NOTES ON

FOUNDATIONS OF CONTINUUM

MECHANICS

for
Mechanics of Deformable Bodies
720-D17

by

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1.1 Scalars and Vectors

Once a suitable system of physical units is selected, certain physical quantities can be placed into one-to-one correspondence with the real numbers. Quantities of this kind are called **scalars**. The length $l$ of a bar, the mass $m$ of a piece of metal, and the distance $d$ between two points in space are familiar examples of scalar quantities. There are other physical quantities, however, such as force, state of stress in a deformed solid, etc., whose mathematical description requires more than the field of real numbers. A subclass of such quantities comprises those physical or geometrical quantities which can be represented by vectors, i.e., directed line elements in space. The magnitude of the vector quantity is represented by the length of the line segment whose direction indicates the direction of the vector quantity. With this definition, two line segments of equal lengths that are parallel and have the same directions represent the same vector. We use bold-faced letters, such as $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, to designate vectors and denote their magnitudes by $A$, $B$, and $C$, respectively. Examples of vector quantities are force, velocity, and acceleration, in particle dynamics.

If $\mathbf{A}$ is a vector and $m$ a real number, then $m \mathbf{A}$ is a new vector that is collinear with $\mathbf{A}$ and has a length $|mA|$, where $|m|$ denotes the absolute value of $m$. If $m$ is negative, then the direction of the new vector $mA$ is opposite to that of $\mathbf{A}$. The following vector operations should be familiar to the reader:

a) **Vector Summation.** The sum $\mathbf{C}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{C} = \mathbf{A} + \mathbf{B}$, is defined by the parallelogram law, see Fig. 1.1. Note
that, for two vectors \( \mathbf{E} \) and \( \mathbf{F} \), the difference \( \mathbf{D} \) is defined by 
\[
\mathbf{D} = \mathbf{E} - \mathbf{F} = \mathbf{E} + (-\mathbf{F})
\]
and that the vector addition is commutative, i.e.,
\[
\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}
\]
and associative, i.e.,
\[
(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})
\]

b) **Scalar Multiplication.** Consider two vectors \( \mathbf{A} \) and \( \mathbf{B} \).

The scalar (or dot) product of \( \mathbf{A} \) and \( \mathbf{B} \), denoted by \( \mathbf{A} \cdot \mathbf{B} \), is a real number \( c \) obtained by multiplying the magnitude of \( \mathbf{A} \), namely \( A \), by the orthogonal projection of \( \mathbf{B} \) on \( \mathbf{A} \), i.e.,
\[
c = \mathbf{A} \cdot \mathbf{B} = A B \cos \theta
\]
where \( \theta \) is the angle formed by the directions of \( \mathbf{A} \) and \( \mathbf{B} \) (see Fig. 1.2).

Note that the scalar product is commutative, i.e., \( \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \), and distributive, i.e.,
\[
\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}
\]
If the directions of \( \mathbf{A} \) and \( \mathbf{B} \) are orthogonal, then \( \mathbf{A} \cdot \mathbf{B} = 0 \).

c) **Vector Multiplication.** The vector product of two non-parallel vectors \( \mathbf{A} \) and \( \mathbf{B} \) is a vector \( \mathbf{C} \) whose direction is normal to the directions of both \( \mathbf{A} \) and \( \mathbf{B} \), the triad \( \mathbf{A}, \mathbf{B}, \mathbf{C} \), being right-handed, and whose magnitude is given by \( A B \sin \theta \) (see Fig. 1.3). We denote the vector product by the symbol \( \times \), i.e., we write \( \mathbf{A} \times \mathbf{B} = \mathbf{C} \), and note that \( \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \), and \( \mathbf{A} \times (\mathbf{B} + \mathbf{D}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{D} \). For any two non-zero vectors, \( \mathbf{A} \neq \mathbf{0}, \mathbf{B} \neq \mathbf{0} \), the relation \( \mathbf{A} \times \mathbf{B} = \mathbf{0} \) is valid if and only if \( A \) and \( B \) are parallel, where \( \mathbf{0} \) denotes the null vector, a vector with zero magnitude and arbitrary direction.

It is important to note that the vector operations (a), (b), and (c) above are defined without reference to a coordinate system.

1.2 **Coordinate System and Index Notation**

In a three-dimensional Euclidean space, a vector may be defined by three numbers which are called its components with respect to a system
of coordinates. Consider a right-handed system of rectangular Cartesian coordinates \( x_1, x_2, x_3 \). A point \( P \) can be represented by its position vector \( \vec{z} \), that is, the directed line segment \( OP \). Let \( \vec{e}_1, \vec{e}_2, \) and \( \vec{e}_3 \) be three base vectors, that is, vectors of unit length in the positive coordinate directions. By the definitions of vector summation and scalar multiplication we have (see Fig. 2.1):

\[
\vec{z} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 = \sum_{i=1}^{3} x_i \vec{e}_i
\]

(2.1)

where the components of \( \vec{z} \) are

\[
x_1 = \vec{z} \cdot \vec{e}_1 , \quad x_2 = \vec{z} \cdot \vec{e}_2 , \quad x_3 = \vec{z} \cdot \vec{e}_3
\]

Accordingly, (2.1) may be written in the form

\[
\vec{z} = \sum_{i=1}^{3} \left( \vec{z} \cdot \vec{e}_i \right) \vec{e}_i
\]

(2.2)

Equations (2.1) and (2.2) are more compactly written using the following summation convention, which was introduced by Einstein:

**Summation Convention.** A repeated subscript in a monomial stands for the sum of the three terms obtained by successively giving to this subscript the values \( 1, 2, \) and \( 3 \). Note that, for this rule to be meaningful, a subscript can at most occur twice in each monomial.

Using this rule, we may rewrite (2.1) and (2.2) as follows:

\[
\vec{z} = x_i \vec{e}_i = \left( \vec{z} \cdot \vec{e}_i \right) \vec{e}_i \quad ; \quad i = 1, 2, 3
\]

(2.3)

Let \( A_i \) and \( B_j \) denote the components of vectors \( \vec{A} \) and \( \vec{B} \), respectively, i.e., \( \vec{A} = A_i \vec{e}_i \), and \( \vec{B} = B_j \vec{e}_j \); \( i, j = 1, 2, 3 \). The dot product of \( \vec{A} \) and \( \vec{B} \) then is
\mathbf{A} \cdot \mathbf{B} = (A_i \mathbf{e}_i) \cdot (B_j \mathbf{e}_j)
\hspace{1cm} = (A_i B_j) \mathbf{e}_i \cdot \mathbf{e}_j \hspace{0.5cm} ; \hspace{0.5cm} i, j = 1, 2, 3 \hspace{1cm} (2.4)

From the definition of unit base vectors, we have

\[ \mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases} \]

It is convenient to introduce a new symbol \( \delta_{ij} \), called the Kronecker delta, defined by

\[ \delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases} \hspace{1cm} (2.5) \]

Equation (2.4) then becomes

\[ \mathbf{A} \cdot \mathbf{B} = (A_i B_j) \delta_{ij} = A_i B_i \]
\[ = A_1 B_1 + A_2 B_2 + A_3 B_3 \hspace{1cm} (2.6) \]

The magnitude \( A \) of the vector \( \mathbf{A} \) is given by

\[ A^2 = A_i A_i \hspace{1cm} (2.7) \]

The cross product of \( \mathbf{A} \) and \( \mathbf{B} \) is

\[ \mathbf{A} \times \mathbf{B} = (A_i \mathbf{e}_i) \times (B_j \mathbf{e}_j) \]
\[ = (A_i B_j) \mathbf{e}_i \times \mathbf{e}_j \hspace{1cm} (2.8) \]

From the definitions of the cross product and the unit base vectors we have
\[ \varepsilon_1 \times \varepsilon_2 = \varepsilon_3 \quad \varepsilon_2 \times \varepsilon_3 = \varepsilon_1 \quad \varepsilon_3 \times \varepsilon_1 = \varepsilon_2 \]

\[ \varepsilon_2 \times \varepsilon_1 = -\varepsilon_3 \quad \varepsilon_3 \times \varepsilon_2 = -\varepsilon_1 \quad \varepsilon_1 \times \varepsilon_3 = -\varepsilon_2 \]

(2.9)

It is convenient to introduce a new symbol \( \varepsilon_{ijk} \), called the permutation symbol, defined as follows:

\[ \varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } 1,2,3 \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } 1,2,3 \\ 0 & \text{if } (i,j,k) \text{ do not form a permutation} \end{cases} \]

(2.10)

Note that according to this definition the relations (2.9) may be concisely written as follows:

\[ \varepsilon_i \times \varepsilon_j = \varepsilon_{ijk} \varepsilon_k \]

(2.9a)

Accordingly, (2.8) yields

\[ A \times B = (A_i B_j) \varepsilon_{ijk} \varepsilon_k \]

\[ = \varepsilon_1 (A_2 B_3 - A_3 B_2) + \varepsilon_2 (A_3 B_1 - A_1 B_3) + \varepsilon_3 (A_1 B_2 - A_2 B_1) \]

(2.11)

If \( C = C_k \varepsilon_k = A \times B \), it follows from (2.11) that

\[ C_k = A_i B_j \varepsilon_{ijk} = \varepsilon_{kij} A_i B_j \]

(2.12)

The following relations are direct consequences of the summation convention and the definitions (2.5) and (2.10):

\[ \varepsilon_{ijk} \varepsilon_{ijk} = 6 \]

\[ \varepsilon_{ijk} \varepsilon_{jkl} = 2 \delta_{kl} \]

(2.13)

\[ \varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \]
The repeated subscripts are called **dummy subscripts**. A dummy subscript may be replaced by any other subscript letter which is not otherwise used in the same relation. The subscripts that occur only once in each monomial of an equation are called **live subscripts**. The first equation in (2.13), for example, contains only dummy subscripts, while in the second equation \( \ell \) and \( k \) are both live subscripts and \( i \) is a dummy subscript.

### 1.3 Coordinate Transformations

Consider now the transformation of coordinates. Let \( x_1, x_2, x_3 \) and \( x'_1, x'_2, x'_3 \) be the coordinates of one and the same point \( P \) with the position vector \( \vec{x} \) in two systems of right-handed rectangular Cartesian coordinates that have the same origin \( O \). Let \( \vec{e}_i \) and \( \vec{e}'_i \), \( i = 1, 2, 3 \), denote the base vectors of the unprimed and primed coordinates, respectively. The vector \( \vec{x} \) may be expressed as

\[
\vec{x} = x_i \vec{e}_i = x'_i \vec{e}'_i ; \quad i = 1, 2, 3 . \tag{3.1}
\]

Taking the dot product of this equation first with \( \vec{e}_j \) and then with \( \vec{e}'_j \), we obtain

\[
x_j = (\vec{e}'_i \cdot \vec{e}_j) x'_i , \tag{3.2}
\]

\[
x'_j = (\vec{e}'_i \cdot \vec{e}_j) x_i ; \quad i, j = 1, 2, 3 . \tag{3.3}
\]

Since \( \vec{e}'_i \) and \( \vec{e}_j \) are unit vectors, \( (\vec{e}'_i \cdot \vec{e}_j) \) denotes the cosine of the angle formed by the \( x'_i \) and \( x_j \) axes. Denoting the cosine of this angle by \( c_{ij} \), i.e., \( c_{ij} = \vec{e}'_i \cdot \vec{e}_j \), we rewrite (3.2) and (3.3) as
\[ x_j = c_{ij} x'_i \]  \hspace{1cm} (3.4)

\[ x'_j = c_{ji} x_i \quad ; \quad i, j = 1, 2, 3 \]  \hspace{1cm} (3.5)

Note that, in accord with the summation convention, for fixed \( j \), the terms on the right side of (3.4) and (3.5) are to be summed on \( i \), for \( i = 1, 2, 3 \).

Substitution of (3.4) into (3.5) yields

\[ x'_j = c_{ji} c_{ki} x'_k \]

which must be an identity, leading to

\[ c_{ji} c_{ki} = \delta_{jk} \]  \hspace{1cm} (3.6)

Similarly, substitution of (3.5) into (3.4) results in the following relation:

\[ c_{ij} c_{ik} = \delta_{jk} \]  \hspace{1cm} (3.7)

Consider a vector \( \vec{A} \) with components \( A_i \) and \( A'_i \) in the unprimed and primed coordinate systems, respectively, i.e.,

\[ \vec{A} = A_i \vec{e}_i = A'_i \vec{e}'_i \]  \hspace{1cm} (3.8)

Taking the dot product of both sides of (3.8) first with \( \vec{e}_j \) and then with \( \vec{e}'_j \), we obtain

\[ A_j = c_{ij} A_i \]  \hspace{1cm} (3.9)

\[ A'_j = c_{ji} A_i \]  \hspace{1cm} (3.10)
With reference to the unprimed coordinate system, the vector \( \mathbf{A} \) is specified by its components \( A_i \). These components change to \( A'_i \) as a new (primed) coordinate system is introduced. The rule (3.10) links the components of \( \mathbf{A} \) in the two coordinate systems. Therefore, a vector may be also defined by a triple of components \( A_i \) which transform in accordance with the rule (3.10) under the coordinate transform (3.5). It is customary to refer to vectors as tensors of order 1.

1.4 Scalar and Vector Fields

Let a scalar function \( \varphi = \varphi(x) \) be defined and continuous in a finite region \( R \) of the three-dimensional Euclidean space whose points are referenced by means of a rectangular Cartesian coordinate system. Consider a generic point \( x_i^o \) in \( R \) and denote the value of \( \varphi \) at this point by \( \varphi^o \). The locus of the points in \( R \) at which the scalar function \( \varphi(x) \) attains the value of \( \varphi^o \) is a surface, called the level surface, with the following equation:

\[
\varphi(x) = \varphi^o = \text{constant} \quad .
\]

(4.1)

The rate of change of the scalar function \( \varphi \) in the direction of the \( x_1 \)-axis at point \( x_i^o \) is given by \( \left[ \frac{\partial \varphi}{\partial x_1} \right]_{x_1^i = x_i^o} \). Thus the quantity \( \frac{\partial \varphi}{\partial x_1} \) is a new scalar field which defines the rate of change of the field \( \varphi \) in the direction of the \( x_1 \)-axis in \( R \). Similarly, \( \frac{\partial \varphi}{\partial x_2} \) and \( \frac{\partial \varphi}{\partial x_3} \) are the scalar fields describing the rate of change of the scalar \( \varphi \) in the \( x_2 \)- and \( x_3 \)-directions, respectively. The triple \( \frac{\partial \varphi}{\partial x_i} \), \( i = 1, 2, 3 \), may, therefore,
be viewed as components of a vector field in \( R \), called the gradient of \( \phi \). 

The symbol \( \nabla \), pronounced "del," is commonly used to denote the gradient, i.e.,

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x_i} \right) e_i ; \quad i = 1, 2, 3 \tag{4.2a}
\]

The operator \( \nabla \equiv e_i \frac{\partial}{\partial x_i} \) operates on a scalar field \( \phi(x) \) and yields the vector field (4.2a). The notation \( \text{grad} \ \phi \) is also employed to denote \( \nabla \phi \).

The rate of change of the scalar \( \phi \) in any given direction is now defined by the projection of \( \nabla \phi \) into that direction. Let \( \mu \) be a unit vector with components \( \mu_i \). The gradient of \( \phi \) in the direction of \( \mu \) at point \( x_i^0 \) is

\[
\phi(\mu)(x_i^0) = \left[ \mu \cdot \nabla \phi \right]_{x_i = x_i^0} = \left[ \mu_i \frac{\partial \phi}{\partial x_i} \right]_{x_i = x_i^0}
\]

If \( \mu \) is tangent to the level surface \( \phi = \phi^0 \), then \( \phi(\mu)(x_i^0) \) is zero. Thus \( \nabla \phi \) is normal to the level surface \( \phi = \phi^0 \). Moreover, \( \phi(\mu)(x_i^0) \) attains its maximum value when \( \mu \) and \( \nabla \phi \) are collinear and possess the same direction. Thus, at each point in \( R \), \( \nabla \phi \) points toward the direction of maximum rate of increase of the scalar \( \phi \), and is in magnitude equal to this rate.

Introducing the following notation for partial differentiation:

\[
\frac{\partial \phi}{\partial x_i} = \phi, i \tag{4.3a}
\]
where a comma followed by subscript letter is to be interpreted as partial 
differentiation with respect to the corresponding coordinate, we may write 
(4.2a) as

$$\nabla \varphi = e_i \varphi, i \quad ; \quad i = 1, 2, 3$$

(4.2b)

Note that the introduction of this notation is purely a matter of convenience.

Some authors use the notation

$$\frac{\partial \varphi}{\partial x_i} \equiv \partial_i \varphi$$

(4.3b)

which has certain advantages. Here we shall use a comma followed by 
a subscript to denote partial differentiation with respect to the respective 
coordinate.

Relations (4.2) suggest that we may view the "del operator,"

$$\vec{\nabla} = e_i \frac{\partial}{\partial x_i}$$

as a vector and form its dot and cross products with 
vector fields which are continuous. Let

$$\vec{\nabla} \cdot \vec{\varphi} = v_i e_i$$

be a vector function of $$\vec{x}$$ in a finite region $$R$$. Assume that the components $$v_i = v_i(x)$$
are continuous functions of their arguments in $$R$$. The dot product of 
$$\vec{\nabla}$$ with $$\vec{\varphi}$$ is

$$\vec{\nabla} \cdot \vec{\varphi} = \left( \frac{\partial v_j}{\partial x_i} e_i \right) \cdot (v_j e_j)$$

$$= \left( \frac{\partial v_j}{\partial x_i} \right) e_i \cdot e_j$$

$$= v_{j, i} \delta_{ij} = v_{i, i}$$

$$= v_{1, 1} + v_{2, 2} + v_{3, 3}$$

(4.4a)

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which is called divergence of \( \mathbf{\nabla} \), and is usually written as

\[
\text{div } \mathbf{\nabla} \equiv v_i, i \equiv \mathbf{\nabla} \cdot \mathbf{\nabla} \quad .
\]

(4.4b)

Similarly, we may consider the cross product of \( \mathbf{\nabla} \) and \( \mathbf{\nabla} \),

\[
\mathbf{\nabla} \times \mathbf{\nabla} = \left( \frac{\partial}{\partial x_1} \mathbf{e}_i \right) \times (v_j \mathbf{e}_j)
\]

\[
= \left( \frac{\partial v_i}{\partial x_1} \right) \mathbf{e}_i \times \mathbf{e}_j
\]

\[
= (v_{j, i}) e_{ijk} e_k
\]

\[
\begin{vmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\
v_1 & v_2 & v_3
\end{vmatrix}
\]

\[
= \mathbf{e}_1 (v_{3,2} - v_{2,3}) + \mathbf{e}_2 (v_{1,3} - v_{3,1}) + \mathbf{e}_3 (v_{2,1} - v_{1,2})
\]

(4.5a)

which is called curl of \( \mathbf{\nabla} \), and is usually denoted by

\[
\text{curl } \mathbf{\nabla} = \mathbf{\nabla} \times \mathbf{\nabla} \quad .
\]

(4.5b)

Note that, according to (4.3b), (4.5a) is written as

\[
\mathbf{\nabla} \times \mathbf{\nabla} = (\partial_i v_j) e_{ijk} e_k
\]

(4.5c)

which possesses the symmetry of usual cross product of vectors (compare with Eq. (2.12)).
Forming the dot product of $\mathbf{\nabla}$ with itself, we obtain

$$\mathbf{\nabla} \cdot \mathbf{\nabla} = \left( \frac{\partial^2}{\partial x_i \partial x_j} \right) \mathbf{e}_i \cdot \mathbf{e}_j$$

$$= \frac{\partial^2}{\partial x_i \partial x_j} \delta_{ij} = \frac{\partial^2}{\partial x_1 \partial x_1}$$

$$= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$= \nabla^2 = \Delta \quad (4.6)$$

which is the usual Laplace's operator. As an exercise, the reader may verify the following identities:

$$\text{div grad } \varphi = \nabla^2 \varphi$$

$$\text{div curl } \mathbf{\nabla} = 0$$

$$\text{curl curl } \mathbf{\nabla} = \text{grad div } \mathbf{\nabla} - \nabla^2 \mathbf{\nabla}$$

where $\varphi$ and $\mathbf{\nabla}$ are twice differentiable scalar and vector fields, respectively.

Since at each point in $\mathbb{R}$ the quantity $\mathbf{\nabla} \varphi$ defines a vector, its components $\varphi_i$ must transform according to the law (3.9). From the chain rule of differentiation, we have

$$\frac{\partial \varphi}{\partial x_i} = \frac{\partial \varphi}{\partial x'_j} \frac{\partial x'_j}{\partial x_i} = \frac{\partial \varphi}{\partial x'_i} c_{ji}$$

where $\partial x'_j/\partial x_i = c_{ji}$ by Eq. (3.4).
1.5 **Tensors**

In Section 1, a scalar quantity was associated with a real number, while a vector quantity was represented by an oriented line segment. There are other physical and geometrical quantities such as stress and strain fields in a deformed continuum, which cannot be represented by real numbers or be vectors, that is, more is needed for their mathematical description. Quantities of this kind are called **tensors**.

Let \( \sim u \) and \( \sim v \) be two independent vectors, and consider a quantity \( \sim w \) defined by

\[
\sim w = \sim u \sim v
\]  

(5.1a)

Note that \( \sim w \) is neither the dot product, nor the cross product of \( \sim u \) and \( \sim v \). It is simply defined by a pair of vectors \( \sim u \) and \( \sim v \) that are ordered as in (5.1a). For example, \( \sim u \) may be the del operator \( \sim u = e_i \frac{\partial}{\partial x_i} \), in which case \( \sim w \) is the gradient of the vector field \( \sim v = e_j \sim v_j(x) \), i.e.,

\[
\sim w = \sim v \sim v = (v_{i,j}) e_j e_i
\]

Note that (5.1a) defines a definite order for \( \sim u \) and \( \sim v \). Thus, in general, \( \sim u \sim v \neq \sim v \sim u \). In the present example, \( \sim v \sim v \) is not, as yet, defined.

