at least during the early stages of deformation, although under very large pressures negative $\frac{\partial G}{\partial I}$ can be expected (collapse of microvoids). Indeed, for granular materials, $\frac{\partial G}{\partial I}$ changes sign during the normal course of deformation, whereas $-\frac{\partial F}{\partial I}$ remains strictly positive, characterizing the overall frictional resistance.

Based on the experimental data of Spitzig et al., values of $-\frac{\partial F}{\partial I}$ and $\frac{\partial G}{\partial I}$ for several steels are given in Table 1. These values support the use of a nonassociative flow rule.

Since $\frac{\partial F}{\partial I}$ is always negative ($F$ decreases with increasing tension), it is expected that ductile flow should occur at higher stress levels in compression than in tension. However, essentially the same strain patterns are involved in both cases. This has been observed experimentally by Anand and Spitzig (1980), where localized deformations have been seen to occur essentially at the same strain levels ($\approx 0.034$ axial strain) in tension and compression, but at different stress levels. The patterns of these shear bands are the same in tension and compression, forming an approximately $38^\circ$ angle with the maximum principal stress axis. Similar results have been reported by other investigators, as discussed, for example, by Asaro (1979) and Goods and Brown (1979). Even for nickel-base superalloys, slip bands are an integral part of the microstructure of plastic flow. Kikuchi and Weertman (1980) have observed that slips emanate from carbides at grain

<table>
<thead>
<tr>
<th>Steel</th>
<th>$-\frac{\partial F}{\partial I}$ (%)</th>
<th>$\frac{\partial G}{\partial I}$ (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maraging (unaged)</td>
<td>1.8</td>
<td>&lt;0.03</td>
</tr>
<tr>
<td>4310</td>
<td>2.8</td>
<td>0.15</td>
</tr>
<tr>
<td>4330</td>
<td>3.0</td>
<td>0.15</td>
</tr>
<tr>
<td>Maraging (aged)</td>
<td>3.5</td>
<td>0.23</td>
</tr>
</tbody>
</table>
boundaries that are parallel to the direction of extension in tensile specimens strained about 10% at room temperature, and then annealed for two hours at 810°C.

The existence of slip bands gives rise to the macroscopic concept of corner for the yield surface. This implies that further plastic strain rates will be affected both in their orientation and magnitude by the corresponding additional stress rates, i.e. the plastic strain rate tensor is noncoaxial with the stress tensor. All theories which consider a smooth flow potential depending on stress invariants, lead to coaxiality, and hence cannot accurately account for the effects of the formation of microscopic localized slip bands.

In the present theory the difficulty may be circumvented by the addition of a term linear in \( \delta_{ij} \) to the right-hand side of Eq. (4). If this term is chosen as

\[
A \left\{ \delta_{ij} - \frac{1}{2 \bar{\sigma}^2} \sigma_{kl} \delta_{kl} \delta_{ij} \right\},
\]

(12)

then it does not contribute to the rate of plastic work. Moreover, the physical meanings of the dilatancy parameter, \( \delta G/\delta I \), and the pressure-sensitivity parameter, \(-\delta F/\delta I \), are unchanged. It is interesting to note that such a term naturally emerges in the double-slip theory of granular materials; Spencer (1964), de Josselin de Jong (1971), and Mehrabadi and Cowin (1978, 1980). With the addition of this term and upon separation of the strain rate into the distortional and dilatational parts, it follows that

\[
D_{ij} = \frac{\sigma_{ij}'}{2H} \left( \frac{\sigma_{kl}}{2 \bar{\sigma}} - \frac{\delta F}{\delta I} \delta_{kl} \right) \delta_{ij} + A \left\{ \delta_{ij}' - \frac{1}{2 \bar{\sigma}^2} \sigma_{kl} \delta_{kl} \delta_{ij} \right\},
\]

\[
D_{kl} = \frac{3}{2H} \frac{\partial G}{\partial I} \left( \frac{\sigma_{kl}'}{2 \bar{\sigma}} - \frac{\delta F}{\delta I} \delta_{kl} \right) \delta_{kl}.
\]

(13)

As it stands, it is difficult to interpret the parameter \( A \) on a physical basis. Some interpretations in terms of a secant modulus have been made by Rudnicki and Rice (1975) and Stören and Rice (1975), using a generalization of linear elasticity in line with deformation plasticity theory. It is, however, desirable to arrive at modifications of this kind by a physical microstructural modeling. This is done in the next section for a two-dimensional case, although the basic approach admits generalization to three-dimensional problems.