Referring to a rectangular Cartesian coordinate system \( x_i \), we let \( u_i \) and \( v_i \), \( i = 1, 2, 3 \), denote the respective components of \( \sim u \) and \( \sim v \). Relation (5.1a) may then be expressed as

\[
\sim w = (u_i e_i)(v_j e_j)
\]

\[
= (u_i v_j) e_i e_j , \quad i,j = 1, 2, 3 , \quad (5.1b)
\]

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where, as usual, the summation convention is implied. If \( \mathbf{\nabla} \) is to represent a physical quantity, it should be invariant under all coordinate transformations; in particular, under the coordinate transformation (3.5). We thus must have

\[
\mathbf{\nabla} = (u_i \ v_j) \ \mathbf{e}_i \ \mathbf{e}_j = (u'_i \ v'_j) \ \mathbf{e}'_i \ \mathbf{e}'_j ,
\]

where the primed quantities refer to the primed coordinate system. Taking the dot products of (5.2) with \( \mathbf{e}'_k \) and with \( \mathbf{e}'_\ell \), consecutively, we obtain

\[
u'_k \ v'_\ell = c_{ki} \ c_{\ell j} \ u_i \ v_j , \quad i, j, k, \ell = 1, 2, 3,
\]

which defines the law of transformation of the components of \( \mathbf{\nabla} \) under the coordinate transformation (3.5). Similarly, dotting both sides of (5.2) with \( \mathbf{e}_k \) and \( \mathbf{e}_\ell \), consecutively, we arrive at

\[
u_k \ v_\ell = c_{ik} \ c_{j\ell} \ u'_i \ v'_j .
\]

Hence, for \( \mathbf{\nabla} \) to represent a physical quantity which exists independently of a particular rectangular Cartesian coordinate system that is used for the description of its components, these components must transform according to (5.3) under the coordinate transformation (3.5). For \( i, j = 1, 2, 3 \), the nine scalar quantities

\[
(u_1 \ v_1) \quad (u_1 \ v_2) \quad (u_1 \ v_3) \\
(u_2 \ v_1) \quad (u_2 \ v_2) \quad (u_2 \ v_3) \\
(u_3 \ v_1) \quad (u_3 \ v_2) \quad (u_3 \ v_3)
\]

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define the components of \( \mathbf{w} \) in the \( x_i \) coordinate system. Introducing the following notation

\[
\mathbf{w} = w_{ij} \mathbf{e}_i \mathbf{e}_j
\]

where

\[
w_{ij} = u_i v_j,
\]

we may speak of \( \mathbf{w} \) as a second order tensor with components \( w_{ij} \) in the \( x_i \) coordinate system. Thus any physical quantity which can be described in a given rectangular Cartesian coordinate system \( x_i \) by its nine scalar components, say \( w_{ij} \), which transform according to the law.

\[
w'_{ij} = c_{ik} c_{j\ell} w_{k\ell}
\]

(5.3c)

under the coordinate transformation (3.5), is called a second order tensor.

From this terminology, it is clear why vectors are called tensors of first order. Moreover, a slight amount of imagination would reveal that scalars do really deserve the zero\(^{th}\) order of tensor-ship; they are called tensors of order zero.

By dotting both sides of (5.1c) with \( \mathbf{e}_k \) and with \( \mathbf{e}_\ell \), we obtain

\[
w_{\ell k} = \mathbf{e}_\ell \cdot \mathbf{w} \cdot \mathbf{e}_k
\]

(5.4)

which formally defines the components of the second order tensor \( \mathbf{w} \) in the rectangular Cartesian coordinate system \( x_i \).

With the above preliminaries, we shall now generalize the definitions of tensors of order zero (scalars) and one (vectors) to tensors.
of higher order. A tensor of order 2 is a geometrical or physical quantity which may be specified by its $3^2$ components with respect to a given coordinate system. Let $\mathcal{J}$ be a second order tensor with components $T_{ij}$ and $T'_{ij}$ in the unprimed and primed coordinate systems, respectively, i.e., let

$$\mathcal{J} = T_{ij} \hat{e}_i \hat{e}_j = T'_{ij} \hat{e}'_i \hat{e}'_j \quad \ldots \quad (5.5)$$

Taking the dot product of (5.5) first with $\hat{e}_k$ and then with $\hat{e}_k'$, we obtain

$$T_{k\ell} = c_{ik} c_{j\ell} T'_{ij} \quad \ldots \quad (5.6)$$

Similarly, the dot product of (5.5) with $\hat{e}'_k$ and $\hat{e}'_k$ yields

$$T'_{k\ell} = c_{ki} c_{\ell j} T_{ij} \quad \ldots \quad (5.7)$$

Equations (5.6) and (5.7) are now used to define a tensor of second order as follows: with reference to the coordinate system $x_i$, the $3^2$ components $T_{ij}$ define a second order tensor if these components transform according to (5.7) under the coordinate transformation (3.5). Similarly, a tensor of order 3 is specified by $3^3$ components $T_{ijk}$ that transform according to

$$T'_{ijk} = c_{im} c_{jn} c_{kp} T_{mnp} \quad \ldots \quad (5.8)$$

In general, a tensor of order $n$ may be expressed as

$$\mathcal{J} = T_{i_1 i_2 \ldots i_n} \hat{e}_{i_1} \hat{e}_{i_2} \ldots \hat{e}_{i_n}$$

$$= T'_{i_1 i_2 \ldots i_n} \hat{e}'_{i_1} \hat{e}'_{i_2} \ldots \hat{e}'_{i_n} \quad , \quad i_1, i_2, \ldots, i_n = 1, 2, 3,\ldots$$

$$\ldots \ldots \quad (5.9)$$

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where the $3^n$ components $T_{j_1 j_2 \ldots j_n}$ in the unprimed coordinates are linked with $T'_{i_1 i_2 \ldots i_n}$ in the primed coordinates by the following transformation:

$$T'_{i_1 i_2 \ldots i_n} = c_{i_1 j_1} c_{i_2 j_2} \ldots c_{i_n j_n} T_{j_1 j_2 \ldots j_n} ;$$

$$i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n = 1, 2, 3 \quad (5.10)$$

Let $\mathbf{v} = v_i \mathbf{e}_i$ be a vector and $\mathcal{J} = T_{ij} \mathbf{e}_i \mathbf{e}_j$ a second order tensor. The dot product of $\mathbf{v}$ and $\mathcal{J}$ is defined as

$$\mathbf{v} \cdot \mathcal{J} = v_i (\mathbf{e}_i \cdot \mathbf{e}_j) e_k T_{jk}$$

$$= (v_i T_{jk}) (\mathbf{e}_i \cdot \mathbf{e}_j) e_k$$

$$= (v_i T_{jk}) \delta_{ij} e_k = v_j T_{jk} e_k \quad (5.11)$$

(continued on next page)
which is a vector. The dot product of \( \mathbf{T} \) with \( \mathbf{v} \), on the other hand, is defined as

\[
\mathbf{T} \cdot \mathbf{v} = T_{jk} \mathbf{e}_j \cdot (\mathbf{e}_k \cdot \mathbf{e}_i) \mathbf{v}_i = T_{jk} \mathbf{v}_i \delta_{ki} \mathbf{e}_j = T_{jk} \mathbf{v}_k \mathbf{e}_j
\]

which also is a vector. Note that (5.11) and (5.12) represent different vectors unless \( \mathbf{T} \) is a symmetric tensor, that is, unless \( T_{ij} = T_{ji} \mathbf{e}_i \mathbf{e}_j \)
is equal to its transpose \( \mathbf{T}^T = T_{ji} \mathbf{e}_i \mathbf{e}_j \). Since \( \mathbf{T} \cdot \mathbf{v} \) in (5.12) is a vector, say \( \mathbf{u} \), we may write

\[
\mathbf{u}_i = T_{ij} \mathbf{v}_j \quad ; \quad i, j = 1, 2, 3
\]

This equation expresses a linear transformation in the three-dimensional Euclidean space; a vector \( \mathbf{v} \) in this space is transformed into another vector \( \mathbf{u} \) in the same space. Such a transformation is called orthogonal if \( \mathbf{T} \) is an orthogonal tensor, that is, if

\[
T_{ij} T_{kj} = \delta_{ik}
\]

The tensor \( \mathbf{T} = c_{ij} \mathbf{e}_i \mathbf{e}_j \) which satisfies condition (3.7) is an orthogonal tensor. In this context, Eq. (3.10) may be interpreted as rotation of a vector \( \mathbf{A} \) with components \( A_i \) into another vector \( \mathbf{A}' \) with components \( A'_j \) when these vectors are referred to one and the same coordinate system \( x_i \).

An orthogonal tensor that transforms a vector \( \mathbf{A} \) into itself is called the unit tensor. From the definition of the Kronecker delta, we have \( A_i = \delta_{ij} A_j \) and, therefore,

\[
\mathbf{T} = \delta_{ij} \mathbf{e}_i \mathbf{e}_j = \mathbf{e}_i \mathbf{e}_i
\]

is the unit tensor.
The tensor product of a second order tensor \( \mathbf{J} \) and a vector \( \mathbf{v} \) is a tensor of order 3, defined as \( \mathbf{J} \mathbf{v} = T_{ij} v_k e_i \otimes e_j \otimes e_k \). Setting \( j = k \) in the expression \( T_{ij} v_k \), we obtain the dot product of \( \mathbf{J} \) and \( \mathbf{v} \), namely \( \mathbf{u} \). This is called **contraction**. In general, making two letter indices of the components of a tensor of order \( p \) identical results in the components of a tensor of order \( p-2 \). Contraction of the indices of a second order tensor \( T_{ij} \) results in a scalar which is called the **trace** of \( \mathbf{J} \) and is denoted by \( \text{tr} \mathbf{J} = T_{ii} \).

As was mentioned before, a second order tensor is called **symmetric** if it is equal to its transpose. In this connection, a tensor \( \mathbf{J} \) is called **antisymmetric** if we have \( \mathbf{J} = -\mathbf{J}^T \). In general, any second order tensor \( \mathbf{J} \) can be written as a sum of two parts, a symmetric and an antisymmetric part, i.e.,

\[
T_{ij} = \frac{1}{2} (T_{ij} + T_{ji}) + \frac{1}{2} (T_{ij} - T_{ji})
\]

\[
= T_{(ij)} + T_{[ij]},
\]

where the components of the symmetric and antisymmetric parts of \( \mathbf{J} \) are denoted by \( T_{(ij)} \) and \( T_{[ij]} \), respectively. The **dual vector** of a non-symmetric second order tensor is defined by

\[
\mathbf{\tilde{u}} = t_k \mathbf{e}_k = T_{ij} \mathbf{e}_i \times \mathbf{e}_j
\]

\[
= e_{ijk} T_{ij} \mathbf{e}_k
\]

or

\[
t_k = e_{ijk} T_{ij}
\]  

(5.1)

Note that the dual vector of a symmetric tensor is the zero vector, since, by definition (2.10), \( e_{ijk} \) in (5.18) is antisymmetric with respect to the exchange of \( i \) and \( j \). Solving (5.18) for \( T_{ij} \), we obtain
\[ T_{ij} = \frac{1}{2} \varepsilon_{ijk} t_k \quad . \]  

A tensor of second order may be viewed as a vector in a 9-dimensional Euclidean space. Let \( \mathcal{J} = T_{ij} \varepsilon_i \varepsilon_j \) and \( \mathcal{V} = V_{ij} \varepsilon_i \varepsilon_j \) be tensors of second order that are connected through a tensor of fourth order \( \mathfrak{A} = B_{ijkl} \varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l \) as follows:

\[ T_{ij} = B_{ijkl} V_{lk} \quad . \]  

or in the invariant notation

\[ \mathcal{J} = \mathfrak{A} : \mathcal{V} \quad , \]  

where the double dot designates two contractions. Equation (5.20) defines a linear transformation of a vector in 9-dimensional Euclidean space to another vector in the same space.

Since we have viewed a tensor as an invariant quantity that exists independently of any coordinate system which may be used to represent its components, the generalization of our results to more general coordinate systems would only require a proper change of the base vectors. We shall consider this later.

1.6 Matrix and Determinant

Referred to a rectangular Cartesian coordinate system, the components \( T_{ij} ; i, j = 1, 2, 3 \), of a second order tensor \( \mathcal{J} \) may be represented collectively in a form called matrix as follows:
\[ \tilde{T} = [T_{ij}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}. \]  \hspace{1cm} (6.1)

Note that the first subscript letter in the element \( T_{ij} \) of the matrix \( \tilde{T} = [T_{ij}] \) denotes the row-location, and the second subscript letter the column-location of this element in the single entity \( \tilde{T} \). Note also that a definite coordinate system is implied when matrices are used to represent tensors. We use the notation \( \tilde{T} \) to denote the matrix representation of the tensor \( \tilde{\mathcal{T}} \) with respect to the rectangular Cartesian coordinate system \( x_i \).

A vector \( \mathbf{v} = v_1 \mathbf{e}_1 \) can be represented by a column matrix:

\[ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \]

whose transpose, \( \mathbf{v}^T \), is a row vector

\[ \mathbf{v}^T = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \{v_1, v_2, v_3\}. \]  \hspace{1cm} (6.2a)

If \( \tilde{A} = [A_{ij}] \) and \( \tilde{B} = [B_{ij}] \), \( i, j = 1, 2, 3 \), are 3 \( \times \) 3 matrices, then the sum \( \tilde{C} = [C_{ij}] \) and product \( \tilde{D} = [D_{ij}] \) of these matrices are defined by

\[ \tilde{C} = [C_{ij}] = [A_{ij} + B_{ij}], \]

and

\[ \tilde{D} = [D_{ik}] = \tilde{A} \tilde{B} = [A_{ij}][B_{jk}] = \begin{bmatrix} A_{ij} & B_{jk} \\ \end{bmatrix} = [A_{ij} B_{jk}] = [A_{i1} B_{1k} + A_{i2} B_{2k} + A_{i3} B_{3k}], \]  \hspace{1cm} (6.3)
respectively. Note that, for an orthogonal matrix \( \tilde{Q} \), we have
\[
\tilde{Q} \tilde{Q}^T = \tilde{\delta},
\]
where \( \tilde{\delta} \) is the identity matrix; \( \tilde{\delta} = [\delta_{ij}] \).

The transformation (5.13) may now be written in the
matrix notation as
\[
\tilde{T} \tilde{\varphi} = [T_{ij}] \{v_j\} = \{u_i\}. \tag{6.4}
\]
This system of linear equations has a unique solution for \( \tilde{\varphi} = \{v_i\} \)
if and only if the determinant \( \det|T_{ij}| \) of the coefficients of the
unknowns \( v_i, i = 1, 2, 3 \), is nonvanishing. The expansion of the
determinant \( \det|T_{ij}| \) may be written as
\[
\det|T_{ij}| = \frac{1}{6} \epsilon_{ijk} e_{rst} T_{ir} T_{js} T_{kt} \tag{6.5}
\]
which can be checked by direct expansion. Assuming that
\( \det|T_{ij}| \neq 0 \), equations (6.4) can be solved for \( \tilde{\varphi} \), and one obtains
\[
\tilde{\varphi} = \tilde{T}^{-1} \tilde{u} \tag{6.6}
\]
where the inverse matrix \( \tilde{T}^{-1} \) is a 3 x 3 matrix which satisfies
the following equation:
\[
\tilde{T} \tilde{T}^{-1} = \tilde{T}^{-1} \tilde{T} = \tilde{\delta}. \tag{6.7}
\]
Thus the elements of \( \tilde{T}^{-1} \) are the reduced cofactors of the corres-
ponding elements in \( \tilde{T} \). The cofactor \( t_{ij} \) of the element \( T_{ij} \) is
given by
\[
t_{ij} = \frac{1}{2} \epsilon_{ipq} \epsilon_{jrs} T_{pr} T_{qs}, \tag{6.8}
\]
and the reduced cofactor of $T_{ij}$ is obtained by dividing $t_{ij}$ by the $\det|T_{ij}|$. We, therefore, have

$$\tilde{T}^{-1} = \left[ \frac{t_{ij}}{\det|T_{pq}|} \right] = \left[ \frac{1}{\begin{array}{c} e_{jrs} e_{ipq} T_{pr} T_{qs} \\ e_{klm} e_{ghf} T_{kg} T_{lh} T_{mf} \end{array}} \right] \quad (6.9)$$

which explicitly defines the inverse $\tilde{T}^{-1}$ of the matrix $\tilde{T}$.

Using the matrix notation, the coordinate transformation (3.5) may be expressed as

$$\{x'_i\} = [c_{ji}] \{x_i\}$$

or

$$\tilde{x}' = \tilde{c} \tilde{x} \quad (6.10)$$

Let us now superpose another transformation, defined by the orthogonal matrix $\tilde{c}'$, and obtain, from (6.10)

$$\tilde{x}'' = \tilde{c}' \tilde{x}' = \tilde{c}' \tilde{c} \tilde{x} = \tilde{D} \tilde{x} \quad (6.11)$$

where the matrix $\tilde{D}$ is also an orthogonal matrix. Thus two successive rotations of the coordinate system are equivalent to a single rotation defined by $\tilde{D} = \tilde{c}' \tilde{c}$. As an example, consider three successive rotations of the coordinate system that correspond to the Eulerian angles as follows:

1) a rotation through angle $\varphi$ about $x_3$ axis ($\tilde{x} \rightarrow \tilde{x}'$),

2) a rotation through angle $\theta$ about $x'_1$ axis ($\tilde{x}' \rightarrow \tilde{x}''$), and

3) a rotation through angle $\psi$ about $x''_3$ axis ($\tilde{x}'' \rightarrow \tilde{x}'''$).

We can find a single rotation, defined by $\tilde{x} = \tilde{E} \tilde{x}'''$, which is equivalent to
these three successive rotations. Let \( \tilde{x} = \tilde{c} \tilde{x}', \tilde{x}' = \tilde{c}' \tilde{x}'', \) and \( \tilde{x}'' = \tilde{c}'' \tilde{x}'''. \) Then \( \tilde{x} = \tilde{E} \tilde{x}''', \) where \( \tilde{E} = \tilde{c} \tilde{c}' \tilde{c}'' \). But we have

\[
\tilde{c} = \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
\tilde{c}' = \begin{bmatrix}
1 & 0 & 0 \\
0 & \sin \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix},
\]

and \( \tilde{c}'' = \begin{bmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{bmatrix}, \)

from which we obtain

\[
\tilde{E} = \tilde{c} \tilde{c}' \tilde{c}''
\]

\[
= \begin{bmatrix}
\cos \psi \cos \phi - \cos \theta \sin \phi \sin \psi \\
-\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi \\
\sin \phi \cos \psi \cos \phi + \cos \theta \cos \phi \cos \psi \\
\sin \theta \sin \phi \\
\sin \theta \cos \psi \\
\cos \theta
\end{bmatrix}
\]

The students are urged to consult standard texts for further discussions on matrices and determinants of a more general nature.

1.7 **Integral Theorems**

A tensor-valued function \( \mathcal{F}(\tilde{x}) \) of position \( \tilde{x} \) in a finite region \( R \) of the three-dimensional Euclidean space is said to be continuous at a point
\( x^0 \) in \( \mathbb{R} \) if the limit \( \lim_{\alpha \to 0} \mathcal{J}(x^0 + \alpha \tilde{x}) \) exists independently of the particular choice of the vector \( \tilde{x} \) of a finite length, \( \alpha \) being a real variable. Such a tensor-valued function defines a continuous tensor field in \( \mathbb{R} \) if it is defined and continuous at every point in \( \mathbb{R} \). Let \( \mathcal{J}(\tilde{x}) = T_{ij}(\tilde{x}) \varepsilon_i \varepsilon_j \) be a continuously differentiable tensor field of second order. The gradient of \( \mathcal{J} \) is defined as

\[
\nabla \mathcal{J} = \varepsilon_i \frac{\partial}{\partial x_i} (T_{jk} \varepsilon_j \varepsilon_k)
\]

\[
= \frac{\partial T_{jk}}{\partial x_i} \varepsilon_i \varepsilon_j \varepsilon_k
\]

and thus is a third order tensor-valued function in \( \mathbb{R} \). Equation (7.1) may be written as

\[
\nabla \mathcal{J} = T_{jk},_i \varepsilon_i \varepsilon_j \varepsilon_k
\]

For a continuously differentiable, nth order tensor field \( \mathcal{J}(\tilde{x}) \) in \( \mathbb{R} \), the quantity \( \nabla * \mathcal{J}(\tilde{x}) \) is a tensor of order \( n-1 \), \( n \), or \( n+1 \) according to whether the symbol * defines a dot, a cross, or a tensor product, that is, according to whether we have \( \nabla \cdot \mathcal{J}(\tilde{x}) \), \( \nabla \times \mathcal{J}(\tilde{x}) \), or \( \nabla \mathcal{J}(\tilde{x}) \). Let \( \mathbb{R} \) be a convex region, bounded by a regular surface \( S \) which possesses a piecewise continuously turning tangent plane. The following integral theorem, called the Gauss theorem, then holds identically:

\[
\int_{\mathbb{R}} \nabla * \mathcal{J} \, dV = \int_{S} n * \mathcal{J} \, dS
\]
where $dV$ is the elementary volume of $R$, $dS$ is the elementary surface of $S$, and $\mathbf{n}$ is the exterior unit normal to $S$. The symbol $\ast$ in (7.2a) may, of course, be interpreted as a dot, a cross, or a tensor product according to the considered case. The validity of (7.2a) may be proved as follows: The left side of (7.2a) may be written as

\[
\int_{R} \mathbf{n} \ast \mathbf{f} \, dV = \int_{R} \frac{\partial}{\partial x_i} (\mathbf{e}_i \ast \mathbf{f}) \, dV = \]

\[
= \int_{R} \left[ \frac{\partial}{\partial x_1} (\mathbf{e}_1 \ast \mathbf{f}) + \frac{\partial}{\partial x_2} (\mathbf{e}_2 \ast \mathbf{f}) + \frac{\partial}{\partial x_3} (\mathbf{e}_3 \ast \mathbf{f}) \right] \, dx_1 \, dx_2 \, dx_3. \tag{7.3}
\]

Since the region $R$ is assumed to be convex, every straight line parallel to the coordinate axes intersects the surface $S$ in, at most, two points. Consider now the last term in the right side of equation (7.3), and noting that $(\mathbf{n} \cdot \mathbf{e}_3) \, dS^u = dx_1 \, dx_2$ on $S^u$ and $(\mathbf{e}_3 \cdot \mathbf{f}) \, dS^f = -dx_1 \, dx_2$ on $S^f$ (see Fig. 7.1), obtain

(continued on next page)
\[ \int_R \frac{\partial}{\partial x_3} (e_3 \star \mathcal{J}) \, dx_1 \, dx_2 \, dx_3 \]

\[ = \int_{S_{12}} \int_{x_3}^{x_3^u} \frac{\partial}{\partial x_3} (e_3 \star \mathcal{J}) \, dx_3 \]

\[ = \int_{S_1} (e_3 \star \mathcal{J}) (n \cdot e_3) \, dS \]

\[ + \int_{S_2} (e_3 \star \mathcal{J}) (n \cdot e_3) \, dS \]

\[ = \int_S (e_3 \star \mathcal{J}) (n \cdot e_3) \, dS . \]  

(7.4a)
Similarly, for the first and the second integrals in the right side of equation (7.3), we obtain

\[
\int_R \frac{\partial}{\partial x_1} (e_1 \times \mathcal{J}) \, dx_1 \, dx_2 \, dx_3 = \int_S (e_1 \times \mathcal{J}) \, (\mathbf{n} \cdot e_1) \, dS ,
\]  

(7.4a)

and

\[
\int_R \frac{\partial}{\partial x_2} (e_2 \times \mathcal{J}) \, dx_1 \, dx_2 \, dx_3 = \int_S (e_2 \times \mathcal{J}) \, (\mathbf{n} \cdot e_2) \, dS .
\]  

(7.4b)

Noting that \( \mathbf{n} = (\mathbf{n} \cdot e_1) e_1 \), substitution from (7.4) into (7.3) yields

\[
\int_R \mathcal{J} \times \mathcal{J} \, dV = \int_S \left[ (\mathbf{n} \cdot e_1) e_1 \times \mathcal{J} + (\mathbf{n} \cdot e_2) e_2 \times \mathcal{J} + (\mathbf{n} \cdot e_3) e_3 \times \mathcal{J} \right] \, dS
\]

\[
= \int_S \mathbf{n} \times \mathcal{J} \, dS
\]  

(7.2b)

which proves the theorem of Gauss.

It should be noted that, although the Gauss theorem was proven for a convex region and a continuously differentiable tensor field \( \mathcal{J}(\mathbf{x}) \), it is also valid when \( R \) can be decomposed into a finite number of such regions, or if \( R \) can be obtained as the limit of a sum of such parts. Moreover, if \( \mathcal{J}(\mathbf{x}) \) is a piecewise differentiable tensor-valued function in \( R \), the Gauss theorem can be written for each sub-region in which \( \mathcal{J} \) is continuously differentiable. When adding, the surface integral over a common boundary of two such subregions vanishes only if \( \mathcal{J} \) is continuous across this surface, otherwise this contribution must also be accounted for in the final sum.
For a scalar field $\varphi(x)$ and a vector field $\vec{v}(x)$, the Gauss theorem (7.2a) may be written in the following more familiar forms:

\[ \int_{R} \vec{v} \cdot \nabla \varphi \, dV = \int_{S} \varphi \, n \cdot \vec{v} \, dS \]  
(7.5a)

\[ \int_{R} \vec{v} \cdot \vec{v} \, dV = \int_{S} n \cdot \vec{v} \, dS \]  
(7.5b)

\[ \int_{R} \vec{v} \times \vec{v} \, dV = \int_{S} n \times \vec{v} \, dS \]  
(7.5c)

We now consider a tensor-valued function which is defined and continuously differentiable on an orientable, open surface $S$ that is bounded by a closed curve $C$. With $\vec{n}$ denoting the unit normal to $S$, we assign a positive sense to the curve $C$ by means of its unit tangent vector $\vec{t}$ that turns according to a right-handed screw which is pointing in the positive direction of $\vec{n}$. With these preliminaries, the theorem of Stokes can be written as

\[ \int_{S} (\vec{n} \times \vec{v}) \ast \vec{I} \, dS = \oint_{C} \vec{t} \ast \vec{I} \, dC, \]  
(7.6)

where $\vec{n} \times \vec{v} = (n_i \, e_i) \times (e_j \, \frac{\partial}{\partial x_j}) = (\varepsilon_{ijk} \, n_i \, \frac{\partial}{\partial x_j}) \, e_k$; the symbol "*", as before, may stand for a dot, or a cross, or a tensor product; $dC$ is the elementary length of the curve $C$. In (7.6), the line integral is to be taken in the positive direction of $C$. 

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For the proof of the Stokes theorem, one may first express the equation defining the surface $S$ in a parametric form:

$$x_i = f_i(\alpha, \beta), \quad i = 1, 2, 3,$$

(7.7)

where $\alpha$ and $\beta$ are two real variables. Then the quantity $n \, dS$ can be written as

$$n \, dS = \frac{\partial x}{\partial \alpha} \times \frac{\partial x}{\partial \beta} \, d\alpha \, d\beta,$$

(7.8)

where $\mathbf{x}$ is the position vector of a generic point of $S$. Now, substitution of (7.8) into the left side of (7.6) and some formal manipulations eventually yield the expression in the right side of (7.6). The students are urged to carry out the details, or else consult one of the relevant texts mentioned at the end of this chapter.

In closing this section, we obtain from the Gauss theorem (7.5b) three identities which are known as Green's identities.

Let $\mathbf{\nabla} = u \, \mathbf{\nabla} \, w$ in equation (7.5b) and obtain

$$\int_R u \, \nabla^2 \, w \, dV + \int_R \mathbf{\nabla} u \cdot \mathbf{\nabla} \, w \, dV = \int_S u(\mathbf{\nabla} \cdot n) \, dS,$$

(7.9a)

where the scalar $u$ is once continuously differentiable in $R$ and continuous on $S$, and the scalar $w$ is twice continuously differentiable in $R$ and once on $S$. Equation (7.9a) is known as Green's first identity. If the scalar $u$ also possesses the required degree of continuity in $R$ and on $S$, we may exchange $u$ and $w$ in (7.9a) and then subtract the resulting equation from (7.9a) to obtain
\[
\int_R (u \nabla^2 w - w \nabla^2 u) \, dV = \int_S (u \nabla w - w \nabla u) \cdot \mathbf{n} \, dS \tag{7.9b}
\]

which is known as Green's second identity.

We now put \( u = 1 \) and \( \nabla^2 w = 0 \) in \( R \), and from (7.9b) obtain Gauss' integral theorem for any harmonic function \( w \) (a harmonic function is a function whose Laplacian vanishes) in \( R \)

\[
\int_S \nabla w \cdot \mathbf{n} \, dS = 0 . \tag{7.10}
\]

Putting \( u = w \) in (7.9a), we obtain Dirichlet's integral

\[
D[u] = \int_R (\nabla u \cdot \nabla u) \, dR = \int_S u \nabla u \cdot \mathbf{n} \, dS \tag{7.11}
\]

for every regular harmonic \( u \) with continuous first derivatives in \( R + S \).

In the potential theory, a significant role is played by what is known as Green's third identity. Consider a point \( \vec{x} \) in \( R \) and let the variable point in equations (7.9) be denoted by \( \vec{x} \). With the scalar \( w \) given by

\[
w = \frac{1}{r} + \psi \quad ; \quad r^2 = (x_i - \xi_i) (x_i - \xi_i) ,
\]

apply equations (7.9) to a region obtained from \( R \) by deleting a sphere of radius \( \epsilon \) about point \( \vec{x} \), and obtain, as \( \epsilon \to 0 \),

\[
\int_R (\nabla u \cdot \nabla w) \, dV + \int_R u \nabla^2 w \, dV = p u + \int_S u \nabla w \cdot \mathbf{n} \, dS , \tag{7.12a}
\]

\[
\int_R (u \nabla^2 \psi - w \nabla^2 u) \, dV = p u + \int_S (u \nabla w - w \nabla u) \cdot \mathbf{n} \, dS , \tag{7.12b}
\]
where
\[ p = \begin{cases} 
4\pi & \text{for } \mathcal{X} \text{ in } R \\
2\pi & \text{for } \mathcal{X} \text{ on } S; \text{ if } S \text{ has a continuously turning tangent plane at } \mathcal{X} \\
0 & \text{for } \mathcal{X} \text{ outside of } R.
\end{cases} \]

Setting \( \psi = 0 \), we arrive at Green's third identity;
\[ 4\pi u = -\int_R \frac{\nabla^2 u}{r} \, dV + \int_S \left( \frac{1}{r} \nabla u - \frac{1}{r^2} \vec{n} \right) \cdot \vec{n} \, dS \]  
(7.13a)

which, for an unbounded region \( R \) with \( u \) regular at infinity, reduces to
\[ 4\pi u = -\int_R \frac{\nabla^2 u}{r} \, dV. \]  
(7.13b)

Let us now choose the function \( \psi \) in (7.12b) in such a manner as to satisfy the following conditions:

a) \( \nabla^2 \psi = 0 \) in \( R \)
b) \( \psi = -\frac{1}{r} \) on \( S \)
c) \( \psi \) is regular everywhere in \( R + S \).

With this selection, equation (7.12b) reduces to
\[ -\int_R w \nabla^2 u \, dV = pu + \int_S u \nabla w \cdot \vec{n} \, dS. \]

It is customary to use \( G(\mathcal{X}; \mathcal{X}) \) to denote the function \( \frac{1}{r} + \psi(\mathcal{X}) \), which is called Green's function, and write the above equation as
\[ pu(\mathcal{X}) = -\int_R G(\mathcal{X}; \mathcal{X}) \nabla^2 u(\mathcal{X}) \, dV - \int_S u(\mathcal{X}) \nabla G(\mathcal{X}; \mathcal{X}) \cdot \vec{n} \, dS. \]  
(7.14)

This equation is of great importance in the potential theory. It defines the value of a regular function \( u \) at a generic point in \( R \) in terms of the Laplacian of the function and its boundary values.
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PROBLEMS I

1. Four vectors \( \mathbf{A} \), \( \mathbf{B} \), \( \mathbf{C} \), and \( \mathbf{D} \) are said to be linearly dependent if there exist three scalars \( a \), \( b \), and \( c \), not all equal to zero, such that

\[
\mathbf{D} = a \mathbf{A} + b \mathbf{B} + c \mathbf{C}
\]

a) Show that, in a three-dimensional Euclidean space, any four vectors are linearly dependent.

b) Let \( \mathbf{I}_1 \), \( \mathbf{I}_2 \), and \( \mathbf{I}_3 \) be three linearly independent vectors. Show that these vectors cannot be co-planar, and that any vector \( \mathbf{F} \), in this space, can be expressed as a linear combination of \( \mathbf{I}_1 \), \( \mathbf{I}_2 \), and \( \mathbf{I}_3 \) as follows:

\[
\mathbf{F} = a_1 \mathbf{I}_1 + a_2 \mathbf{I}_2 + a_3 \mathbf{I}_3
\]

where \( a_1 \), \( a_2 \), and \( a_3 \) are scalars. Find these coefficients in terms of \( \mathbf{I}_1 \), \( \mathbf{I}_2 \), \( \mathbf{I}_3 \), and \( \mathbf{F} \).

2. Using the index notation and definitions (2.5), (2.9), and the identities (2.13) establish the validity of the following relations:

\[
\mathbf{T} \times (\mathbf{U} \times \mathbf{V}) = (\mathbf{V} \cdot \mathbf{T}) \mathbf{U} - (\mathbf{T} \cdot \mathbf{U}) \mathbf{V}
\]

\[
(\mathbf{S} \times \mathbf{T}) \cdot (\mathbf{U} \times \mathbf{V}) = (\mathbf{S} \cdot \mathbf{U})(\mathbf{T} \cdot \mathbf{V}) - (\mathbf{S} \cdot \mathbf{V})(\mathbf{T} \cdot \mathbf{U})
\]

\[
(\mathbf{S} \times \mathbf{T}) \times (\mathbf{U} \times \mathbf{V}) = (\mathbf{V} \cdot \mathbf{S} \times \mathbf{T}) \mathbf{U} - (\mathbf{U} \cdot \mathbf{S} \times \mathbf{T}) \mathbf{V}
\]

\[
(\mathbf{V} \times \mathbf{T}) \cdot (\mathbf{T} \times \mathbf{U}) \times (\mathbf{U} \times \mathbf{V}) = (\mathbf{T} \cdot \mathbf{U} \times \mathbf{V})^2
\]

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3. **Quotient rule.** Let a quantity \( \mathbf{T} \) be defined by 3^2 real numbers \( T_{ij}, \; i, j = 1, 2, 3, \) in a right-handed rectangular Cartesian coordinate system, and assume that \( \mathbf{V} \) and \( \mathbf{R} \) are arbitrary tensors of first and second order, respectively. Show that \( F_{ij} \) transform as the components of a second order tensor under the coordinate transformation (3.5) if

a) \( F_{ij} R_{ij} = \phi \), where \( \phi \) is a scalar, or

b) \( F_{ij} V_{j} = U_{i} \), where \( U_{i} \) are components of a vector.

4. Consider the following transformation:

\[
{u}_i = K_{ijk} {T}_{jk}; \; i, j, k = 1, 2, 3,
\]

which maps a vector \( \mathbf{T} \) in a 9-dimensional Euclidean space into a vector \( \mathbf{u} \) in a three-dimensional Euclidean space. Show that:

a) If this transformation is valid only for symmetric tensors

\( \mathbf{T} = \mathbf{T}^T \), then \( (K_{ijk} + K_{ikj}) \) are components of a third order tensor;

b) If this transformation is valid only for antisymmetric tensors

\( T_{ij} = - T_{ji} \), then \( (K_{ijk} - K_{ikj}) \) are components of a third order tensor.

c) What happens if this transformation is valid for all second order tensors?

5. a) Show that the determinant of a 3 x 3 matrix \( [a_{ij}] \), \( i, j = 1, 2, 3 \), is given by

\[
\det |a_{ij}| = \epsilon_{ijk} a_{1i} a_{2j} a_{3k} = \epsilon_{ijk} a_{i1} a_{j2} a_{k3}
\]

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and
\[ e_{rst} \det |a_{ij}| = e_{ijk} a_{ri} a_{sj} a_{tk} \]

b) If the cofactor of the element \( a_{ij} \) is denoted by \( A_{ij} \), prove that
\[ \delta_{rs} \det |a_{ij}| = A_{kr} a_{ks} \]
c) If \( a_{ij} x_j = b_i \), verify Cramer's rule
\[ x_m = \frac{b_i A_{im}}{\det |a_{ln}|} \]
where it is assumed that \( \det |a_{ln}| \neq 0 \).

6. Let \( \mathbf{K} = [K_{ij}] \), \( i, j = 1, 2, \ldots, n \), be a symmetric \( n \times n \) matrix with real elements. The quadratic form
\[ \sum_{i,j=1}^{n} K_{ij} x_i x_j = k \]
is said to be positive-definite if \( k \geq 0 \), where the equality holds only if \( x_i = 0 \), \( i = 1, 2, \ldots, n \). A matrix of this kind is called positive-definite. The eigenvalues of \( \mathbf{K} \) are the roots of the equation
\[ \det |K_{ij} - x \delta_{ij}| = 0 \]
\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]
a) Show that \( \mathbf{K} \) has \( n \) positive, real eigenvalues.
b) Show that by an orthogonal transformation, \( \mathbf{K} \) can always be written in a diagonal form. Find an orthogonal matrix \( \mathbf{Q} \) which reduces \( \mathbf{K} \) to this form.
7. Verify the following identities:

\[ \nabla \cdot \mathbf{x} = 3 \quad , \quad (\nabla \cdot \mathbf{Y}) \mathbf{x} = \mathbf{Y} \quad . \]

\[ \nabla \cdot (\mathbf{Y} \times \mathbf{F}) = 0 \quad , \quad \nabla \cdot (\varphi \mathbf{Y}) = \varphi (\nabla \cdot \mathbf{Y}) + \mathbf{Y} \cdot \nabla \varphi \quad , \]

\[ \nabla \times (\mathbf{Y} \times \mathbf{F}) = \nabla (\mathbf{Y} \cdot \mathbf{F}) - \nabla^2 \mathbf{Y} \quad . \]

where \( \varphi \) and \( \mathbf{Y} \) are each once continuously differentiable scalar-valued and vector-valued functions, and \( \mathbf{F} \) is a twice continuously differentiable vector-valued function of the position \( \mathbf{x} \) in the considered region.

8. A vector field \( \mathbf{Y} = y_i (\mathbf{x}) \mathbf{e}_i \) is called \textbf{irrotational} if

\[ \nabla \times \mathbf{Y} = 0 \]

everywhere in the regular domain \( \mathbf{R} \) of its definition. Let \( \mathbf{Y} \) be irrotational, and prove the following:

a) \[ \oint_{\mathbf{C}} \mathbf{Y} \cdot \mathbf{t} \, ds = 0 \quad , \]

where \( \mathbf{t} \) is the unit tangent vector and \( ds \) the element of length along the closed curve \( \mathbf{C} \) which lies entirely in \( \mathbf{R} \);

b) The integral

\[ I = \int_{\mathbf{x}^0}^{\mathbf{x}'} \mathbf{Y} \cdot \mathbf{t} \, ds \quad , \]

for \( \mathbf{x}^0 \) and \( \mathbf{x}' \) in \( \mathbf{R} \), is independent of the particular path that lies in \( \mathbf{R} \) and connects these points;
c) The vector field \( \mathbf{v} \) is the gradient of a scalar field \( \phi(x) \), that is,

\[
\mathbf{v}(x) = \nabla \phi(x)
\]

9. Prove Stokes' theorem (7.6).

10. Consider a vector field whose divergence and curl are prescribed in a regular region \( R \) with boundary \( S \). Prove that if the normal component of this vector field is given on \( S \), then this vector field is uniquely defined in \( R + S \).

11. Show that a vector field whose curl and divergence vanish identically in the region of its definition is the gradient of a scalar field \( \phi \) which satisfies Laplace's equation \( \nabla^2 \phi = 0 \) in this region.

12. The instantaneous velocity field of a rigid body can always be expressed as

\[
\mathbf{v} = \mathbf{v}^0 + \mathbf{w} \times \mathbf{x}
\]

where \( \mathbf{v} \) is the velocity of the point \( \mathbf{x} \), \( \mathbf{v}^0 \) and \( \mathbf{w} \) are two vector-valued functions of time. Show that \( \nabla \cdot \mathbf{v} = 0 \) and \( \nabla \times \mathbf{v} = 2 \mathbf{w} \).
Figure 2.1
CHAPTER II
DEFORMATION

Continuum mechanics is concerned with the study and description of the responses of certain mathematical models which are to represent real bodies that are composed of matter and are subjected to external forces. In these mathematical models, the molecular structure of matter is disregarded, and it is assumed that matter is continuously distributed throughout any region of the Euclidean three-dimensional space which is occupied by the body.

The deformations and motions of bodies under forces are assumed to be governed by certain laws which are common to all bodies. The differences and peculiarities of bodies are then recognized through their constitutive equations which are used to characterize the constitution of the material that comprises the bodies. This chapter is concerned with the study and description of deformations of continuous media.
2.1 Bodies and Their Configurations

The concept of body may be described as follows: A body is a collection of certain elements, called particles or material points, that can occupy place in three-dimensional Euclidean space. The body, or any part of it, is endowed with a measure called mass. Moreover, any elementary volume, no matter how small, of the space occupied by the body contains particles of that body; the material points are continuously distributed over the space that the body happens to be occupying.

This definition clearly distinguishes between the particles which comprise the body, and their positions in the Euclidean three-dimensional space. The body is a smooth manifold of material points which may be placed into one-to-one correspondence with the points in the Euclidean space. The collection of all the spatial points that are occupied by the material points at a given instant of time defines the configuration of the body at that time. As the body deforms and moves under the action of external forces, its configuration changes with time. The study of the deformation of a body \( B \) requires a comparison between two distinct configurations of that body, the reference configuration \( C_0 \) and the current configuration \( C \).

In the reference state, the material points \( X \) of the body \( B \) occupy the spatial points \( \mathcal{X} \), whose collection forms the reference configuration \( C_0 \) of the body. Under the action of the external forces, the material points \( X \) of the body \( B \) move and occupy new spatial positions \( \mathcal{X} \), whose collection establishes a new configuration \( C \), called the current configuration of the body. The deformation of the body from \( C_0 \) to \( C \) is thus characterized by the following one-to-one mapping

\[
\mathcal{X} = \mathcal{X}(\mathcal{X}) \quad (1.1a)
\]

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which carries a material point $\mathbf{x}$ of the body $B$ from its position $\mathbf{x}$ in configuration $C_0$ into the position $\mathbf{x}$ in the configuration $C$. Note carefully the meaning of various symbols used in (1.1a). The symbol $\mathbf{x}$ on the right side is a vector-valued function with the domain $C_0$ and range $C$. Given a particle $\mathbf{x}$ at the point $\mathbf{x}^0$ in $C_0$, we can find its position vector $\mathbf{x}$ in $C$, (after the deformation), by evaluating the right side of (1.1a) for $\mathbf{x} = \mathbf{x}^0$ ; $\mathbf{x}^0 = \mathbf{x}(\mathbf{x})\big|_{\mathbf{x} = \mathbf{x}^0}$.