3. Microscopic approach

Plastic deformation of metals involves flow in the form of slip, even during the early stages of loading. For a single crystal, a system of
double-slip develops symmetrically about the direction of maximum principal stress. For polycrystalline metals also, slip systems are activated in a definite pattern in relation to the maximum stress direction; see, for example, Dyson et al. (1976); Kikuchi and Weertman (1980). As pointed out in the introduction, during the final stages of ductile flow, intense localized deformations form between adjacent voids, which may lead to the generation of void sheets. Therefore, both for single crystals and alloys it is reasonable to consider a microstructural model that consists of slip at active slip systems, accompanied by plastic volumetric change associated with the formation and growth of microvoids. The model then applies even to the later stages of ductile flow, where intense localized deformation bands (formed between adjacent voids) are viewed as individual slip systems. Moreover, the process is, to a large extent, pressure sensitive. In view of these observations, a dilatant, pressure-sensitive set of rate constitutive relations will now be developed by a systematic calculation based on the concept of dilatant, active slip systems having a microscopic pressure-sensitive constitutive response.

3.1. Notation

Quantities pertaining to an individual slip system are denoted by the addition of Greek superscripts which, when repeated, are summed over all instantaneously active slip systems. Since two-dimensional flow (plane strain) is envisaged, italic subscripts take on values 1 and 2, and the repeated ones are summed. For convenience, dyadic and indicial notations are used, e.g. the stress tensor \( \sigma \) with components \( \sigma_{ij} \), deformation rate and spin tensors, \( D, W \), with components \( D_{ij}, W_{ij} \), etc. The fourth-order elasticity tensor is denoted by \( L \), having components \( L_{ijkl} \). For application, isotropic elasticity is assumed, i.e.

\[
L_{ijkl} = G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda(\delta_{ij}\delta_{kl}),
\]

where \( G \) (shear modulus) and \( \lambda \) are the Lamé coefficients.

3.2. General theory

The total velocity gradient, \( D + W \), is obtained by plastic distortion giving rise to \( D^p + W^p \), followed by elastic distortion with elastic deformation rate, \( D^e \), which is accompanied by a spin denoted by \( W^* \), so that

\[
D = D^e + D^p, \quad W = W^* + W^p.
\]

as discussed by Hill (1966), Hill and Rice (1972), Asaro and Rice (1977) and Asaro (1979); (15) represents an exact decomposition in the context of continuum plasticity; Nemat-Nasser (1979).

\footnote{Earlier related works are by Taylor (1938), Bishop and Hill (1951), Budiansky and Wu (1962), Mandel (1966), Hill (1967, 1971), Hutchinson (1970), Kocks (1970), Lin (1971), Bui et al. (1972), Zarka (1973), and Havner (1979), where reference to other work can be found.
The plastic parts in (15) are viewed to have stemmed from the plastic slip rates, \( \gamma^a \), and the plastic dilation rates, \( \theta^a \), over all active slips. Let \( \mathbf{n}^a \) be the unit vector normal to the \( \alpha \)th slip band, \( \mathbf{s}^a \) be the unit vector in the direction of the slip, and introduce the following tensors associated with each individual slip system:

\[
\rho_i^a = \frac{1}{2}(s_i^a n_j^a + s_j^a n_i^a) + \tan \nu n_i^a n_j^a, \\
\omega_i^a = \frac{1}{2}(s_i^a n_j^a - s_j^a n_i^a), \quad \text{(no sum on } \alpha),
\]

where \( \tan \nu \) is the dilatancy parameter,

\[
\tan \nu = \frac{\partial \theta^a}{\partial \gamma^a},
\]

assumed, for simplicity, to be the same for all \( \alpha \). Then the plastic constituents in (15) are

\[
D^p = \gamma^a \rho^a, \quad W^p = \gamma^a \omega^a,
\]

where \( \alpha \) is summed over all active slip systems.

The elastic parts in (15) give rise to elastic stress rates corotational with the corresponding spin, according to

\[
\nabla \sigma = L : D^e, \\
\nabla \sigma = \dot{\sigma} - W^* \omega + \omega W^e.
\]