Consider a generic material point $\mathbf{x}^0$ with its spatial positions $\mathbf{x}^0$ in $C_0$ and $\mathbf{x}$ in $C$. Let $\Delta V^0$ be an elementary volume which contains the point $\mathbf{x}^0$, and denote by $\Delta \mathbf{x}$ the set of material points which are contained in $\Delta V^0$ when $B$ is in the configuration $C_0$. The mass-density of the body $B$ at the point $\mathbf{x}$ in the configuration $C_0$ is now defined by

$$\rho^0 \bigg|_{\mathbf{x} = \mathbf{x}^0} = \lim_{\Delta V^0 \to 0} \frac{\Delta M}{\Delta V^0}, \quad (1.2a)$$

where $\Delta M$ is the mass of particles $\Delta \mathbf{x}$. Since the mapping (1.1a) transforms $\Delta V^0$ into $\Delta v$, the mass-density at the image point $\mathbf{x}$ of $\mathbf{x}^0$ is
\[ \rho (\mathbf{x}) = \lim_{\Delta v \to 0} \frac{\Delta M}{\Delta v}. \]  

(1.2b)

Thus, although the mass of the body or any portion of the body, stays unchanged as the body deforms from configuration \( C_0 \) into configuration \( C \), the mass-density distribution varies from one configuration to another. The mass is a measure imputed \textit{a priori} to the body, while the mass-density correlates the body with its spatial configurations.

Using a fixed rectangular Cartesian coordinate system, equation (1.1a) may be written as

\[ x_i = x_i(X_1, X_2, X_3); \ i = 1, 2, 3. \]  

(1.1b)

The triple scalar-valued functions \( x_i \) are assumed to be single-valued and as many times differentiable as required. The \textit{inverse} to the deformation (1.1) can be written as

\[ \mathbf{X} = \mathbf{X}(\mathbf{x}) \]  

(1.3a)

or

\[ X_\alpha = X_\alpha(x_1, x_2, x_3), \ \alpha = 1, 2, 3. \]  

(1.3b)

which is also assumed to be single-valued and as many times differentiable as required. The existence of a single-valued inverse of class \( * \ C^1 \) is actually guaranteed if the deformation (1.1) is once continuously differentiable, and the following determinant

\[ * \text{ A function is said to be of class } \ C^n \text{ in a given region if it is continuous together with its } n \text{th derivatives in that region.} \]
\[ J = \det \left| \frac{\partial x_i}{\partial x_\alpha} \right| = \frac{1}{6} c_{ijk} c_{\alpha \beta \gamma} x_i^\alpha, x_j^\beta, x_k^\gamma \]

\[ i, j, k, \alpha, \beta, \gamma = 1, 2, 3, \]
called the Jacobian, does not vanish at any point of the configuration \( C_0 \) (why?). We shall require that the Jacobian determinant \( J = \det \left| x_i^\alpha \right| \) be neither zero nor infinity; without loss of generality \( J \) will be assumed positive.

The requirements of the single-valuedness and the smoothness imposed on the deformation (1.1) and its inverse (1.3) guarantee that no part of the matter is destroyed by the deformation; permanence of matter. Moreover, two distinct particles, which have distinct positions in one configuration, continue to have distinct places in the other configuration; the impenetrability of matter is also ensured.

In studying the deformation of a body, one may use the positions \( \bar{X} \) of the particles \( X \) in the reference configuration \( C_0 \) as independent variables, and express the positions \( \bar{x} \) of the particles in the current state \( C \) in terms of \( \bar{X} \). Such a formulation of the deformation is known as the Lagrangian formulation. On the other hand, one may reverse this procedure and view \( \bar{x} \) as independent variables, and obtain what is known as the Eulerian formulation. These imputations, while they cannot be justified historically, are commonly employed. In these notes, we shall refer to the configuration \( C_0 \) as the reference, or the initial configuration, and to \( C \) as the current, or the final configuration of the body \( B \). When \( C_0 \) constitutes the undeformed virgin state of the body, we shall also refer to it as the undeformed configuration of \( B \). Note that, in general, the reference configuration \( C_0 \) need not coincide with the undeformed configuration.
2.2 Deformation Gradients

Consider a generic material point $X$ of the body $B$, and let $dX$ be a material line element emanating from $X$. Let $d\hat{X}$ and $dx$ denote the respective spatial line elements occupied by the set of particles $dX$ in the configurations $C_0$ and $C$. Referring to a fixed rectangular Cartesian coordinate system with the base vectors $\hat{e}_1$, we may write

$$d\hat{X} = dX_\alpha \hat{e}_\alpha,$$

$$dx = dx_i \hat{e}_i ; \alpha, i = 1, 2, 3. \quad (2.1)$$

Using the mapping (1.1) and its inverse (1.3), we have

$$dx_i = \frac{\partial x_i}{\partial X_\alpha} dX_\alpha = x_i,\alpha dX_\alpha \quad (2.2a)$$

$$dX_\alpha = \frac{\partial X_\alpha}{\partial x_i} dx_i = X_\alpha, i dx_i. \quad (2.2b)$$

The linear transformations (2.2) characterizes the manner in which a material line element is carried from the initial state $C_0$ into the final state $C$ by the deformation (1.1) and vice-versa. The tensor-valued function

$$\mathfrak{F}(X) = \mathfrak{F}\hat{X}(X) = \frac{\partial}{\partial X_\alpha} \mathfrak{e}_\alpha x_i(X) \mathfrak{e}_i$$

$$= x_i,\alpha \hat{e}_\alpha \hat{e}_i, \quad (2.3)$$

defined on the initial configuration $C_0$, is a fundamental quantity called deformation gradient.

From the chain rule of differentiation, we have

$$\frac{\partial x_i}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial x_j} = x_i,\alpha X_\alpha, j = \delta_{ij}, \quad (2.4a)$$

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and

$$\frac{\partial X_\alpha}{\partial x_i} \frac{\partial x_i}{\partial X_\beta} = X_\alpha, i x_i, \beta = \delta_{\alpha \beta}. \quad (2.4b)$$

Since the Jacobian determinant

$$J = \det | x_i, \alpha | - \frac{1}{6} e_{ijk} \epsilon_{\alpha \beta \gamma} x_i, \alpha x_j, \beta x_k, \gamma$$

is finite, (2.4a) may be solved for $X_\alpha, i$, and one obtains

$$X_\alpha, i = \frac{1}{J} \frac{1}{2} e_{ijk} \epsilon_{\alpha \beta \gamma} x_j, \beta x_k, \gamma \quad (2.5a)$$

Moreover, by forming the determinant of both sides of (2.4a), we have

$$\det | x_i, \alpha X_\alpha, j | = \det | x_i, \alpha | \det | X_\alpha, j | = J^2 = 1. \quad (2.6)$$

Thus the inverse Jacobian determinant $J = \det | X_\alpha, j | = \frac{1}{J}$ is also finite, and we may solve (2.4b) for $x_i, \beta$ and obtain

$$x_i, \alpha = \frac{1}{2} \epsilon_{\alpha \beta \gamma} e_{ijk} X_\beta, j x_j, \gamma \quad (2.5b)$$

Let us denote by $\tilde{F}$ the $3 \times 3$ matrix of deformation gradient $F$:

$$\tilde{F} = [x_i, \alpha]. \quad \text{Since} \quad [x_i, \alpha][X_\alpha, j] = \delta, \quad \text{the inverse matrix of} \quad \tilde{F},$$

therefore, is the matrix defined by $\tilde{F}^{-1} = [X_\alpha, i]$. Consider now two successive deformations of the body, one from the reference state $C_0$ to $C'$ with the deformation gradient matrix $\tilde{F}' = \frac{\partial x_i'}{\partial x_i}$, and the other from $C'$ to $C''$ with the deformation gradient matrix $\tilde{F}'' = \frac{\partial x_i''}{\partial x_i'}$.

From the chain rule of differentiation, one has
\[
\frac{\partial x''_i}{\partial X^\alpha} = \frac{\partial x''_i}{\partial X^j} \frac{\partial x'_j}{\partial X^\alpha} \tag{2.7a}
\]

which yields
\[
\widetilde{F} = \widetilde{F}'' \widetilde{F}' \tag{2.7b}
\]

where \( \widetilde{F} = \left[ \frac{\partial x''_i}{\partial X^\alpha} \right] \) is the deformation gradient matrix defining the transformation of material elements of the body from initial state \( \mathcal{C}_0 \) into the final state \( \mathcal{C}'' \). Note that, in general, \( \widetilde{F}'' \widetilde{F}' \neq \widetilde{F}' \widetilde{F}'' \).

The deformation gradient \( \mathcal{F} = x_i, \alpha e_\alpha e_i \) (and also its inverse \( \mathcal{F}^{-1} = X_\alpha, i e_i e_\alpha \)) is a fundamental quantity in the mechanics of continua.

As we shall see in these notes, it can be used to describe the manner by which material line elements at a neighborhood of a particle change their relative positions and elongate or shorten as the body deforms from the reference configuration \( \mathcal{C}_0 \) into a deformed configuration \( \mathcal{C} \).

Although only one and the same rectangular Cartesian frame is used to identify points in Euclidean space, we have adopted the convention of designating the coordinates of the reference (or initial) positions of particles \( X \) by majuscule letters having Greek indices, while the positions of these particles in the current configuration are denoted by minuscule letters having italic indices. Unless otherwise stated explicitly, this convention will be followed throughout these notes.
2.3 Measures of Deformation

When a neighborhood of a generic particle X moves from its initial state in $C_0$ into its final state in C such that the relative positions of the material particles in this neighborhood change, we say that this neighborhood is deformed. The change of the length of a material line element and the change of angle between two such elements are commonly used as measures of the deformation.

Let a material line element $dX$, at a generic particle X, have the spatial positions $d\bar{x}$ in $C_0$ and $d\bar{x}$ in C, respectively. The squared length of this element is given by

$$ (dS)^2 = d\bar{x} \cdot d\bar{x} = dX_\alpha dX_\alpha $$  \hspace{1cm} (3.1a) 

in configuration $C_0$, and by

$$ (ds)^2 = d\bar{x} \cdot d\bar{x} = dx_i dx_i $$  \hspace{1cm} (3.1b) 

in configuration C. The quantities

$$ \lambda = \frac{ds}{dS}, $$  \hspace{1cm} (3.2) 

and

$$ \delta = \frac{ds - dS}{dS} = \lambda - 1, $$  \hspace{1cm} (3.3) 

called, respectively, stretch and extension of the material line element $DX$, are evidently appropriate measures of the deformation of this element. Note that these quantities are independent of the elementary length of the considered material element, but depend on its direction.

Let $\hat{M}$ denote the unit vector along $d\bar{x}$, $d\bar{x} = (dS)\hat{M}$, and from (2.2) and (3.2) obtain
\[ \Lambda^a = \left( \frac{ds}{ds} \right)^2 = \frac{dx}{d\bar{x}} \cdot \frac{dx}{d\bar{x}} = x_{i, \alpha} x_{i, \beta} M^\alpha M^\beta = C_{\alpha\beta} M^\alpha M^\beta, \]  

where the symmetric, second order tensor \( C_{\alpha\beta} \) is called Green's deformation tensor. (The upper case letters \( \Lambda, M^\alpha \), and \( C_{\alpha\beta} \) are used, since the corresponding quantities are referred to the configuration \( C_0 \)). Equation (3.4a) may also be written as

\[ \chi^a = \left( \frac{ds}{ds} \right)^2 = \frac{X_{\alpha, i}}{X_{\alpha, j}} \frac{dx_i}{dx_j} = \frac{1}{c_{ij}} \frac{1}{\mu_i \mu_j}, \]  

where \( \bar{x}_{i} = \frac{dx_i}{ds} \bar{s}_{i} = \mu_i \bar{s}_i \) is the unit vector along \( dx \), and the symmetric, second order tensor

\[ \bar{c}_{i} = c_{ij} \bar{s}_i \bar{s}_j = x_{\alpha, i} x_{\alpha, j} \bar{s}_i \bar{s}_j \]  

is called Cauchy's deformation tensor. Note that \( \Lambda = \lambda \) if one and the same material element is considered.

From equation (3.4a) we see that the squared stretch in any direction at a given particle is defined by the normal component\(^1\) of the Green's deformation tensor in that direction. The stretch, therefore, depends in general, on the direction of the considered element. To denote this dependency, we write

\[ \Lambda^a(M) = C_{\alpha\beta} M^\alpha M^\beta \]  

---

\(^1\) The normal component of a second order tensor \( \bar{c}_{i} \) in the direction of a unit vector \( M \) is the scalar \( \bar{c} \cdot \bar{M} \).
when the direction of the element is given with respect to the initial configuration $C$. From equation (3.4b), it should also be clear that the normal component of Cauchy's deformation tensor in the direction $\mu$ is equal to the reciprocal of the squared stretch of the element currently tangent to $\mu$ at the considered particle, i.e.,

$$\lambda_{(\mu)}^{-2} = c_{ij} \mu_i \mu_j$$

(3.6b)

Note that the subscript letters in parentheses in (3.6) define the direction of the element at the considered point. Thus the squared stretch of the material line elements initially parallel to the coordinate axes are given by

$$\lambda_{(1)}^2 = C_{11},$$
$$\lambda_{(2)}^2 = C_{22},$$
$$\lambda_{(3)}^2 = C_{33},$$

(3.7a)

and the reciprocal of the squared stretch of those elements which are currently parallel to the coordinate axes are expressed as

$$\lambda_{(1)}^{-2} = c_{11}$$
$$\lambda_{(2)}^{-2} = c_{22}$$
$$\lambda_{(3)}^{-2} = c_{33}$$

(3.7b)

Equations (3.7a) and (3.7b) are the respective normal components of the tensors $C$ and $\xi$ in the directions of the coordinate axes.

A complete description of deformation of a material neighborhood
requires, in addition to stretch and extension, a measure for defining the distortion, i.e., the angle change between material line elements of the neighborhood. Let $\Theta_{(MN)}$ denote the angle between the material elements $\delta X$ and $\delta \tilde{X}$ in the reference state $C$, where $\tilde{M}$ and $\tilde{N}$ are unit vectors along $d\tilde{X}$ and $\delta \tilde{X}$, respectively. As the deformation (1.1) maps $d\tilde{X}$ into $d\tilde{x}$, and $\delta \tilde{X}$ into $\delta \tilde{x}$, the angle $\Theta_{(MN)}$ decreases by an amount $\Gamma_{(MN)}$ called shear of the directions $\tilde{M}$ and $\tilde{N}$. We have

$$\cos \Theta_{(MN)} = \sim \cdot \sim = \left( \frac{d\tilde{X}}{ds} \right) \cdot \left( \frac{\delta \tilde{X}}{\delta s} \right), \quad (3.8a)$$

$$\cos (\Theta_{(MN)} - \Gamma_{(MN)}) = \left( \frac{d\tilde{x}}{ds} \right) \cdot \left( \frac{\delta \tilde{x}}{\delta s} \right)$$

$$= \frac{C_{\alpha \beta} \alpha \beta}{\Lambda(M) \Lambda(N)}, \quad (3.8b)$$

where $\Lambda(M)$ and $\Lambda(N)$ are stretch in the $\tilde{M}$ and $\tilde{N}$ directions, respectively. When these directions are orthogonal, (3.8b) becomes

$$\sin \Gamma_{(MN)} = \frac{C_{\alpha \beta} \alpha \beta}{\Lambda(M) \Lambda(N)} \quad (3.9)$$

which is called orthogonal shear of directions $\tilde{M}$ and $\tilde{N}$. Since $C_{\alpha \beta} \alpha \beta$ is the shear component of the tensor $C$ in $\tilde{M}$ and $\tilde{N}$ directions, (3.9) states that the orthogonal shear of directions $\tilde{M}$ and $\tilde{N}$ is proportional to the orthogonal shear component of the tensor $C$ in these directions. The orthogonal shear of the elements initially along the $X_1$ - and $X_2$ - axes are given by

$$\sin \Gamma_{(12)} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}$$

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and, similarly, the orthogonal shears for other coordinate directions are
\[
\sin \Gamma_{(23)}^{(23)} = \frac{C_{33}}{\sqrt{C_{22} C_{33}}},
\]
and
\[
\sin \Gamma_{(31)}^{(31)} = \frac{C_{31}}{\sqrt{C_{33} C_{11}}},
\]
These equations can be written more concisely as
\[
\sin \Gamma_{(\alpha \beta)}^{(\alpha \beta)} = \frac{C_{\alpha \beta}}{\sqrt{C^{\alpha \alpha} C_{\beta \beta}}}, \quad \alpha \neq \beta,
\]
where the underlined, repeated subscript is not to be summed. Note that, similarly to the stretch, shear is only dependent on the directions of the considered elements at the given material point.

From the continuity of the deformation, it is clear that the stretch \( \Lambda_{(M)} \) can be neither zero nor infinity, \( 0 < \Lambda_{(M)} < \infty \); also the extension is bounded, \(-1 < \delta_{(M)} < \infty \). Therefore, we may seek the direction \( M \) for which the squared stretch \( \Lambda_{(M)}^2 \) attains its extremum. Such extremum values of stretch are clearly useful quantities, since they bound the value of stretch for any other direction at the considered particle.

Consider a generic particle \( X \) and let \( \bar{M} \) be a unit vector defining the direction of a material element \( dX \) in \( G \). The squared stretch in this direction is
\[
\Lambda_{(M)}^2 = C_{\alpha \beta} M_\alpha M_\beta,
\]
which is to be maximized (minimized) subject to the constraint
\[ M_\alpha M_\beta \delta_{\alpha\beta} - 1 = 0. \]

With \( C \) denoting the Lagrangian \(^1\) multiplier, we seek the stationary values of
\[ (C_{\alpha\beta} - C \delta_{\alpha\beta}) M_\alpha M_\beta + C \] (3.11)
by setting its derivative with respect to \( M_\gamma \) equal to zero, obtaining
\[ (C_{\alpha\beta} - C \delta_{\alpha\beta}) M_\alpha = 0 \; ; \; \alpha, \beta = 1, 2, 3. \] (3.12)

System (3.12) is a set of three linear, homogeneous equations in \( M_\alpha ; \alpha = 1, 2, 3 \). Nontrivial solutions exist if and only if the determinant of the coefficient of \( M_\alpha \) is zero, i.e.,
\[ \det |C_{\alpha\beta} - C \delta_{\alpha\beta}| = 0 \] (3.13)
which defines a polynomial of third degree in \( C \). Expanding (3.13), we obtain
\[ C^3 - I_C C^2 + II_C C - III_C = 0, \] (3.14)
where the coefficients \( I_C \), \( II_C \), and \( III_C \) are scalars; they are called \underline{basic invariants} of the second order, symmetric tensor \( C \), and are given by
\[ I_C = \text{tr} C = C_{\alpha\alpha} = C_{11} + C_{22} + C_{33}, \]

\(^1\) The same notation is used for current configuration of the body. This, of course, should not cause any confusion.
\[ \Pi_C = \frac{1}{2} e^{\alpha \beta \gamma} e^{\alpha \zeta \eta} C_{\beta \zeta} C_{\gamma \eta} = \frac{1}{2} (C_{\alpha \alpha} C_{\beta \beta} - C_{\alpha \beta} C_{\beta \alpha}) , \]

\[ \Pi_C = \det |C_{\alpha \beta}| = \frac{1}{6} e^{\alpha \beta \gamma} e^{\alpha \zeta \eta} C_{\alpha \zeta} C_{\beta \eta} C_{\gamma \theta} \]

\[ = \frac{1}{6} \left( 2 C_{\alpha \beta} C_{\beta \gamma} C_{\gamma \alpha} - 3 C_{\alpha \beta} C_{\beta \alpha} C_{\gamma \gamma} + C_{\alpha \alpha} C_{\beta \beta} C_{\gamma \gamma} \right) , \tag{3.15} \]

where \( \text{tr} C \) denotes the trace of \( C \), and \( \Pi_C \) may also be viewed as representing the trace of a matrix whose elements are the respective cofactors of the elements \( C_{\alpha \beta} \) in the matrix \( \widetilde{C} = [C_{\alpha \beta}] \).

Equation (3.14) has three roots \( C_I, C_{\Pi}, \) and \( C_{\text{III}} \). These roots are called principal values (or proper numbers, or characteristic values) of the second order tensor \( C \). To each principal value \( C_J, J = I, \Pi, \text{III}, \) there corresponds a principal direction \( M^J = M^J_{\alpha} e_\alpha \).

For example, for \( C_{\Pi} \) we have

\[ C_{\alpha \beta} M^\Pi_{\alpha} - C_{\Pi} M^\Pi_{\beta} = 0 \tag{3.16} \]

which defines the second principal direction.

Since \( C_{\alpha \beta} \) is a symmetric tensor, \( C_{\alpha \beta} = C_{\beta \alpha} \), all its principal values are real. This can be shown as follows: The polynomial (3.14) has, at least, one real root and the other two roots are either both real or they are complex conjugates of each other. Let us assume the latter, and show the contradiction. If \( C_I \) is the real root, then \( C_{\Pi} = A + i B \) and \( C_{\text{III}} = A - i B = \overline{C_{\Pi}} \), \((i = \sqrt{-1})\). The corresponding principal directions thus are \( M^\Pi_{\alpha} \) and \( \overline{M^\Pi_{\alpha}} \), and from (3.12) we get

\(^1\) Note that \( C_J \) does not denote components of a vector.

\(^2\) A superposed bar denotes complex conjugate.
\[ C_{\alpha\beta} M^{\alpha}_{\beta} = C_{\alpha} M^{\alpha}_{\beta}, \]

and

\[ C_{\alpha\beta} \overline{M}^{\alpha}_{\beta} = \overline{C_{\alpha}} \overline{M}^{\alpha}_{\beta}. \]

Multiplying the first equation by \( M^{\alpha}_{\beta} \) and the second equation by \( M^{\alpha}_{\beta} \), and then taking their difference, we obtain

\[ 2iB \left[ M^{\alpha}_{\beta} \overline{M}^{\alpha}_{\beta} \right] = 0, \quad i = \sqrt{-1}, \]

which is satisfied if and only if \( B \equiv 0 \).

Multiplying both sides of (3.16) by \( M^{\alpha}_{\beta} \) and summing on \( \beta \), we obtain

\[ C_{\alpha\beta} M^{\alpha}_{\beta} M^{\beta}_{\beta} = \Lambda^{a}_{(\Pi)} = C_{\Pi} > 0 \]

This equation and similar results for other principal directions show that the principal values \( C_{\Pi} \), \( C_{\Pi} \), and \( C_{\Pi} \) are the extremum values of the squared stretch.

The principal directions that correspond to distinct principal values are orthogonal. To see this, we first write

\[ C_{\alpha\beta} M^{\alpha}_{\alpha} M^{\beta}_{\beta} = 0, \]

\[ C_{\alpha\beta} M^{\alpha}_{\alpha} M^{\beta}_{\beta} = 0 \]

which define the principal directions \( \overline{M}^{\alpha} \) and \( \overline{M}^{\alpha} \), respectively. Now, multiplying the first equation by \( M^{\alpha}_{\beta} \) and the second equation by \( M^{\alpha}_{\beta} \), and then taking their difference, we obtain

\[ (C_{I} - C_{II}) M^{\alpha}_{\beta} M^{\beta}_{\beta} = (C_{I} - C_{II}) (\overline{M}^{\alpha} \cdot \overline{M}^{\alpha}) = 0. \]  

(3.17)

Therefore, if \( C_{I} \neq C_{II} \), then \( \overline{M}^{\alpha} \) is normal to \( \overline{M}^{\alpha} \). Similar results
hold for other principal directions.

Although we stated the above results in terms of Green's deformation tensor \( \mathcal{C} \), they are valid for any second order, symmetric, positive-definite tensor; in particular, they can be immediately applied to Cauchy's deformation tensor \( \varepsilon = \varepsilon_{ij} \varepsilon_i \varepsilon_j \). We denote the principal values of this tensor by \( c_I, c_{II}, \) and \( c_{III} \), and the unit vectors in the corresponding principal directions by \( \mu^I, \mu^{II}, \) and \( \mu^{III} \), respectively. We note that, when \( c_I \neq c_{II} \neq c_{III} \), then we have \( \mu^I \parallel \mu^{II} \parallel \mu^{III} \).

Also, the principal values of \( \mathcal{C} \) are the reciprocals of the squared stretch of elements which are currently directed along the principal directions. We now explore relations between the principal directions and principal values of tensors \( \mathcal{C} \) and \( \varepsilon \) as follows:

We start with Green's deformation tensor \( \mathcal{C} \), and note that the principal values are given by

\[
C_{\alpha\beta} M^J_\alpha = c_J M^J_\beta, \quad J = I, II, III, \quad \text{(no sum on } J \text{)}, \quad (3.18)
\]

where

\[
M^J_\alpha M^K_\alpha = \delta_{JK} = \begin{cases} 
1 & \text{if } J = K \\
0 & \text{if } J \neq K 
\end{cases}
\]

We then define an orthonormal basis at the image point \( \mu \) in \( C \) as follows:

\[
\mu^J_i = x_{i, \alpha} M^J_\alpha / \sqrt{C_J}, \quad J = I, II, III, \quad \text{(no sum on } J \text{)}, \quad (3.19a)
\]

To show that \( \mu^J_i, \quad J = I, II, III, \) define an orthogonal unit triad, we consider

\[
\mu^J_i \mu^K_i = x_{i, \alpha} x_{i, \beta} M^J_\alpha M^K_\beta / \sqrt{C_J C_K}
\]

\[
= C_{\alpha\beta} M^J_\alpha M^K_\beta / \sqrt{C_J C_K} \quad \text{(no sum on } J, k \text{)}
\]

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\( = \delta^{JK}; J, K = I, II, III \).

The normal component of \( \xi \) in the direction \( \mu^J \) is

\[
c_{ij} \mu_i^J \mu_j^J = \frac{X_{\alpha, i} X_{\alpha, j} x_{i, \beta} x_{j, \gamma} M_{\beta}^J M_{\gamma}^J}{C_J} = \frac{1}{C_J},
\]

and we also have

\[
c_{ij} \mu_i^J = \frac{X_{\alpha, i} X_{\alpha, j} x_{i, \beta} M_{\beta}^J \sqrt{C_J}}{C_J^{-1} C_{\gamma} \beta M_{\gamma}^J x_{j, \beta}} = \frac{1}{C_J} \mu_j^J,
\]

which reveals that the principal values of \( \xi \) are reciprocal of those of \( \xi \), i.e.,

\[
C_J = \frac{1}{C_J}, J = I, II, III,
\]

and the respective principal directions are related by (3.19a) and its inverse

\[
M_{\alpha}^J = \frac{X_{\alpha, i} \mu_i^J \sqrt{C_J}}{C_J^{-1} C_{\gamma} \beta M_{\gamma}^J x_{j, \beta}}.
\]

Consider a sphere of infinitesimal radius \( ds \) at point \( \mathbf{x} \) in the current state \( C \);

\[
(ds)^2 = d\mathbf{x} \cdot d\mathbf{x} = C_{\alpha \beta} dX_{\alpha} dX_{\beta}.
\]

The quadric \( C_{\alpha \beta} dX_{\alpha} dX_{\beta} = (ds)^2 > 0 \), called \underline{Cauchy's deformation quadric}, defines an ellipsoid with center at \( \mathbf{X} \), called \underline{strain ellipsoid} at \( \mathbf{x} \) in the initial state \( C_0 \). The material points that are contained in this ellipsoid are mapped, by the deformation (1.1), into a sphere.
\[(ds)^2 = dx_i \cdot dx_i\] with center at \( \mathbf{X} \) in the deformed state \( C \). Similarly, the material points in a sphere of infinitesimal radius \( dS \) at \( \mathbf{X} \) in configuration \( C_0 \),

\[
(ds)^2 = d\mathbf{X} \cdot d\mathbf{X} = c_{ij} dx_i dx_j \tag{3.23b}
\]

are mapped into an ellipsoid \( (ds)^2 = c_{ij} dx_i dx_j \) called strain ellipsoid at \( \mathbf{X} \) in \( C \).

Let \( d\mathbf{X} \) and \( \delta\mathbf{X} \) define two orthogonal directions at \( \mathbf{X} \) in \( C_0 \), i.e.,

\[
d\mathbf{X} \cdot \delta\mathbf{X} = d\mathbf{X}_\alpha \delta\mathbf{X}_\alpha = (c_{ij} dx_i) \delta x_j = 0.
\]

Since the vector \( (c_{ij} dx_i) e_j \) is normal to the vector \( \delta x_j e_j \), and since the former vector is directed along the gradient of the strain ellipsoid at the terminus of \( dx_i e_i \), two vectors \( (dx_i e_i) \) and \( (\delta x_j e_j) \) are, by definition, conjugate diameters of the strain ellipsoid at \( \mathbf{X} \). Thus, orthogonal elements at \( \mathbf{X} \) are mapped into conjugate diameters of the strain ellipsoid at \( \mathbf{X} \). This is known as Cauchy's first fundamental theorem.

An ellipsoid has three orthogonal axes which are normal to their conjugate planes and, therefore, mutually conjugates of each other. At point \( \mathbf{X} \) in \( C \), these axes correspond to the principal directions \( \mu^J \), \( J = I, II, III \), of Cauchy's deformation tensor \( \xi = c_{ij} e_i e_j \); the ratio of the squared lengths of these axes are the inverse of the ratios of the corresponding principal values \( c_J \). Similarly, at point \( \mathbf{X} \) in \( C_0 \), the principal axes of the strain ellipsoid are the principal directions of the tensor \( \xi = C_{\alpha\beta} e_\alpha e_\beta \); the ratios of the squared lengths of these axes are the inverse of the ratios of the corresponding principal values.

\footnote{This is commonly called reciprocal strain ellipsoid.}

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Thus elements along the principal directions at \( \mathbf{x} \) are mapped into the principal directions at \( \mathbf{x} \); the principal axes of strain ellipsoid at \( \mathbf{x} \) rotate into the principal axes of strain ellipsoid at \( \mathbf{\bar{x}} \). The squared stretches in these directions are

\[
\Lambda(J) = \lambda^2(J) = c_J = \frac{1}{c_J}, \quad J = I, II, III
\]

which show that the longest axis of the strain ellipsoid at \( \mathbf{\bar{x}} \) rotates into the shortest axis at \( \mathbf{x} \).

We see that to each point \( \mathbf{x} \) in \( C_0 (\mathbf{\bar{x}} \) in \( C \), we may associate a strain ellipsoid whose principal axes are along the principal directions of the strain tensor \( \mathbf{\bar{C}} \) (\( \mathbf{\bar{c}} \)), which directions are orthogonal and rotate into the principal axes of the strain ellipsoid at the image point \( \mathbf{\bar{x}} \) in \( C (\mathbf{\bar{x}} \) in \( C_0 \). The stretch \( \lambda(J) \), \( J = I, II, III \), of the elements along the principal directions are extremum; if \( \lambda(I) \geq \lambda(II) \geq \lambda(III) \), then \( \lambda(I) \) is the maximum, and \( \lambda(III) \) is the minimum value of stretch at \( \mathbf{\bar{x}} \), while \( \lambda(II) \) is a minimax. These principal stretches \( \lambda(J) \) have the same ratios as the lengths of the corresponding axes of the strain ellipsoid at \( \mathbf{x} \), and inversely the same as those of the ellipsoid at \( \mathbf{\bar{x}} \). These facts constitute Cauchy's second fundamental theorem.

So far we assumed that Green's (or Cauchy's) deformation tensor \( \mathbf{\bar{C}} \) (or \( \mathbf{\bar{c}} \)) possesses distinct principal numbers. If two of these numbers are equal at a material point, then there is only one unique principal direction at this point; this corresponds to the distinct principal value. Normal to this direction, any two orthogonal directions may be taken as the
principal directions; the strain ellipsoid degenerates to a spheroid. If all three principal values are equal, then the ellipsoid reduces to a sphere, and any orthogonal triad at this point constitutes a principal triad. In this case, either the deformation is locally rigid \( (C_{\alpha\beta} = \delta_{\alpha\beta}, \text{at that point}) \), or it is an isotropic deformation \( (C_{\alpha\beta} = \Lambda^2 \delta_{\alpha\beta}, \text{at that point}) \).

Since the principal values of \( \zeta \) are the roots of equation (3.14), this equation can also be written as

\[
C^3 - I_C C^2 + II_C C - III_C = (C - C_1) (C - C_{II}) (C - C_{III}) = 0 .
\]  

(3.24)

Thus the basic invariants of \( \zeta \) may also be expressed in terms of the principal values and principal stretches as

\[
I_C = C_1 + C_{II} + C_{III} = \Lambda^2_{(I)} + \Lambda^2_{(II)} + \Lambda^2_{(III)} ,
\]

\[
II_C = C_1 C_{II} + C_{II} C_{III} + C_{III} C_1 = \Lambda^2_{(I)} \Lambda^2_{(II)} + \Lambda^2_{(II)} \Lambda^2_{(III)} + \Lambda^2_{(III)} \Lambda^2_{(I)} ,
\]

\[
III_C = C_1 C_{II} C_{III} = \Lambda^2_{(I)} \Lambda^2_{(II)} \Lambda^2_{(III)} .
\]  

(3.25)

Note that, since \( C_j = \frac{1}{C_j} \), \( j = I, II, III \), the basic invariants of Cauchy's deformation tensor \( \zeta \) may be expressed as

\[
I_c = c_1 + c_{II} + c_{III} = \frac{1}{C_1} + \frac{1}{C_{II}} + \frac{1}{C_{III}} = \frac{II_C}{III_C} ,
\]
\[
\Pi_c = c_I c_{II} + c_{II} c_{III} + c_{III} c_I = \frac{I_C}{III_C},
\]
\[
III_c = c_I c_{II} c_{III} = \frac{1}{c_I c_{II} c_{III}} = \frac{1}{III_C}
\]

(3.26)

Let \( \tilde{M}_J \), \( J = I, II, III \), denote the principal directions of tensor \( \tilde{C} \). This tensor may then be written as
\[
\tilde{C} = C_{\alpha\beta} \tilde{e}_\alpha \tilde{e}_\beta = \sum_{J=I}^{III} C_J \tilde{M}_J^\alpha \tilde{M}_J^\beta \tilde{e}_\alpha \tilde{e}_\beta = \sum_{J=I}^{III} C_J \tilde{M}_J \tilde{M}_J,
\]
and with \( \tilde{M}_J \) as coordinate axes, the Cauchy strain quadric at \( \tilde{X} \) becomes
\[
(ds)^2 = C_{\alpha\beta} dX_\alpha dX_\beta = \sum_{J=I}^{III} C_J (dX_J)^2,
\]
(3.28)

where \( dX_J \), \( J = I, II, III \), measure distances along the respective directions \( \tilde{M}_J \). Similarly, the matrix \( \tilde{C} \) may be written in the following diagonal form:
\[
\tilde{C} = [C_{\alpha\beta}] = \begin{bmatrix}
C_I & 0 & 0 \\
0 & C_{II} & 0 \\
0 & 0 & C_{III}
\end{bmatrix}
\]
(3.29a)

from which it is seen that any power of \( \tilde{C} \) has principal values of \( \tilde{C} \) that are raised to the same power, i.e.,
\[
\tilde{C}^n = \begin{bmatrix}
C_I^n & 0 & 0 \\
0 & C_{II}^n & 0 \\
0 & 0 & C_{III}^n
\end{bmatrix}
\]
(3.29b)

and we have
\[
\tilde{C}^n = \sum_{J=I}^{III} (C_J)^n \tilde{M}_J^\alpha \tilde{M}_J^\beta \tilde{e}_\alpha \tilde{e}_\beta = \sum_{J=I}^{III} C_J^n \tilde{M}_J \tilde{M}_J
\]
(3.30)
where $n$ may be any positive or negative integer or fraction. Since these results are valid for any symmetric tensor of second order that possesses distinct principal values, they can be immediately applied to Cauchy's deformation tensor $\mathbf{C}$. Note that, since $\mathbf{C}$ is positive definite, $C_J$, $J = I, II, III$, are all real and positive, and for $n = \frac{1}{2}$ we have

$$\mathbf{C}^{\frac{1}{2}} = \begin{bmatrix}
\Lambda(I) & 0 & 0 \\
0 & \Lambda(II) & 0 \\
0 & 0 & \Lambda(III)
\end{bmatrix}$$

(3.21)

which is sometimes denoted by $\mathbf{U}$. Note also that the proper numbers of the deformation gradient matrix $\mathbf{F} = [x_i, \alpha]$ are the same as those of $\mathbf{U}$ but, in general, their principal directions are different. The former assertion follows from the fact that, in their diagonal forms, $\mathbf{F}$ and $\mathbf{C}^{\frac{1}{2}}$ are identical, since, by definition, we have

$$\mathbf{C} = [x_i, \alpha]^T [x_i, \beta] = \mathbf{F}^T \mathbf{F}.$$ (3.32)

Thus the deformation gradient matrix can be written as

$$\mathbf{F} = \mathbf{R} \mathbf{U},$$ (3.33a)

where $\mathbf{R}$ is a $3 \times 3$ proper orthogonal matrix; $\det |\mathbf{R}| = 1$, $\mathbf{R}^{-1} = \mathbf{R}^T$. From (3.32) and (3.33a), we obtain
\[ \tilde{C} = (\tilde{R} \tilde{U})^T (\tilde{R} \tilde{U}) = \tilde{U}^T \tilde{R}^T \tilde{R} \tilde{U} = \tilde{U}^2, \]  

(3.33b)

and

\[ \tilde{R} = \tilde{F} \tilde{U}^{-1} \]  

(3.33c)

which actually define \( \tilde{U} \) and \( \tilde{R} \). The decomposition (3.33a) is known as polar decomposition, which is unique (why?). It may also be written as

\[ \tilde{F} = \tilde{V} \tilde{R} \]  

(3.34a)

where

\[ \tilde{V} = \tilde{R} \tilde{U} \tilde{R}^T \]  

(3.34b)

and

\[ \tilde{U} = \tilde{R}^T \tilde{V} \tilde{R} \]  

(3.34c)

The second order tensor \( \tilde{y} \), whose matrix is \( \tilde{U} \), is often called right stretch tensor, and the second order tensor \( \tilde{y} \) is called left-stretch tensor of the deformation.
2.4 Deformation of Surface and Volume Elements

Consider a material point \( X \) and let an elementary material parallelogram be specified by material line elements \( dX \) and \( \delta X \) at this point. With \( \tilde{dX} \) and \( \tilde{\delta X} \) denoting the positions of these elements in configuration \( C_0 \), we define the elementary vector \( \tilde{dA} \) as

\[
\tilde{dA} = \tilde{dX} \times \tilde{\delta X} = e_{\alpha \beta \gamma} \tilde{dX}_\alpha \tilde{\delta X}_\beta \tilde{e}_\gamma = dA_\gamma \tilde{e}_\gamma
\]

which is normal to the plane formed by \( \tilde{dX} \) and \( \tilde{\delta X} \), and has a magnitude equal to the area of the elementary parallelogram at \( \tilde{X} \). The square of this elementary area is

\[
(dA)^2 = \tilde{dA} \cdot \tilde{dA} = dA_\alpha dA_\alpha 
\]

(4.1a)

The deformation (1.1) which carries the particle \( X \) from \( \tilde{X} \) in \( C_0 \) into \( x \) in \( C \), maps \( \tilde{dX} \) into \( dx \), \( \tilde{\delta X} \) into \( \delta x \), and \( \tilde{dA} \) into \( da \) such that

\[
da = dx \times \delta x = e_{ijk} dx_i \delta x_j \tilde{e}_k = da_k \tilde{e}_k
\]

(4.1b)

\[
(da)^2 = da_\alpha \cdot da_\alpha = da_k da_k
\]

(4.2b)

The squared elementary area \( (dA)^2 \) can be written as

\[
(dA)^2 = dA_\alpha dA_\alpha = (e_{\alpha \beta \gamma} dX_\beta \delta X_\gamma)(e_{\alpha \eta \zeta} dX_\eta \delta X_\zeta)
\]

\[
= e_{\alpha \beta \gamma} e_{\alpha \eta \zeta} X_\beta, i X_\gamma, j X_\eta, k X_\zeta, \ell dx_i \delta x_j dx_k \delta x_\ell
\]
\[
\begin{align*}
&= (e_{ijm} x_m, \alpha \det |X_\alpha, p| \, dx_i \delta x_j) \\
&= (e_{kln} x_n, \alpha \det |X_\gamma, q| \, dx_k \delta x_l) \\
&= x_m, \alpha x_n, \alpha \det |c_{pq}| \, da_m \, da_n \\
&= \frac{b_{mn}}{III_C} \, da_m \, da_n = III_C b_{mn} \, da_m \, da_n, \tag{4.3a}
\end{align*}
\]

where (4.1b) is also used, \( III_C = \frac{1}{III_c} \) is the third invariant of Green's deformation tensor \( \sim \), and the second order, symmetric tensor

\[
\sim = x_i, \alpha x_j, \alpha \varepsilon_i \varepsilon_j = b_{ij} \varepsilon_i \varepsilon_j \tag{4.4a}
\]
is commonly known as Finger's strain tensor. A comparison between (4.3a) and (3.23b) immediately reveals that the tensor \( \sim III_C \) transforms elements of area in exactly the same manner as the tensor \( \sim \) transforms elements of length. An inverse relation to (4.3a) can be obtained by starting with (4.2b) and proceeding in a similar manner as in (4.3a) to get

\[
(da)^2 = X_\alpha, i X_\beta, i \det |C_\gamma \delta| \, dA_\alpha \, dA_\beta \\
= \frac{B_{\alpha \beta}}{III_C} \, dA_\alpha \, dA_\beta = III_C B_{\alpha \beta} \, dA_\alpha \, dA_\beta, \tag{4.3b}
\]

where the second order, symmetric tensor

\[
\sim = X_\alpha, i X_\beta, i \varepsilon_\alpha \varepsilon_\beta = B_{\alpha \beta} \varepsilon_\alpha \varepsilon_\beta \tag{4.4b}
\]
is also called Finger's strain tensor. Note the analogy between $b_{ij}$ and $\underline{C}$ by comparing (4.3b) with (3.23a).

The tensors $b_{ij}$ and $\underline{C}$ are inverse tensors to $\underline{c}$ and $\underline{C}$, respectively, since we have

$$c_{ij} b_{jk} = \left( \frac{\partial x^\alpha}{\partial x_i} \frac{\partial x^\alpha}{\partial x_j} \right) \left( \frac{\partial x^\beta}{\partial x_j} \frac{\partial x^k}{\partial x_\beta} \right) = \delta_{ik} \quad , \quad (4.5a)$$

and

$$C_{\alpha\beta} B_{\beta\gamma} = (X^i, \alpha \times_j, \beta) (X^\beta, k X^\gamma, k) = \delta_{\alpha\gamma} \quad . \quad (4.5b)$$

From these equations we can conclude that, while the principal directions of $\underline{c}$ and $\underline{b}$ are the same, their principal values are reciprocal to one another. Therefore, the principal values of $\underline{b}$ are the squares of principal stretches at $\underline{c}$ in the current state. The tensor $\underline{b}$ is most commonly used in finite elasticity. Note that the matrix $\tilde{b} = [b_{ij}]$ is given in terms of the deformation gradient matrix $\tilde{F} = [x_i, \alpha]$ as

$$\tilde{b} = \tilde{F} \tilde{F}^T = [x_i, \beta] \ [x_j, \beta]^T = \tilde{\nu}^2 \quad . \quad (4.6a)$$

The tensor $\underline{b}$ is also called left Cauchy-Green tensor. This is in comparison with $\underline{C}$, whose matrix $\tilde{C}$ can be written as

$$\tilde{C} = \tilde{F}^T \tilde{F} = \tilde{\nu}^2 \quad . \quad (4.6b)$$

which is also known as right Cauchy-Green tensor. Note that the principal directions of $\underline{C}$ and $\underline{b}$ are the same while their proper
numbers are reciprocal to one another. Thus the basic invariants of \( \sim_1 \) and \( \sim_2 \) are equal; and, similarly, those of \( \sim_1 \) and \( \sim_3 \) are the same.