For a typical active slip, \( \alpha \), the shear rate is assumed to be governed by the following constitutive relation:

\[
\dot{\gamma}^a = \tan \eta \dot{\theta}^a = h^{a\beta} \dot{\gamma}^\beta,
\]

where \( \gamma^a = \sigma_{ij} s_i^a n_j^a \) and \( \sigma^a = \sigma_{ij} n_i^a n_j^a \) (\( \alpha \) not summed) are the shear and normal stresses transmitted over the \( \alpha \)th slip plane, \( h^{a\beta} \) is the symmetric work-hardening matrix, and \( \tan \eta \) is the pressure-sensitivity parameter. In addition to (21), it is necessary to introduce constitutive assumptions that describe time variations of the unit vectors, \( \mathbf{n}^a \) and \( \mathbf{s}^a \). Various possibilities may be entertained, but the final judgment is dictated by the physical modeling associated with actual microstructural behavior. Here, for simplicity, Asaro’s (1979) assumption will be used,

\[
\dot{n}_i^a = W_i^* s_j^a, \quad \dot{s}_i^a = W_i^* n_j^a.
\]

Then it follows from (21) that

\[
\nabla \sq_j^{\alpha} = h^{a\beta} \dot{\gamma}^\beta,
\]

where

\[
q_i^a = \frac{1}{2}(s_i^a n_j^a + s_j^a n_i^a) + \tan \eta n_i^a n_j^a \quad \text{(no sum on } \alpha)
\]

represents the normal to the yield surface associated with \( \alpha \)th slip system.
Let

\[ N^{ab} = h^{ab} + q^a : L : p^b, \quad (25) \]

and denote the inverse of matrix \( N^{ab} \) by \( M^{ab} \). Then, with the aid of (15)–(24) and in view of (5), it can easily be shown that

\[ \dot{\sigma} = L : D + M^{ab}(\sigma^{ba} - \omega^a \sigma - L : p^a)(q^b : L : D). \quad (26) \]

In this equation, the superscripts \( \alpha \) and \( \beta \) are summed over all instantaneously active slip systems.

The rate constitutive relations represented by (26) include pressure sensitivity and dilatancy, but nevertheless have a rather simple structure.

3.3. Special cases

To bring the basic structure of rate constitutive Eqs. (26) to light, consider a special case of only two active slip systems, symmetrically oriented with respect to the maximum principal stress, \( \sigma_1 \) at the angle \( \Phi / 4 + (\phi / 2) \); the other principal stress is \( \sigma_2 \). Assume further,

\[ h^{11} = h^{22} = h \quad \text{and} \quad h^{12} = h^{21} = h_1, \quad (27) \]

and introduce the following parameters:

\[ M = \frac{\sin \eta}{\cos(\Phi - \eta)}, \quad B = \frac{\sin \nu}{\cos(\Phi - \nu)}, \]

\[ 2H_1 = (h - h_1)(1 - M \sin \phi)(1 - B \sin \phi), \]

\[ 2\mu^* \cos^2 \phi = (h + h_1)(1 - M \sin \phi)(1 - B \sin \phi), \quad (28) \]

where \( M \) is the pressure sensitivity parameter, and \( B \) the dilatancy parameter proportional to \( \partial G / \partial I \) of Section 2, as is seen from

\[ B = (D_{11}^p + D_{22}^p)/(D_{11}^p - D_{22}^p). \]

(29)

It is now easy to deduce from (26) the following explicit rate equations:

\[ \dot{\sigma}_{ij} = a_1 D_{ij} + a_2 D_{22}, \quad \dot{\sigma}_{ij} = b_1 D_{11} + b_2 D_{22}, \quad \dot{\sigma}_{ij} = 2\mu D_{12}, \quad (30) \]

where

\[ a_1 = K[2G + \lambda)(\mu^* + G)(1 - M)(1 - B)], \]

\[ a_2 = K[2G + \lambda)(\mu^* + G)(1 + M)(1 + B)], \]

\[ b_1 = K[2G + \lambda)(\mu^* + G)(1 + M)(1 - B)], \]

\[ b_2 = K[2G + \lambda)(\mu^* + G)(1 + M)(1 + B)], \]

\[ \mu = \frac{G[H_1 + \tau(M - \sin \phi)(1 - B \sin \phi)]}{H_1 + G(M - \sin \phi)(B - \sin \phi)}, \]

\[ K = [\mu^* + G + MB(G + \lambda)]^{-1}, \quad \tau = \frac{1}{2}(\sigma_1 - \sigma_2). \quad (31) \]

\(^2\)Note that the angle \( \phi \) used here relates to the angle \( \Phi \) used by Asaro (1979); \( \Phi = (\pi/4) \pm (\phi/2) \) for tension and compression, respectively.
Further specialization results if elastic incompressibility is imposed. Care is required when obtaining the limiting form of Eqs. (31), where the quantity $\lambda/2(G+\lambda)$ which equals the Poisson ratio, must approach 1/2. This results in a new set of parameters, denoted by superposed bar, as follows:

$$\bar{a}_1 = \frac{1}{MB} \left[ \mu^* + G(1-M)(1-B) \right],$$

$$\bar{a}_2 = \frac{1}{MB} \left[ \mu^* + G(1-M)(1+B) \right],$$

$$\bar{b}_1 = \frac{1}{MB} \left[ \mu^* + G(1+M)(1-B) \right],$$

$$\bar{b}_2 = \frac{1}{MB} \left[ \mu^* + G(1+M)(1+B) \right],$$

(32)

where $\mu$ remains unchanged. Observe that the material is still plastically compressible as displayed by the presence of dilatancy parameter $B$.