Denoting the principal numbers of \( \sim_1 \) and \( \sim_2 \) by \( b_J \) and \( B_J \), \( J = I, II, III \), respectively, we can write

\[
c_J = \frac{1}{C_J} = \frac{1}{b_J} = B_J ; \quad J = I, II, III .
\] (4.7)

Note that, since \( III_b = b_I b_{II} b_{III} \), and \( III_B = B_I B_{II} B_{III} \), where \( III_b \) and \( III_B \) are the third invariant of \( \sim_1 \) and \( \sim_2 \), respectively, equations (4.3) may also be written as

\[
(dA)^2 = III_b^{-1} b_{mn} da_m da_n \quad (4.3c)
\]

\[
(da)^2 = III_B^{-1} B_{\alpha\beta} dA_\alpha dA_\beta . \quad (4.3d)
\]

Let \( dX, \delta X, \) and \( \Delta X \) be three distinct material elements, emanating from a typical material point \( X \), with \( d\bar{X}, \delta \bar{X}, \) and \( \Delta \bar{X} \) their respective positions in \( C_0 \). The element of volume \( dV \) at \( \bar{X} \) is given by

\[
dV = d\bar{\alpha} \cdot \Delta \bar{X} = (d\bar{X} : \delta \bar{X}) \cdot \Delta \bar{X}
\]

\[
= e_{\alpha\beta\gamma} dX_\alpha \delta X_\beta \Delta X_\gamma . \quad (4.8)
\]

The deformation (1.1) maps the material points in \( dV \) at \( \bar{X} \) in the reference state \( C_0 \), into \( dv \) at \( x \) in the current state \( C \), and we have
\[ dV = e_{\alpha \beta \gamma} X_\alpha, i X_\beta, j X_\gamma, k \, dx_i \, \delta x_j \, \Delta x_k \]

\[ = \det |X_\alpha, \ell| e_{ijk} \, dx_i \, \delta x_j \, \Delta x_k \]

\[ = \det |X_\alpha, \ell| (dx_\xi \times \delta x_\zeta) \cdot \Delta x_\zeta \]

\[ = \det |X_\alpha, \ell| \, da \cdot \Delta x_\zeta \]

\[ j \, dv = \frac{dv}{j}, \quad (4.9a) \]

where \( dv \) is the elementary volume at \( \xi \) and \( j = \det |X_\alpha, i| \)

\[ = \frac{1}{\det |x_i, \alpha|} = \frac{1}{j} \] is the Jacobian.

Since we have \( \det |x_i, \alpha| = \left[ \det |C_{\alpha \beta}| \right]^{1/2} = \sqrt{\lambda_3 C} \) \( \), equation (4.9)

may also be written as

\[ dV = \sqrt{\lambda_3 C} \quad dv = \frac{dv}{\sqrt{\lambda_3 C}}, \quad (4.9b) \]

and in terms of the principal stretches, it becomes

\[ dV = \frac{dv}{\Lambda(1) \, \Lambda(II) \, \Lambda(III)} \quad . \quad (4.9c) \]

In defining the mass-density, equations (1.2), we stated that the mass of the body, or any portion of the body, stays unchanged as the body deforms from one configuration into another configuration. With \( \rho_0 \) and \( \rho \) denoting the mass-densities at \( \xi \) and \( \zeta \), respectively, we thus have
\[ \rho_0 \, dV = \rho \, dv \quad (4.10) \]

which expresses conservation of mass. Equations (4.9) and (4.10) now yield

\[ \rho = j \rho_0 = \frac{\rho_0}{J} \]

\[ = \frac{\rho_0}{\sqrt{III_C}} \quad (4.11a) \]

or

\[ \frac{\rho_0}{\rho} = J = \sqrt{III_C} \quad (4.11b) \]

A deformation for which \( J = \sqrt{III_C} = 1 \) is called isochoric.
2.5 Displacement Vector and Other Strain Tensors

The displacement $u$ of a particle $X$, which is displaced from its position $\bar{x}$ in $C_0$ into a new position $\bar{x}$ in $C$, is

\[ u(x) = x - \bar{x}(x) \]  \hspace{1cm} (5.1a)

or

\[ u(X) = x(X) - \bar{x}. \]  \hspace{1cm} (5.1b)

where in (5.1a) the Eulerian variables are used, while (5.1b) is expressed in terms of the Lagrangian variables. Referring to a common rectangular Cartesian frame, we may express (5.1) as

\[ u_{\alpha}(x) = x_{\alpha} - X_{\alpha}(x) \delta_{\alpha i} \]  \hspace{1cm} (5.2a)

\[ u_{\alpha}(X) = x_{i}(X) \delta_{i \alpha} - X_{\alpha} \]  \hspace{1cm} (5.2b)

which, after differentiation, yield

\[ u_{i, j}(x) = \delta_{i j} - X_{\alpha, j}(x) \delta_{\alpha i} \]  \hspace{1cm} (5.3a)

\[ u_{\alpha, \beta}(X) = x_{i, \beta}(X) \delta_{i \alpha} - \delta_{\alpha \beta} \]  \hspace{1cm} (5.3b)

The displacement gradients $u_{i,j}$ and $u_{\alpha, \beta}$ may also be written as
\[ u_{\alpha, \beta} = \frac{\partial}{\partial x_{\beta}} (\delta_{\alpha i} u_{i}) = \delta_{\alpha i} u_{i, j} x_{j, \beta} \] (5.4a)

\[ u_{i, j} = \delta_{i \alpha} u_{\alpha, \beta} x_{\beta, j} \] (5.4b)

Thus Cauchy's and Green's deformation tensors, \( \mathcal{D} \) and \( \mathcal{C} \), may be expressed in terms of the displacement gradients as follows:

\[ c_{ij} = (\delta_{L i} - u_{L, i}) (\delta_{k j} - u_{k, j}) \delta_{L k} \] (5.5a)

\[ = \delta_{ij} - \left[ \frac{\partial u_{i}}{\partial x_{i}} + \frac{\partial u_{i}}{\partial x_{j}} - \frac{\partial u_{k}}{\partial x_{i}} \frac{\partial u_{k}}{\partial x_{j}} \right] , \] (5.5a)

\[ C_{\alpha \beta} = \delta_{\alpha \beta} + \left[ \frac{\partial u_{\alpha}}{\partial x_{\beta}} + \frac{\partial u_{\beta}}{\partial x_{\alpha}} + \frac{\partial u_{\gamma}}{\partial x_{\alpha}} \frac{\partial u_{\gamma}}{\partial x_{\beta}} \right] , \] (5.5b)

where in (5.5a) the Eulerian variables are used, while (5.5b) is expressed in terms of the Lagrangian variables.

There are two strain measures, known as **Almansi's strain tensor** and **Lagrangian strain tensor**, which are commonly used.

Almansi's strain tensor \( \varepsilon \), denoted by \( \varepsilon = e_{ij} e_{i} e_{j} \), is defined by

\[ e_{ij} = \frac{1}{2} \left[ \frac{\partial u_{i}}{\partial x_{j}} + \frac{\partial u_{j}}{\partial x_{i}} - \frac{\partial u_{k}}{\partial x_{i}} \frac{\partial u_{k}}{\partial x_{j}} \right] \] (5.6a)

\[ ^1 \text{This is also known as Eulerian strain tensor.} \]
\frac{1}{2} \left( \mathcal{g} - \mathcal{c} \right), \\
(5.6b)

from which one obtains

\begin{align*}
(ds)^2 - (dS)^2 &= dx_i \, dx_i - dX_\alpha \, dX_\alpha \\
&= (\delta_{ij} - c_{ij}) \, dx_i \, dx_j \\
&= 2 \, e_{ij} \, dx_i \, dx_j \\
(5.6c)
\end{align*}

which provides a motivation for defining such a tensor. (The factor of $\frac{1}{2}$ in (5.6a) is used in order that the linear approximation of $\mathcal{g}$ reduce to the usual strain tensor. To see this we write

\begin{align*}
2 \, e_{ij} \, \mu_i \, \mu_j &= 1 - \lambda^{-2} \\
&= 1 - (1 + \delta(\mu))^2 \\
&\approx 2 \, \delta(\mu) + o(\delta^2(\mu)) , \\
(5.7a)
\end{align*}

where $\delta(\mu)$ is the extension in the direction defined by $\mu$ at point $X$. When $\mu$ is along the $x_i$-axis, we have

\begin{align*}
e_{ii} &= \delta(i) , \quad i = 1, 2, 3 , \\
(5.8a)
\end{align*}

where repeated but underlined indices are not to be summed. Equation (5.8a) states that the normal components of $\mathcal{g}$ in the directions of the coordinate axes are, to the first order of approximation in the corresponding extension, the extensions in the directions of these axes.)
The Lagrangian strain tensor, denoted by \( \mathbf{\varepsilon} = E_{\alpha \beta} \mathbf{e}_\alpha \mathbf{e}_\beta \), is given by

\[
E_{\alpha \beta} = \frac{1}{2} \left[ \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial u_\gamma}{\partial x_\alpha} \frac{\partial u_\gamma}{\partial x_\beta} \right],
\]

or

\[
\mathbf{\varepsilon} = \frac{1}{2} (\mathbf{\varepsilon} - \mathbf{\gamma})
\]

which yields the change in the square of an element of length as follows:

\[
(ds)^2 - (dS)^2 = dx_i \, dx_i - dX_\alpha \, dX_\alpha = 2 E_{\alpha \beta} \, dX_\alpha \, dX_\beta.
\]

(Note that, similarly to equations (5.7a) and (5.8a), we have)

\[
2 E_{\alpha \beta} M_\alpha M_\beta = \Lambda(M)^2 - 1
\]

\[
= (1 + \Delta(M))^2 - 1
\]

\[
\approx 2 \Delta(M),
\]

where \( \Delta(M) \) is the extension in the direction defined by \( \mathbf{M} \) at point \( \mathbf{X} \). When \( \mathbf{M} \) is along the \( X_\alpha \)-axis, we have

\[
E_{\alpha \alpha} \approx \Delta(\alpha), \quad \alpha = 1, 2, 3
\]

which states that, to the first order of approximation in the corresponding extension, the normal components of \( \mathbf{\varepsilon} \) along the coordinate axes are the extensions of elements in the directions of these axes.)

Note that the strain tensors \( \mathbf{\varepsilon} \) and \( \mathbf{\gamma} \) are both symmetric.
their principal values are all real. Denoting these principal values by \( E_J \) and \( e_J \), \( J = I, II, III, \) respectively, we have

\[
e_J = \frac{1}{2} (1 - c_J) = \frac{1}{2} (1 - \frac{\Lambda^2}{(J)}) , \tag{5.10a}
\]

\[
E_J = \frac{1}{2} (C_J - 1) = \frac{1}{2} (\Lambda^2 - 1) , \tag{5.10b}
\]

from which we obtain

\[
\Lambda^2(J) = \left(\frac{1 + 2E_J}{1 - 2e_J}\right) = \frac{1}{1 - 2e_J} , \tag{5.11a}
\]

and

\[
E_J = \frac{e_J}{1 - 2e_J} ; \quad J = I, II, III, \quad \text{(no sum on } J) . \tag{5.11b}
\]

Note also that the invariants of \( \zeta \) and \( \zeta \) can be expressed in terms of the invariants of \( \zeta \), or those of \( \zeta \), using the following formulae:

\[
c_{ij} = \delta_{ij} - 2e_{ij} \tag{5.12a}
\]

\[
C_{\alpha\beta} = \delta_{\alpha\beta} + 2E_{\alpha\beta} . \tag{5.12b}
\]

This way, we obtain

\[
I_C = 3 + 2I_E ,
\]

\[
II_C = 3 + 4(I_E + II_E) ,
\]

\[
III_C = 1 + 2(I_E + 2II_E + 4III_E) . \tag{5.13a}
\]
\[ I_c = 3 - 2 I_e, \]
\[ II_c = 3 - 4 (I_e - II_e), \]
\[ III_c = 1 - 2 (I_e - 2 II_e + 4 III_e). \]  

(5.13b)

Before closing this section, we introduce a measure, called elongation, which, although not a strain measure, is often employed in elasticity. The material element \( dX \) at particle \( X \) is mapped from \( d\tilde{X} \) (with unit vector \( \tilde{M} \)) at \( \tilde{X} \) into \( dx \) (with unit vector \( \mu \)) at \( x \). The elongation \( \epsilon(M) \) in the direction \( \tilde{M} \) at point \( \tilde{X} \) is defined by

\[
\epsilon(M) = \frac{(dx - d\tilde{X})}{d\tilde{X}} \cdot \frac{d\tilde{X}}{dx} = \frac{dx_i \delta_{i\alpha} M_\alpha}{(dS)} - 1
\]

\[
= \Lambda(M) \mu_i \delta_{i\alpha} M_\alpha - 1. \]  

(5.14a)

Note that elongation is not necessarily a measure of distortion, since even for rigid rotations of the body, elongations of all elements are not zero.

Let us split the displacement gradients \( u_{i,j} \) and \( u_{\alpha,\beta} \) into symmetric and skew-symmetric parts as follows:

\[
u_{i,j} = u_{(i,j)} + u_{[i,j]} = e_{ij}^* + r_{ji}^*, \]  

(5.15a)

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\[ u_{\alpha, \beta} = u(\alpha, \beta) + u[\alpha, \beta] = E^*_{\beta \alpha} + R^*_\beta \alpha . \] (5.15b)

Note that \( \varepsilon^* \) and \( \zeta^* \) (or \( \varepsilon^* \) and \( \zeta^* \)) are the infinitesimal strain and rotation tensors of linear theories.

We now show that the elongation in the direction of \( \mathbf{M} \) at \( X \) is equal to the normal component of \( \varepsilon^* \) in this direction. To this end we write, from (5.2),

\[
dX_\beta \delta_i \alpha = (\delta_\alpha \beta + u_{\alpha, \beta}) dX_\beta
\]

\[
= (\delta_\alpha \beta + E^*_\beta \alpha + R^*_\beta \alpha) dX_\beta ,
\] (5.16)

and substitute into (5.14a) to obtain

\[
\epsilon(M) = u_{\alpha \beta} M_\alpha M_\beta = E^*_\alpha \beta M_\alpha M_\beta .
\] (5.14b)

Thus \( E^*_{11} \), \( E^*_{22} \), \( E^*_{33} \) are the respective elongations of elements initially along the coordinate axes at \( X \).

In terms of the tensors \( \varepsilon^* \) and \( \zeta^* \) (or \( \varepsilon^* \) and \( \zeta^* \)), the Lagrangian strain tensor \( \varepsilon \) (or Almansi's strain tensor \( \varepsilon \)) may be written as

\[
E_{\alpha \beta} = E^*_{\alpha \beta} + \frac{1}{2} (E_{\alpha \gamma}^* + R^*_{\alpha \gamma}) (E^*_{\beta \gamma} + R^*_{\beta \gamma})
\]

\[
= E^*_{\alpha \beta} + \frac{1}{2} \left[ E^*_{\alpha \gamma} E^*_{\beta \gamma} + R^*_{\alpha \gamma} E^*_{\gamma \beta} - E^*_{\alpha \gamma} R^*_{\beta \gamma} - R^*_{\alpha \gamma} R^*_{\beta \gamma} \right] .
\] (5.17)
which suggests various ways of approximating $\overset{\sim}{\xi}$. Note that the vanishing of $\overset{\sim}{\xi}^*$ does not necessarily imply the vanishing of $\overset{\sim}{\xi}$. Such an implication is, in general, only valid in the linearized cases.
2.6 Rotation

The deformation (1.1) that maps the element $d\tilde{X}$ at $\tilde{X}$ into the element $d\tilde{x}$ at $\tilde{x}$, rotates this element by an angle $\nu$ defined by

\[
\cos \nu = \frac{d\tilde{X} \cdot d\tilde{x}}{(dS)(dS)}
\]

\[
= \frac{x_{i,\beta}\delta_{\alpha i}^\beta}{\Lambda(M)} M_\alpha M_\beta
\]

\[
= X_{\alpha, j} \delta_{\alpha i}^\lambda(\mu) \mu_i \mu_j
\]  

(6.1a)

(6.1b)

where $\tilde{M} = M_{\alpha} e^\alpha$ is a unit vector along $d\tilde{X}$, $\mu = \mu_i e_i$ is a unit vector along $d\tilde{x}$, and since the same material element is considered

$\Lambda(M) = \lambda(\mu)$. In particular, the rotations of the principal directions $\tilde{M}^J$, $J = I, II, III$, may be expressed as

\[
\cos \nu_J = (C_J)^{-\frac{3}{2}} C_{\alpha \gamma} \delta_{\alpha i}^\gamma x_{i, \beta} M_\beta^J M_\gamma^J
\]

\[
= (c_J)^{-\frac{3}{2}} c_{ik} \delta_{i\alpha} X_{\alpha, j} \mu_j^J \mu_k^J
\]

$J = I, II, III$, (no sum on $J$),  

(6.2)

where $\mu^J$, $J = I, II, III$, are the principal directions at $\tilde{x}$.

When (1.1) is such that the principal directions do not suffer
rotation, i.e. $\nu_J = 0$, the deformation is called a pure strain. The principal axes of the strain ellipsoids at $\tilde{X}$ and $\tilde{x}$, in this case, are parallel to each other. Note that, in pure strain, the rotations of elements other than those along the principal directions are, in general, non-zero.

Since the deformation (1.1) rotates the principal directions at $\tilde{X}$ into those at $\tilde{x}$, the deformation of a material neighborhood of the particle $X$ may be specified as follows: A rigid translation of this neighborhood from $\tilde{X}$ to $\tilde{x}$, a rigid rotation which matches the shifted principal triad with the corresponding triad at $\tilde{x}$, and finally stretches along the principal directions. If $R_{\alpha\beta}$ is the proper orthogonal tensor that rotates shifted $\tilde{M}^J$, $J = I, II, III$, into the principal directions $\tilde{\mu}^J$ at $\tilde{x}$, we must have

$$\mu_i^J = R_{ij} \delta_{j\alpha} \tilde{M}_\alpha^J; \quad J = I, II, III$$  \hspace{1cm} (6.3a)

Note that, since one and the same rectangular Cartesian coordinate system is used, the components $M^J_{\alpha}$ and $M^J_{\alpha} \delta_{j\alpha}$ represent the same vector. The operator $\delta_{j\alpha}$ simply designate the translation of $\tilde{M}^J$ from point $\tilde{X}$ to point $\tilde{x}$.

Instead of translation from $\tilde{X}$ to $\tilde{x}$, we may shift $\tilde{\mu}^J$ from $\tilde{x}$ to $\tilde{X}$ and then rotate them into $\tilde{M}^J$. This way we obtain

$$M^J_{\alpha} = R_{\alpha\beta}^{-1} \delta_{i\beta} \mu_i^J$$ \hspace{1cm} (6.3b)

* Note that the superscript -1 in $R_{\alpha\beta}$ is part of the symbol and does not imply $1/R_{\alpha\beta}$.
where $\theta^{-1} = R_{\alpha\beta} \ e^\alpha \ e^\beta$ defines the inverse rotation to that specified by $\theta = R_{ij} \ e_i \ e_j$. Introducing the notation

$$R_{i\alpha} = R_{ij} \delta_j^{\alpha} \quad ,$$  \hspace{1cm} (6.4a)

$$R_{\alpha i} = R_{\alpha\beta} \delta_{\beta i} \quad ,$$  \hspace{1cm} (6.4b)

we reduce (6.3) to

$$\mu_i^J = R_{i\alpha} M_{\alpha}^J \quad ,$$  \hspace{1cm} (6.5a)

$$M_{\alpha}^J = R_{\alpha i} \mu_i^J \quad .$$  \hspace{1cm} (6.5b)

Multiplying both sides of (6.5a) by $M_{\beta}^J$, and summing on $J$ for $J = I, II, III$, we obtain

$$R_{i\alpha} \sum_{J=I}^{III} M_{\alpha}^J M_{\beta}^J = R_{i\beta}$$

$$= \sum_{J=I}^{III} \mu_i^J M_{\beta}^J \quad .$$  \hspace{1cm} (6.6a)

Similarly, (6.5b) yields

$$R_{\beta i}^{-1} = \sum_{J=I}^{III} M_{\beta}^J \mu_i^J \quad .$$  \hspace{1cm} (6.6b)

From Eq. (3.19) we have

$$\left(\frac{\mu_i^J}{\sqrt{c_J}}\right)_{x_i, \alpha} = M_{\alpha}^J \quad ,$$  \hspace{1cm} (6.7a)

$$M_{\alpha}^J \sqrt{c_J} = X_{\alpha, i} \mu_i^J \quad ,$$  \hspace{1cm} (6.7b)
which together with (6.5) now yield

\[ \sqrt{c} R_{i\alpha} M^J_{\alpha} = x_{i,\alpha} M^J_{\alpha} \quad (6.8a) \]

\[ \sqrt{c} R_{i\alpha} \mu^J_i = x_{\alpha, i} \mu^J_i \quad (6.8b) \]

Multiplying both sides of (6.8a) by \( M^J_{\beta} \), summing on \( J \) for \( J = I, II, III \), and using (3.30) with \( n = \frac{1}{2} \), we finally arrive at

\[ x_{i,\beta} = R_{i\alpha} C^{\alpha\beta}_{\beta} \]

\[ = R_{ij} \delta_{j\alpha} C^{\alpha\beta}_{\beta} \quad (6.9a) \]

where \( C^{\alpha\beta}_{\beta} \) are the components of the tensor \( C^{\frac{1}{2}} \) defined by (3.30) with \( n = \frac{1}{2} \). The right side of (6.9a) states that the deformation of a material neighborhood at a point \( \vec{X} \) in \( C_0 \) consists of a rigid-body rotation to match the principal axes of strain at \( \vec{X} \) with those at \( \vec{x} \), a translation from \( \vec{X} \) to \( \vec{x} \), and finally stretches \( \sqrt{C} \) along the principal axes. Note that (6.9a) expresses the polar decomposition already stated by (3.33a) in the matrix form.

If we start with (6.8b), and employ a similar procedure as above, we obtain

\[ x_{\alpha, j} = R_{\alpha i} c_{ij} \]

\[ = \delta_{\alpha k} R_{ki} c_{ij} \quad (6.9b) \]

where \( c_{ij} \) denotes components of \( \frac{1}{2} c_0 \).
For an infinitesimal deformation, in which the displacement
gradients \( u_{\alpha,\beta} \) and \( u_{i,j} \) are so small that the quadratic terms in
the strain tensors (5.5) and (5.6) can be neglected in comparison with
the linear terms, the rigid-body rotation of a material neighborhood is
completely defined by the antisymmetric tensor
\[
\varepsilon^* = R^*_\alpha \varepsilon_\alpha \varepsilon_\beta = u_{[\beta, \alpha]} \varepsilon_\alpha \varepsilon_\beta .
\]
For large deformations, this tensor is a measure of mean rotation, as has been shown by Novozhilov, and will be con-
sidered subsequently.

We let \( \mathbf{M}^{(3)} \) denote a unit vector perpendicular to the \( X_3 \)-
axis, making an angle \( \phi \) with the \( X_1 \)-axis, i.e.,
\[
\mathbf{M}^{(3)} = e_1 \cos \phi + e_2 \sin \phi .
\]  
(6.10)

The deformation (1.1) maps the material line element tangent to \( \mathbf{M}^{(3)} \)
in configuration \( C_0 \) into an element whose unit tangent is \( \mathbf{\nu}^{(3)} \). The
projection of \( \mathbf{\nu}^{(3)} \) upon the \( X_1, X_2 \)-plane makes an angle \( \varphi \) with
the \( X_1 \)-axis. We let
\[
\mathbf{\nu}^{(3)} = \varphi - \phi ,
\]  
(6.11)
and using the relations
\[
tan \varphi = tan (\phi + \nu^{(3)})
\]
\[
= \frac{tan \phi + tan \nu^{(3)}}{1 - tan \phi tan \nu^{(3)}}
\]

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\[
\frac{\partial x_2}{\partial x_1} = \frac{\partial x_2}{\partial x_1} \frac{dx_1}{dx_2} + \frac{\partial x_2}{\partial x_2} \frac{dx_2}{dx_1} \frac{dx_1}{dx_2},
\]

obtain

\[
\tan \nu^{(3)} = \frac{\frac{\partial x_2}{\partial x_1} \cos^2 \Phi + \left( \frac{\partial x_2}{\partial x_2} - \frac{\partial x_1}{\partial x_1} \right) \sin \Phi \cos \Phi - \frac{\partial x_1}{\partial x_2} \sin^2 \Phi}{\frac{\partial x_1}{\partial x_1} \cos^2 \Phi + \frac{\partial x_2}{\partial x_2} \sin^2 \Phi + \left( \frac{\partial x_2}{\partial x_1} + \frac{\partial x_1}{\partial x_2} \right) \sin \Phi \cos \Phi}
\]

\[= \frac{R_{12}^* + E_{12}^* \cos 2 \Phi + \frac{1}{2} \left( E_{22}^* - E_{11}^* \right) \sin 2 \Phi}{1 + E_{11}^* \cos^2 \Phi + E_{22}^* \sin^2 \Phi + E_{12}^* \sin 2 \Phi}. \tag{6.12} \]

The right-hand side of (6.12) is a periodic function of \( \Phi \) with period \( \pi \). Following Novozhilov, we now average (6.12) over the interval 0 to \( 2\pi \), and obtain

\[
\tan \nu^{(3)} = \frac{1}{2\pi} \int_0^{2\pi} \tan \nu^{(3)}(\Phi) \, d\Phi
\]

\[= \frac{R_{12}^*}{\left[ (1 + E_{11}^*)(1 + E_{22}^*) - E_{12}^* \right]^\frac{1}{2}}, \tag{6.13} \]

where a superposed bar denotes the average value. Equation (6.13) shows that \( R_{12}^* \) is proportional to a measure of the average rotation suffered by the elements initially in the \( X_1, X_2 \)-plane. Similar remarks apply to \( R_{23}^* \) and \( R_{31}^* \). Hence, \( \Phi^* \) may be regarded as a
measure of the mean rotation suffered by a material neighborhood.
For this reason, $\mathbf{\omega}^*$ is often referred to as the mean rotation tensor.
The deformation for which $\mathbf{\omega}^* = \mathbf{0}$ is called potential deformation,
since, in this case, the displacement field may be expressed as gradient
of a scalar field, i.e.,

$$u_\alpha = \frac{\partial U}{\partial X_\alpha}, \quad U = U(X_1, X_2, X_3). \quad (6.14)$$
References


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PROBLEMS

1. Show that

\[ \frac{\partial J}{\partial (x_i, \alpha)} = X_{\alpha, i} J, \]

\[ (J X_{\alpha, i}), \alpha = 0, \]

\[ \left( \frac{x_i}{j} \right), i = 0. \]

\[ \checkmark b) \]

\[ x_i, \alpha = - x_{\beta, j} \frac{\partial (x_i, \beta)}{\partial (x_{\alpha, i})}, \]

\[ \frac{\partial (x_k, \beta)}{\partial (x_{\alpha, i})} = - x_k, \alpha x_i, \beta \]

\[ c) \]

\[ c_{ij} = - \frac{\partial (x_i, \alpha)}{\partial (x_j, \alpha)}, \]

\[ C_{\alpha \beta} = \frac{\partial (x_j, \alpha)}{\partial (x_{\beta, j})}, \]

\[ b_{ij} = - \frac{\partial (x_i, \alpha)}{\partial (x_{\alpha, j})} \]

\[ B_{\alpha \beta} = - \frac{\partial (x_{\alpha, j})}{\partial (x_j, \beta)} . \]

\[ \checkmark 2. \] Prove that the ratios of the principal stretches are inversely the same as the ratios of the lengths of the corresponding axes of the strain ellipsoid at \( X \).
3. Prove that transformation (3.19) rotates the principal triad at \( \overset{\sim}{X} \) into that at \( \overset{\sim}{X} \). Find the proper orthogonal tensor \( \overset{\sim}{p} \) of this rotation.

4. Show that \( \overset{\sim}{C} = \overset{\sim}{J} \) implies \( I_C = II_C \) and \( III_C = 1 \), which are necessary and sufficient conditions for the deformation to be that of a rigid-body.

5. Find necessary and sufficient conditions for vanishing of the shear of non-orthogonal directions \( \overset{\sim}{M} \) and \( \overset{\sim}{N} \) at \( \overset{\sim}{X} \).

6. Show that the extremum values of \( \sin \Gamma_{(MN)} \), \( \overset{\sim}{M} = \overset{\sim}{N} \), are

\[
\pm \frac{C_I - C_{III}}{C_I + C_{III}}, \quad \pm \frac{C_{II} - C_{III}}{C_{II} + C_{III}}, \quad \pm \frac{C_I - C_{II}}{C_I + C_{II}}.
\]

What are the corresponding directions?

7. Show that the stretch \( \Lambda_{(M)} \) in the direction of \( \overset{\sim}{M} \) at \( \overset{\sim}{X} \) can be written as

\[
\Lambda_{(M)} = \left\{ \sum_{J=1}^{III} \lambda_{(J)}^2 \cos^2 (\overset{\sim}{M}, \overset{\sim}{M}^J) \right\}^{1/3},
\]

where \( (\overset{\sim}{M}, \overset{\sim}{M}^J) \) denotes the angle between the direction \( \overset{\sim}{M} \) and the principal direction \( \overset{\sim}{M}^J \). Write the corresponding equation at \( \overset{\sim}{X} \).

8. Prove that the strain ellipsoid at \( \overset{\sim}{X} \) degenerates to a spheroid if and only if that at \( \overset{\sim}{X} \) does.

9. Show that a deformation which doubles all lengths leads to the following values of the basic invariants:

\[
I_C = 12, \quad II_C = 48, \quad III_C = 64,
\]

and

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\[ I_c = \frac{3}{4}, \quad \Pi_c = \frac{3}{16}, \quad \Pi = \frac{1}{64}. \]

10. Prove that a second order, symmetric tensor has only three basic invariants. If the square of \( \overset{2}{\sim} \) is defined by
\[ \overset{2}{\sim} = C_{\alpha \gamma} C_{\gamma \beta} \sim_{\alpha} \sim_{\beta} \]
with similar definitions for the higher powers of \( \overset{2}{\sim} \), show that all positive integer powers of \( \overset{2}{\sim} \) can be expressed as linear combinations of \( \overset{2}{\sim} \), \( C \) and \( \mathcal{J} \), where the coefficients in these linear relations are polynomials in the three basic invariants of \( \overset{2}{\sim} \).

11. Show that \( d a_{i} = J X_{\alpha, i} \frac{dA}{\alpha} \), where \( d a_{i} \) and \( \frac{dA}{\alpha} \) are components of element of area vectors at \( x \) and \( X \), respectively.

12. If \( a_{i} = X_{\alpha, i} A_{\alpha} \), show that \( a_{[i, j]} = X_{\alpha, i} X_{\beta, j} A_{[\alpha, \beta]} \), where \( \sim = a_{i}(\sim) \sim_{i} \) and \( \sim = A_{\alpha}(X) \sim_{\alpha} \) are vector fields defined on \( C \) and \( C_{0} \), respectively.

13. Show that the greatest (least) change of area occurs in an element normal to the direction of the least (greatest) stretch.

14. Show that the strain tensors \( \varepsilon \) and \( \overset{2}{\varepsilon} \) are related by
\[ E_{\alpha \beta} = e_{ij} X_{i, \alpha} X_{j, \beta}, \quad e_{ij} = E_{\alpha \beta} X_{\alpha, i, \beta, j}. \]

15. Prove that, in any deformation,
\[ \left( \frac{1}{3} I_c \right)^{3/2} \leq \frac{dv}{dV} \leq \left( \frac{1}{3} I_c \right)^{3/2}. \]
When do the equalities hold?
16. Prove the theorem of Kelvin and Tait, which states that in any deformation at least one direction is left unaltered. Discuss cases where more than one direction are rotationless.

17. Show that the stretch $\Lambda$, the extension $\Delta$, the rotation $\cos \nu$, and the elongation $\epsilon$ of an element are related by the following equation:

$$
\epsilon = \Lambda \cos \nu - 1 = \Delta \cos \nu - 2 \sin^2 \frac{1}{2} \nu.
$$

18. Verify the validity of the following relation:

$$
b_{ij}^n = R_{i\alpha}^n C_{\alpha\beta}^{-1} R_{\beta j}^n,
$$

where $b_{ij}^n$ and $C_{\alpha\beta}^n$ are, respectively, components of the tensors $b^n$ and $C^n$.

19. Consider an isochoric deformation.

a) Show that

$$
I_c = \Pi_c, \quad \Pi_c = I_c,
$$

and that

$$
I_c', \quad \Pi_c', \quad I_b', \quad \Pi_b \geq 3.
$$

b) Let $\delta^2 = \sum_{J=I}^{\Pi} \delta(J)$, where $\delta(J)$, $J = I, \Pi, \Pi$, are the principal extensions. Show that

$$
I_c - 3 = 0(\delta^2), \quad \Pi_c - 3 = 0(\delta^2).
$$

Compare these results with the general case, for which we may only conclude that

$$
I_c - 3 = 0(\delta), \quad \Pi_c - 3 = 0(\delta), \quad \text{and} \quad \Pi_c - 1 = 0(\delta).
$$
CHAPTER III

MOTION

In studying the deformation of a continuum, we distinguished two configurations of the body, the initial configuration \( C_0 \) and the final configuration \( C \). When the body moves and deforms under the action of external forces, its material points at each instant of time \( t \) occupy points in space that form the configuration \( C_t \) of the body at that instant of time (the subscript \( t \) is to stress that the configuration of the body is changing with time). We shall assume that the body \( B \) is in the initial configuration \( C_0 \) at the initial time \( t = t_0 = 0 \), and agree to identify the particles \( X \) of the body with their positions in this reference state. Hence, we may speak of material points \( \tilde{X} \) rather than the material points \( X \) which in \( C_0 \) were at points \( \tilde{X} \). The students must keep in mind the differences between particles and their positions as they were discussed in the preceding chapter. In line with the convention stated above, we may use Lagrangian or material variables \( \tilde{X} \) and \( t \) as independent variables, obtaining the so-called material description of motion. On the other hand, the use of Eulerian or spatial variables \( \tilde{x} \) and \( t \) yields a spatial description of motion.
3.1 Continuous Motion, Velocity, and Acceleration

A continuous motion of a body $B$ is characterized by the following single-valued, smooth, one-parameter family of transformations:

\[ \mathbf{x} = \mathbf{x}(\mathbf{x}, t), \ 0 \leq t < \infty, \quad (1.1a) \]

which possess unique inverse,

\[ \mathbf{x} = \mathbf{x}(\mathbf{x}, t), \ 0 \leq t < \infty, \quad (1.1b) \]

and which uniquely define the positions $\mathbf{x}$ of the particles $\mathbf{X}$ at each instant of time $t$. For a fixed $t$, the collection of the spatial points $\mathbf{x}$ that are occupied by the particles $\mathbf{X}$ constitutes the configuration $C_t$ of the body $B$ at the time $t$. We shall require that the motion (1.1a) and its inverse (1.1b) possess as many partial derivatives with respect to their arguments as needed, and that the Jacobian determinant of (1.1a), for each fixed $t$, be finite and positive at every particle $\mathbf{X}$.

Consider a particle $\mathbf{X}$ that at time $t$ is at point $\mathbf{x}$. The time-rate of change of its position vector $\mathbf{x}$ at $t$, i.e.,

\[ \dot{\mathbf{x}} = \dot{\mathbf{x}}(\mathbf{x}, t) = \frac{\partial \mathbf{x}}{\partial t} = \frac{\partial x_i}{\partial t} \mathbf{e}_i 
\]

\[ = x_i(\mathbf{x}, t) \mathbf{e}_i, \quad (1.2a) \]

\[ ^1 \text{As was agreed, the particles are referred to by their positions } \mathbf{X} \text{ in } C_0 \text{ at } t = 0. \]

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is called the velocity of the particle \( \mathbf{X} \) at time \( t \). Substitution from (1.1b) into (1.2a) yields

\[
\dot{\mathbf{X}} = \dot{\mathbf{X}}(\mathbf{X}(\mathbf{x}, t), t) = \mathbf{\dot{x}}(\mathbf{x}, t) = \dot{x}_i(\mathbf{x}, t) \mathbf{e}_i
\]

(1.2b)

which defines, for each \( t \), a vector field called the velocity field at time \( t \), and which gives the velocity of the particle that at time \( t \) is at point \( \mathbf{x} \). Equation (1.2a), on the other hand, yields the velocity of a given particle as a function of time \( t \). This equation is expressed in terms of the material variables \( \mathbf{X} \) and \( t \), while in (1.2b) spatial variables \( \mathbf{x} \) and \( t \) are used. Note that the right side of equations (1.2a) and (1.2b) are actually two distinct functions, although the same notation, namely \( \mathbf{x}(\cdot) \), is used for their representation.

A motion is called steady if its velocity field (1.2b) is independent of time;

\[
\dot{\mathbf{x}} = \mathbf{x}(\mathbf{x})
\]

(1.3a)

Note that the velocities of the particles, i.e.,

\[
\dot{\mathbf{X}} = \dot{\mathbf{X}}(\mathbf{X}, t)
\]

(1.3b)

are, in general, time-dependent, even if the motion is steady. In general, any quantity of motion that in its spatial description appears independently of time, i.e., is a function of place \( \mathbf{x} \) only, is said to be steady.
A motion is said to be plane if, with a suitable choice of the coordinate frame, its velocity field can be written as

$$
\dot{x}_1 = x_1(x_1, x_2, t) \\
\dot{x}_2 = x_2(x_1, x_2, t) \\
\dot{x}_3 = 0 .
$$

(1.3c)

A point at which $\dot{x} = 0$ is called a stagnation point.

The acceleration of a particle is the time-rate of change of its velocity. Thus, fix the particle $\mathbf{x}$. The partial time-derivative of (1.2a) then yields the acceleration $a(\mathbf{x}, t)$ of the material point $\mathbf{x}$, i.e.,

$$
a(\mathbf{x}, t) = \frac{\partial \mathbf{x}(\mathbf{x}, t)}{\partial t} \bigg|_{\mathbf{x} = \text{const}} = \frac{\partial \mathbf{x}(\mathbf{x}, t)}{\partial t} ,
$$

(1.4a)

where $\frac{\partial}{\partial t}$, as usual, defines partial derivative with respect to $t$ with all other variables held fixed. When the spatial description (1.2b) of the velocity is used, a simple partial time-derivative does not yield the acceleration, since it is the particle which is held fixed and not its position $\mathbf{x}$. The latter is a function of time through equation (1.1a). The time-derivative of any quantity of motion that is taken with material point $\mathbf{x}$ held fixed, is called the material derivative of that quantity. In particular, acceleration is obtained by taking the material derivative of the velocity. When Lagrangian or material description is used, the material derivative reduces to the usual partial time-derivative. In Eulerian or spatial description, on the other
hand, we must account for variation of $\tilde{x}$ with respect to $t$ when forming a material derivative. To distinguish between partial time-derivative and material derivative, we shall use a superposed dot or the symbol $\frac{d}{dt}$ to denote the material derivative. Hence, (1.4a) may also be written as

$$a(\tilde{x}, t) = \ddot{x}(\tilde{x}, t) = \frac{d^2 \tilde{x}(\tilde{x}, t)}{dt^2} = \frac{\partial^2 \tilde{x}(\tilde{x}, t)}{\partial t^2}$$

since material description is used. From (1.2b), on the other hand, we obtain

$$a(\tilde{x}, t) = \ddot{x}(\tilde{x}, t) = \frac{d}{dt} \left[ \dot{x}(\tilde{x}, t) \right]$$

$$= \frac{\partial \dot{x}(\tilde{x}, t)}{\partial t} + \dot{x}(\tilde{x}, t) \cdot \text{grad } \dot{x}(\tilde{x}, t)$$

$$= \left[ \frac{\partial \dot{x}_i}{\partial t} + \dot{x}_{i,j} \dot{x}_j \right] e_i \tag{1.4b}$$

for the acceleration field. The first term in the right side of the second or third line in (1.4b) defines the local acceleration while the second term corresponds to the convected acceleration. For a steady motion, the former is zero.

Using the notation $v(\tilde{x}, t) = v_i(\tilde{x}, t) e_i$, or $v(\tilde{x}, t) = v_i(\tilde{x}, t) e_i$, for the velocity, i.e.,

$$\dot{x}(\tilde{x}, t) = v(\tilde{x}, t)$$

we express (1.4) as

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\[ \ddot{z}(X, t) = \frac{\partial v(X, t)}{\partial t} \]  
\[ \quad , \quad (1.5a) \]

and also as

\[ \ddot{z}(\bar{x}, t) = \frac{\partial v(\bar{x}, t)}{\partial t} + v(\bar{x}, t) \cdot \text{grad} v(\bar{x}, t) \]  
\[ \quad , \quad (1.5b) \]

Note that \( \ddot{z}(\bar{x}, t) \) is the acceleration of the particle which at time \( t \) is at point \( \bar{x} \), while \( \ddot{z}(X, t) \) defines the acceleration of the particle \( X \) as a function of time \( t \). These are actually two distinct functions, and it will certainly be clearer to employ different notation for their representation. For example, we may use \( \ddot{z}(X, t) \) when material variables are employed and obtain

\[ \ddot{z}(\bar{x}, t) \equiv \ddot{z}(X(\bar{x}, t), t) \]  
while

\[ \ddot{z}(\bar{x}, t) \neq \ddot{z}(X, t) . \]

It is quite unfortunate that no such notation is commonly used, and--more unfortunate--we feel compelled to adhere to the common usage in these notes.

Consider now a vector-valued function \( \mathbf{F} = F(X, t) \) of material variables \( X \) and \( t \). It can be written as

\[ F(\bar{x}, t) \Rightarrow F(X(\bar{x}, t), t) = f(\bar{x}, t) \]

in terms of the spatial variables. The material derivative of this vector is.
\[ \dot{F} = \frac{\partial F}{\partial t} = \frac{\partial F_i(X, t)}{\partial t} e_i \]  

or

\[ \ddot{f} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \text{grad } \mathbf{f} \]

\[ = \left[ \frac{\partial f_i(X, t)}{\partial t} + v_j(X, t) f_{i, j}(X, t) \right] e_i \]

\[ = f_i(X, t) e_i \]  

Note that the vector-valued functions \( \dot{F}(X, t) \) and \( \ddot{f}(X, t) \) are related to one another through (1.1) by  

\[ \dot{F}(X, t) = \ddot{f}(X(t), t) \]  

and

\[ \ddot{f}(X, t) = \dot{F}(X(t), t) \]  

For a fixed particle \( X^0 \), Eq. (1.1a) becomes

\[ \dot{X} = \dot{X}(X^0, t) \]

which is the parametric equation of a curve called the path of the particle \( X^0 \). With a spatial point \( X^0 \) fixed in Eq. (1.1b), we obtain

\[ \frac{d}{dt} \mathbf{f} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \text{grad } f \]
\[ \chi = \chi(\chi^0, t') \]  

(1.9a)

which identifies the particle \( \chi \) that at time \( t' \) is at point \( \chi^0 \).

With \( t' \) as a parameter, (1.9a) yields all the particles which at some instant occupy the place \( \chi^0 \). Substitution from (1.9a) into (1.1a) yields

\[ \chi = \chi(\chi^0(\chi^0, t'), t) \]  

(1.9b)

which, with \( t' \) as the parameter and \( t \) fixed, defines a curve in space that passes through \( \chi^0 \). This curve is the locus of the spatial points which at time \( t \) are occupied by particles which either have already passed through \( \chi^0 \) or will do so in the future. This curve is called the streak line of \( \chi^0 \) at \( t \).

At any time \( t \), the velocity field is given by (1.2b) which defines a vector field. The field lines (which are curves tangent to these vectors) of the velocity-vector field at a given instant \( t \) are called the stream lines at that time. For \( t \) fixed, the integral curves of the system

\[ \frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} \]  

(1.10)

are the stream lines at \( t \). We note that the stream line through point \( \chi \) at time \( t \), the path line of the particle which is located at \( \chi \) at time \( t \), and the streak line of the point \( \chi \) at \( t \) all possess
a common tangent at $x$. In a steady motion all these curves are coincident, but, in general, they are distinct.