In the absence of pressure sensitivity, $M$ vanishes, and then the pressure rate is indeterminate. It follows that

$$\frac{G}{G+\mu^*} (D_{11} - D_{22}) - \frac{1}{B} (D_{11} + D_{22}) = 0.$$

(33)

The deviatoric part of the stress rate, $\dot{\sigma}^i$, is now obtained from (30) and (31) by setting $M=0$.

Finally, when pressure insensitive, incompressible plastic flow is assumed, the indeterminate quantity, $(D_{11} + D_{22})/B$, is eliminated, using (33). The resulting rate equation for the deviatoric part of the stress tensor is summarized by

$$\dot{\sigma}^i = \frac{G\mu^*}{G+\mu^*} (D_{11} - D_{22}) f,$$

(34)

where Eq. (30) remains the same.

$$f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(35)

and $\mu^*$ is obtained from (28) with $M=B=0$, i.e.

$$\mu^* = \frac{(h_1 + h_2)}{2 \cos^2 \phi},$$

(36)

leading to Asaro's (1979) equations.

4. Localization

Let $n$ with components $n_1$ and $n_2$ be the unit vector normal to a localized shear band formed in biaxial plane strain extension (or compression). Then, for materials with constitutive relations (30), the orientation of this band.
defined by \( c = n_2 / n_1 \), satisfies,
\[
b_2 (\mu - \tau) c^4 + \left[ a_1 b_2 - a_2 b_1 - a_2 (\mu - \tau) - b_1 (\mu + \tau) \right] c^2 + a_1 (\mu + \tau) = 0.
\]  
(37)

Substitution from (31) results in
\[
(\mu - \tau) \left[ 2(1 - \hat{\nu}) \mu^* + G(1 + M)(1 + B) \right] c^4
+ 2 \left[ G \left[ 2 \mu^* - \tau (M - B) - \mu (1 - MB) \right] \right]
- 2 \hat{\nu} \mu^* c^2 + (\mu + \tau) \left[ 2(1 - \hat{\nu}) \mu^* + G(1 - \hat{\nu})(1 - B) \right] = 0,
\]  
(38)

where \( \hat{\nu} \) is the Poisson ratio.

The rigid plastic limit is obtained when \( G \) is taken to infinity, yielding
\[
(\mu_R - \tau)(1 + M)(1 + B) c^4 + 2 \left[ 2 \mu^* - \mu_R (1 - MB) - \tau (M - B) \right] c^2
+ (\mu_R + \tau)(1 - M)(1 - B) = 0,
\]  
(39)

where \( \mu_R \) is the limiting value of \( \mu \) in Eq. (31), as \( G \to \infty \), i.e.
\[
\mu_R = \frac{H_1 + \tau (M - \sin \phi)(1 - B \sin \phi)}{(M - \sin \phi)(B - \sin \phi)}.
\]  
(40)

It should be noted that Eq. (39), with a different \( \mu_R \), has been given previously by Mehrabadi and Cowin (1980) in connection with soil plasticity.

Equation (39) reduces to Hill and Hutchinson’s (1975) characteristic equation, when \( M \) and \( B \) are set equal to zero, and \( \mu_R \) and \( \mu^* \) are reinterpreted accordingly. This yields
\[
(\mu_R - \tau) c^4 + 2 (2 \mu^* - \mu_R) c^2 + (\mu_R + \tau) = 0,
\]  
(41)

where \( \mu^* \) is given by (36), and
\[
\mu_R = \frac{h - h_1 - 2 \tau \sin \phi}{2 \sin^2 \phi}.
\]  
(42)

The characteristic Eq. (41) is identical with that obtained by Asaro (1979) for the rigid-plastic case. Asaro also considers the more general case of incompressible elastoplastic materials; the corresponding equation is obtained from (38) by setting \( \hat{\nu} = \frac{1}{2} \) and \( M = B = 0 \), arriving at
\[
(\mu - \tau) c^4 + 2 \left( \frac{2 \mu^*}{G + \mu^*} \mu^* - \mu \right) c^2 + (\mu + \tau) = 0,
\]  
(43)

where \( \mu^* \) is given by (36) and \( \hat{\mu} \) is obtained from Eq. (31),
\[
\hat{\mu} = \frac{G (h - h_1 - 2 \tau \sin \phi)}{h - h_1 + 2 G \sin^2 \phi}.
\]  
(44)
4.1. Comparison with experiments

As mentioned before, Anand and Spitzig (1980) report localized shear bands occurring in tension and compression tests of certain maraging steel specimens. The bands are reported to form at angles $\pm (38\pm 2)^\circ$ about the maximum principal stress, the maximum principal stress being zero for the compression test. The localization takes place at about 0.034 axial strain in both compression and tension. The corresponding tangent Young modulus, $E_t$, is reported to be about 391 MPa for tension and 486 MPa for compression. The corresponding secant modulus, $E_s$, is reported to be 58,656 MPa for tension and 59,046 MPa for compression.

Anand and Spitzig assume incompressibility and a plane strain condition, and apply both the $J_i$ flow theory and a deformation theory in order to predict their experimental observations. The flow theory predicts shear band orientation of about $\pm 45^\circ$ at the critical axial strain of the absolute value of 0.184, whereas the deformation theory predicts shear band orientation of about $\pm 42.55^\circ$ (with respect to the maximum stress direction) at the critical strain of the absolute value of 0.085, both for compression and for tension.

The experimental results of Anand and Spitzig are in very good accord with the prediction which can be deduced from the characteristic Eq. (41); this is not reported by the authors who have focused on estimating the critical strain. Indeed, if (41) is rewritten as

$$Ac^4 + 2Be^2 + C = 0,$$

where

$$A = \mu_R - \tau, \quad B = 2\mu^* - \mu_R, \quad C = \mu_R + \tau,$$

then real values for $c$ require $B^2 - AC > 0$ or $\tau^2 > \mu_R^2 - (2\mu^* - \mu_R)^2$, so that the smallest absolute value of $\tau$ is obtained when the critical condition $B^2 = AC$ or

$$\tau^2 = \mu_R^2 - (2\mu^* - \mu_R)^2$$

holds. The corresponding critical value of $c^2$ is given by

$$c^2 = \frac{-B}{A} = \sqrt{\frac{C}{A}} = \sqrt{\frac{\mu_R + \tau}{\mu_R - \tau}},$$

where $2\mu^* < \mu_R$ is assumed, which is almost always the case.

For incompressible materials, $4\mu^* = E_t$, which is measured at the critical state. Therefore, $\mu_R$ in (48) can be eliminated in favor of $\mu^*$, with the aid of Eq. (47). Direct substitution results in

$$c^2 = \frac{\tau + 2\mu^*}{\tau - 2\mu^*} = \frac{2\tau + E_t}{2\tau - E_t}$$

which applies at the critical state independently of the particular interpretation of the instantaneous moduli $\mu^*$ and $\mu_R$, as long as $2\mu^* < \mu_R$. 
To obtain the angle $\theta$ which the direction of the shear band makes with the maximum principal stress, it is observed that $\tan^2 \theta = \frac{1}{c^2}$, so that

$$\tan^2 \theta = \frac{2\tau - E_t}{2\tau + E_t}.$$  \hspace{1cm} \text{(50)}

Entering the recorded magnitude of the stress of 1,777 MPa in tension, and 2,126 MPa in compression, it is immediately deduced from (50) that $\theta = \pm 39.0^\circ$ in tension and $\pm 38.8^\circ$ in compression, which are in excellent agreement with the observed results.

The recorded experimental data can be used to estimate the values of the microscopic parameters, $h$, $h_1$, in terms of $\phi$, at the critical state. From (36), (42), and (47) it follows that

$$h = \frac{\tau (\cos 2\theta + \sin \phi)}{2\cos 2\theta},$$  \hspace{1cm} \text{(51)}

$$h_1 = \tau \cos 2\theta \cos^2 \phi - h,$$  \hspace{1cm} \text{(52)}

which give the values of $h$ and $h_1$ at the critical state.

For a single crystal, $\phi$ is regarded as a crystalline parameter. For a polycrystalline material, on the other hand, $\phi$ must be viewed as a statistically averaged, microscopically preferred, microlocalization orientation parameter. In this context, it is interesting to observe that when the average microlocalization orientation coincides with the orientation of (macro) shear band, as one would expect that within the framework of the present approximation it should, then the hardening parameter $h$ vanishes, as is seen from Eq. (51).

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References