Various sets of material points may be identified as a body moves. A smooth collection of particles is called a **material manifold**. In the initial state $C_0$, let a material curve be defined by its parametric equation

$$\bar{x} = \bar{x}(\psi) , \; \psi_0 \leq \psi \leq \psi_1 ,$$  \hfill (1.11)

where $\psi$ is a real variable. As the body moves, this material curve is carried along and, at each instant of time $t$, is occupying a place in space that is given by

$$\bar{x} = \bar{x}(\bar{x}(\psi), t) , \; \psi_0 \leq \psi \leq \psi_1 .$$  \hfill (1.12)

Note that, by (1.11), we must have

$$\bar{x}(\psi) = \bar{x}(\bar{x}(\psi), 0) .$$

Similarly, we may consider material surfaces and material volumes. A material surface may be expressed by a two-parameter vector-valued function

$$\bar{x} = \bar{x}(\psi, Z)$$  \hfill (1.13)

in the initial state $C_0$, where $\psi$ and $Z$ are real variables. The material points of this material
surface move with the body and, at the instant \( t \), coincide with the following spatial surface

\[
x = x(\bar{x}(u, z), t).
\]  \hspace{1cm} (1.14)

Suppose now a moving spatial surface is given by the equation

\[
g(x, t) = 0.
\]  \hspace{1cm} (1.15a)

The question arises, under what conditions does (1.15a) define a material surface. This question can be resolved using Lagrange's criterion which states that a necessary and sufficient condition for \( g(x, t) = 0 \) to constitute a material surface is the vanishing of its material derivative, i.e.,

\[
\dot{g}(x, t) = \frac{\partial g(x, t)}{\partial t} + \dot{x} \cdot \nabla g(x, t) = 0.
\]  \hspace{1cm} (1.15b)

For proof see Problem 2 at the end of this chapter.
3.2 Instantaneous Pure Deformation, Deformation-Rate Tensor

At a given instant \( t \), let a body \( B \) be in an instantaneous configuration \( C_t \) defined by the collection of the spatial points \( \mathbf{x} \) that are instantaneously occupied by the particles \( \mathbf{X} \). The rate at which these particles move to a neighboring configuration is evidently defined by the velocity field \( \mathbf{v} = \mathbf{v}(\mathbf{x}, t) \).

Consider a generic particle \( \mathbf{x} \) that at time \( t \) is at point \( \mathbf{x} \). The instantaneous pure deformation of the material neighborhood of this particle may clearly be defined by the rate of stretch and shear of the material elements emanating from \( \mathbf{x} \). Let \( d\mathbf{x} \) and \( \delta \mathbf{x} \) be two such material elements with respective spatial representations \( d\mathbf{x} \) and \( \delta \mathbf{x} \) at time \( t \). The material derivative of \( d\mathbf{x} \cdot \delta \mathbf{x} \) is

\[
\frac{d}{dt} (d\mathbf{x} \cdot \delta \mathbf{x}) = \frac{d}{dt} \left( d\mathbf{x}_i \delta \mathbf{x}_i \right) = d\mathbf{v} \cdot \delta \mathbf{x} + d\mathbf{x} \cdot \delta \mathbf{v} = (v_i, j + v_j, i) \, d\mathbf{x}_i \delta \mathbf{x}_j
\]

\[
= 2 \, v_{(i,j)} \, d\mathbf{x}_i \delta \mathbf{x}_j
\]

\[
= 2 \, D_{ji} \, d\mathbf{x}_i \delta \mathbf{x}_j \quad ,
\]

where the second order, symmetric tensor

\[
\mathbf{D}(\mathbf{x}, t) = D_{ij}(\mathbf{x}, t) \, \mathbf{e}_i \mathbf{e}_j
\]

\[
= v_{(j,i)} \, \mathbf{e}_i \mathbf{e}_j
\]
is called the rate of deformation tensor or simply stretching tensor.

When \( \delta \dot{x} = \mu \, ds \), \( \mu \) being a unit vector along \( \delta \dot{x} \), (2.1) reduces to

\[
\frac{d}{dt} (ds)^2 = 2D_{ij} \, dx_i \, dx_j ,
\]

(2.3a)

or to

\[
\frac{(ds)}{ds} = D_{ij} \mu_i \mu_j = D(\mu) ,
\]

(2.3b)

where \( D(\mu) \) is the normal component of \( \delta \dot{x} \) in the direction \( \mu \). Thus the rate of stretch\(^1\) \( \frac{(ds)}{ds} \) in the direction \( \mu \) is equal to the normal component \( D(\mu) \) of the stretching tensor \( \delta \dot{x} \) in this direction. The stretching of the elements instantaneously along the coordinate axes are, therefore, given by the normal components \( D_{11}, D_{22}, D_{33} \), of \( \delta \dot{x} \) in the corresponding direction of these axes.

Let us now take \( \delta \dot{x} = \mu \, ds \), \( \delta x = \nu \, ds \), and let \( \cos \theta = \mu \cdot \nu \), where \( \theta \) is the angle between \( \delta \dot{x} \) and \( \delta x \). Substitution of these values into (2.1) yields

\[
(D(\mu) + D(\nu)) \cos \theta - \hat{\theta} \sin \theta = 2D_{ij} \mu_i \nu_j
\]

(2.3a)

which, for \( \theta = \frac{\pi}{2} \), reduces to

\[
-\frac{1}{2} \hat{\theta} = D_{ij} \mu_i \nu_j ,
\]

(2.4b)

\(^1\) This is also called the rate of extension
where \(-\frac{1}{2} \dot{\theta}\) is called the rate of orthogonal shear or simply the orthogonal shearing\(^1\) of the directions \(\sim\) and \(\sim\). Hence the orthogonal shearing of the material elements with instantaneous directions \(\sim\) and \(\sim\) is equal to the shear component of the stretching tensor for these directions. In particular, the orthogonal shear-rate of the material elements instantaneously along the coordinate directions \(\sim_i\) and \(\sim_j\) is given by \(D_{ij}; i \neq j\).

From the above results, we see that the instantaneous pure deformation of a material neighborhood is completely defined by the deformation-rate tensor \(\sim\). Since \(\sim\) is a real, second order, symmetric tensor, many of its properties are the same as those of Green's deformation tensor \(\zeta\); except for the fact that \(\zeta\) is also positive-definite, while \(\sim\) may or may not be positive-definite. Thus \(\sim\) has three real principal values \(D_J, J = I, II, III\), which are the roots of the equation

\[
\text{det} \left| D_{ij} - D \delta_{ij} \right| = 0, \quad (2.5.1)
\]

or, equivalently, the equation

\[
D^3 - I_D D^2 + II_D D - III_D = 0, \quad (2.5.2)
\]

\(^1\) Note that \(-\dot{\theta}\) is the rate of decrease of the angle of the orthogonal material elements instantaneously at point \(x\) at time \(t\). Many authors define \(-\dot{\theta}\) as the shear-rate rather than \(-\frac{1}{2} \dot{\theta}\).
where
\[ I_D = \text{tr} \mathcal{A} = D_{ii}, \]
\[ II_D = \frac{1}{2} \epsilon_{ijk} \epsilon_{ilm} D_{jkl} D_{km}, \]
and
\[ III_D = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} D_{ijl} D_{jkm} D_{kn} \]  \hspace{1cm} (2.5c)
are the basic invariants of \( \mathcal{A} \). The principal values of \( \mathcal{A} \) may be positive, negative, or zero. The corresponding principal directions \( \nu^J, J = I, II, III, \) are defined by
\[ D_{ij} \nu^J_i = D_{j} \nu^J_j, \quad J = I, II, III, \] (no sum on \( J \)). \hspace{1cm} (2.6)

When the principal values are distinct, then the corresponding principal directions are orthogonal. Note that, since \( \mathcal{A} \) is not necessarily a positive-definite tensor, its principal values are not necessarily all positive, and its quadric, called quadric of deformation-rate (or quadric of stretching), may be an ellipsoid or a hyperboloid or any one of their degenerate forms. For \( \mathcal{A} \) negative-definite, the material neighborhood is instantaneously waning, while for \( \mathcal{A} \) positive-definite, this neighborhood is instantaneously waxing. When \( \mathcal{A} \) is indefinite, on the other hand, some elements of the neighborhood are shrinking while others may be dilating. In this case, the quadric of \( \mathcal{A} \) is a hyperboloid whose asymptotic cone separates the waning elements from those that are waxing. In terms of the principal quantities, the tensor \( \mathcal{A} \) can be written as...
\[ \mathcal{D} = D_{ij} \varepsilon_i \varepsilon_j = \sum_{J=I}^{III} D_J \psi_i^J \psi_j^J \varepsilon_i \varepsilon_j, \]  

(2.7a)

and its quadric as

\[ \pm K^2 = \sum_{J=I}^{III} D_J (dx^J)^2, \]  

(2.7b)

where \( dx^J, J = I, II, III, \) measure distances along the directions \( \psi^J, \) and where the sign of the left-hand side of this equation is to be so selected as to render the quadric a real surface.

The **deviator tensor** \( \mathcal{D}' \) of the deformation-rate tensor \( \mathcal{D} \) is defined by

\[ \mathcal{D}' = D'_{ij} \varepsilon_i \varepsilon_j = (D_{ij} - \frac{1}{3} I D \delta_{ij}) \varepsilon_i \varepsilon_j \]  

(2.8a)

whose trace is zero; \( \text{tr} \, \mathcal{D}' = D'_{ii} = 0. \) The instantaneous deformation is said to be dilatational if the deviator \( \mathcal{D}' \) is zero; in this case \( \mathcal{D} \) is a **spherical tensor** and we have \( \mathcal{D} = \frac{1}{3} I D \mathcal{D}. \) Note that \( \mathcal{D} \) can always be written as a sum of its deviator \( \mathcal{D}' \) and a spherical tensor \( \mathcal{D}'' \) as follows:

\[ \mathcal{D} = \mathcal{D}' + \mathcal{D}'', \]

where \( \mathcal{D}' = \mathcal{D} - \frac{1}{3} I D \mathcal{D}, \) and \( \mathcal{D}'' = \frac{1}{3} I D \mathcal{D}. \)  

(2.8b)

Clearly enough, the deviator \( \mathcal{D}' \) is a measure of instantaneous

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distortion, and, for this reason, it is often called the distortion tensor, while the spherical part $S''$ corresponds to dilatation.

Let us now seek those pairs of orthogonal directions for which the shear-rates have extreme values called principal shear-rates; the corresponding directions are called principal directions of shearing. Let $\mu$ and $\nu$ be two orthogonal unit vectors,

$$\mu_i \mu_i - 1 = 0, \quad \nu_i \nu_i - 1 = 0, \quad \mu_i \nu_i = 0$$  \hspace{1cm} (2.9)

The shear-rate for these directions is $D_{ij} \mu_i \nu_j$, which is to be maximized (minimized) subject to the constraints (2.9). Let \( \ell \), \( \eta \), and \( \zeta \) denote the Lagrangian multipliers, and consider the following expressions:

$$Y = D_{ij} \mu_i \nu_j - \frac{1}{2} \ell (\mu_i \mu_i - 1) - \eta \mu_i \nu_i$$

$$Z = D_{ij} \mu_i \nu_j - \frac{1}{2} \zeta (\nu_i \nu_i - 1) - \eta \mu_i \nu_i$$

Setting $\partial Y / \partial \mu_k$ and $\partial Z / \partial \nu_k$ equal to zero, we obtain

$$D_{kj} \nu_j = \ell \mu_k + \eta \nu_k$$  \hspace{1cm} (2.10)

$$D_{kj} \mu_j = \zeta \nu_k + \eta \mu_k$$

from which it immediately follows that

$$\ell = \zeta \quad \text{and} \quad D_{kj} \mu_k \nu_j = \eta$$  \hspace{1cm} (2.11)

Equations (2.10) now yield

$$D_{ij} (\mu_j + \nu_j) = (\eta + \ell)(\mu_i + \nu_i)$$
\[ D_{ij} (\mu_j - \nu_j) = (n - m)(\mu_i - \nu_i) \]  

(2.12)

which state that \( n + m \) and \( n - m \) are the principal stretch-rates; the corresponding principal directions are given by \( \mu_i + \nu_i \) and \( \mu_i - \nu_i \), respectively. We thus set

\[ n + m = D_J \quad , \quad n - m = D_K \quad , \]

\[ \mu_J = \frac{\mu_i + \nu_i}{|\mu_i + \nu_i|} = \frac{\sqrt{2}}{2} (\mu + \nu) \quad , \]

\[ \mu^K = \frac{\mu_i - \nu_i}{|\mu_i - \nu_i|} = \frac{\sqrt{2}}{2} (\mu - \nu) \quad , \quad J, K = I, II, III \]

\[ J \neq K \quad , \]

and conclude that the principal shear-rates are given by

\[ m = \frac{1}{2} (D_J - D_K) \quad , \quad J \neq K \quad , \]

(2.13)

with the corresponding principal directions of shearing defined by

\[ \mu = \frac{\sqrt{2}}{2} (\mu_J + \mu^K) \quad , \text{and} \quad \nu = \frac{\sqrt{2}}{2} (\mu_J - \mu^K) \quad , \quad J \neq K \quad . \]

(2.14)

The principal directions of shearing \( \mu \) and \( \nu \), therefore, lie in the plane of the principal directions of stretch-rate, \( \mu_J \) and \( \mu^K \), and bisect the angles formed by these latter vectors; half of the difference between the corresponding principal stretch-rates is equal to the principal shearing.
3.3 Instantaneous Rotation, Spin and Vorticity

Let \( \sim n \) be a fixed unit vector. Denote by \( \phi_{(n\nu)} \) the angle that an element \( d\sim X \) with a unit vector \( \sim \nu \) forms with the direction of \( \sim n \). The spin of the material element \( d\sim X \) instantaneously at \( d\sim X \), relative to the direction \( \sim n \), is the angular-rate-\( \dot{\phi}_{(n\nu)} \), where the right-hand rule is to be employed for the sign convention as follows:

if \( \sim m \) is a vector perpendicular to the plane which is parallel to the vectors \( \sim n \) and \( \sim \nu \), the triad \( \sim \nu, \sim n, \sim m \) being right-handed, then positive-\( \dot{\phi}_{(n\nu)} \) corresponds to an angular-rate which has the same sense of rotation as a right-handed screw that progresses in the positive \( \sim m \)-direction. We have

\[
\cos \phi_{(n\nu)} = \sim \nu \cdot \sim n = \frac{dx}{ds} \cdot \sim n,
\]  

where \( ds \) is the instantaneous length of the element. Taking the material derivative of both sides of (3.1a), we obtain

\[
- \dot{\phi}_{(n\nu)} \sin \phi_{(n\nu)} = v_{i,j} n_i \nu_j - D(\nu) \cos \phi_{(n\nu)}
\]  

where \( v_{i,j} = \dot{x}_{i,j} \) is the velocity-gradient at time \( t \), and \( D(\nu) \) is the normal component of the deformation-rate tensor \( D \) in the direction \( \sim \nu \). For \( \phi_{(n\nu)} = \frac{\pi}{2} \), (3.1b) reduces to

\[
- \dot{\phi}_{(n\nu)} = v_{i,j} n_i \nu_j
\]  

which is the spin of the element along \( \sim \nu \) relative to the fixed direction \( \sim n \), to which \( \sim \nu \) is instantaneously perpendicular. In particular, let \( \sim n \)
be the unit vector \( \mathbf{e}_2 \) along the \( x_2 \)-axis, and consider an element instantaneously in the \( x_1 \)-direction. Equation (3.1c) then yields

\[
\dot{\varphi}_{(21)} = v_{2,1}
\]

which is the spin of the element instantaneously along the \( x_1 \)-axis relative to the \( x_2 \)-axis. Similarly, \(-v_{1,2} \) represents the spin of the element along the \( x_2 \)-axis about the \( x_1 \)-axis. Hence, \( \frac{1}{2}(v_{1,2} + v_{2,1}) \) which is the shear-rate of the \( x_1 \), \( x_2 \)-directions, is half of the sum of the negative spins of these directions. Note that positive \( \dot{\varphi}_{(21)} \) denotes a clockwise and positive \( \dot{\varphi}_{(12)} \) a counter clockwise rotation of elements instantaneously along the \( x_1 \), \( x_2 \)-directions, respectively.

The velocity-gradient tensor \( \nabla \sim \nabla = v_{i,j} \mathbf{e}_j \mathbf{e}_i \) can be written as a sum of a symmetric and an antisymmetric part, i.e.,

\[
v_{i,j} = \frac{1}{2}(v_{i,j} + v_{j,i}) + \frac{i}{2}(v_{i,j} - v_{j,i})
\]

\[
= v(i,j) + v[i,j] = D_{ji} + W_{ji}
\]

The skew-symmetric tensor \( \sim = W_{ji} \mathbf{e}_j \mathbf{e}_i \) is called Cauchy's spin tensor; its trace is zero and its components represent halves of the differences of the relative spins of the elements instantaneously along the respective coordinate directions.

1 Since \( \nabla \sim \nabla \) is not symmetric, the order of subscripts must be carefully noted.
The dual vector \( w \) of the spin tensor \( \w \) is called the vorticity vector; it is given by

\[
\w = \nabla \times \w = \text{curl } \w = - (e_{ijk} W_{ji}) \varepsilon_k
\]  
(3.3a)

or

\[
w_k = e_{ijk} W_{ij}
\]  
(3.3b)

Solving for \( W_{ij} \), we obtain from Eq. (3.3b)

\[
W_{ij} = \frac{1}{\varepsilon} e_{ijk} w_k
\]  
(3.3c)

The vector field defined by the vorticity vectors at an instant \( t \) is called the vorticity field at this time. The field lines of this vector field are the vortex lines.

The vorticity vector at each point represents half of the angular velocity of the instantaneous rigid rotation of the material neighborhood at that point. Since the rates of shear of the material elements that are instantaneously along the principal triad of the deformation-rate tensor \( \mathcal{G} \) are zero, these elements, in addition to pure stretching, undergo an instantaneous rotation with the angular velocity \( \frac{1}{2} \w \). To prove this assertion, consider an element \( dx = \nu ds \) and evaluate the material derivative of the unit vector \( \nu \) as follows:

\[
\begin{align*}
\dot{\nu}_i &= \frac{d}{dt} \left( \frac{dx_i}{ds} \right) = \frac{1}{ds} \left[ dx_i - D_{ij} \nu_j \right] \\
&= \frac{dx_i}{ds} \left[ \delta_{ij} - D_{ij} \nu_j \delta_{ij} \right] \\
&= \nu_j \left[ D_{ji} + W_{ji} - D_{ij} \nu \delta_{ij} \right]
\end{align*}
\]  
(3.4a)
Now, if \( \nu \) is taken in the direction of one of the principal axes at \( \mathbf{x} \), it must satisfy the following equation:

\[
D_{ij} \nu^J_j = D_J \nu^J_i, \quad J = \text{I, II, III}, \quad \text{(no sum on } J) \]

In this case, Eq. (3.4a) reduces to

\[
\nu^J_i = W_{ji} \nu^J_j, \quad (3.4b)
\]

which may also be written as

\[
\nu^J_i = \frac{1}{2} e_{jik} W_{kj} \nu^J_j, \quad (3.4c)
\]

or as

\[
\nu^J_i = \frac{1}{2} (\omega \times \nu^J) \quad (3.4d)
\]

which proves the assertion. In obtaining (3.4d) from (3.4a), due attention must be paid to the order of subscripts which occur in the decomposition (3.2) (see the remark that was made following equation (4.5b) of Chapter I).

Using the results of this and the preceding sections, we shall now study the instantaneous motion of a material neighborhood as follows: Let \( v_i(x, t) \) be the velocity of a typical particle \( X \) situated, at the instant \( t \), at the point \( x \). Consider a neighboring particle \( X' \) which is at \( x + dx \) at the same instant \( t \). For a sufficiently smooth velocity field, the velocity of \( X' \) can be expressed as

\[
v_i(x + dx, t) = v_i(x, t) + \frac{\partial v_i(x, t)}{\partial x_j} \, dx_j + \ldots
\]
Hence, to within the first order of approximation in \( dx_i \), the velocity of \( X' \) is given by

\[
v_i(x + dx, t) = v_i(x, t) + v_{i,j}(x, t) \, dx_j
\]

\[
= v_i(x, t) + dx_j(D_{ji} + W_{ji})
\]  

(3.5a)

where \( D_{ji} \) and \( W_{ji} \) are defined by (3.2). This equation may also be written as

\[
v(x + dx, t) = v(x, t) + d\overset{\sim}{x} \cdot \overset{\sim}{\omega} + dx \cdot \overset{\sim}{\omega}
\]

\[
= v(x, t) + dx \cdot \overset{\sim}{\omega} + \frac{1}{2} \overset{\sim}{\Omega} \times dx
\]  

(3.5b)

which states that the instantaneous motion of a typical material neighborhood, in general, consists of a rigid-body translation, a pure deformation, and a rigid-body rotation.

A material neighborhood will instantaneously undergo only rigid-body translation and rotation if the deformation-rate tensor \( \overset{\sim}{\omega} \) vanishes there. The instantaneous rotation is completely defined by the antisymmetric tensor \( \overset{\sim}{\Omega} \). If, on the other hand, the spin tensor \( \overset{\sim}{\Omega} \) is zero at a neighborhood of the particle \( X \), this neighborhood, in general, performs an instantaneous rigid-body translation and pure deformation; in this case, the velocity of the particle \( X' \) relative to \( X \) is given by \( dx_j D_{ji} \). Thus an instantaneous pure deformation, in general, involves instantaneous change in the direction of material line elements;
the directions of material elements instantaneously in the principal
directions of the stretch-rate tensor \( \mathbf{\alpha} \) are, however, left unaltered
by an instantaneous pure deformation.

We shall close this section by deriving Beltrami's diffusion
equation for the vorticity vector \( \mathbf{\gamma} \). To this end, we first note the
following vector identity:

\[
\mathbf{\gamma} \cdot \text{grad} \mathbf{\gamma} = \mathbf{\gamma} \times \mathbf{\gamma} + \text{grad} \frac{1}{2} \mathbf{v}^2
\]

where \( \mathbf{v}^2 = \mathbf{\gamma} \cdot \mathbf{\gamma} = \gamma_i \gamma_i \) is the square of the speed. We then
take the curl of both sides of Eq. (1.5b) and using (3.6) obtain

\[
curl \mathbf{\gamma} = \frac{\partial \mathbf{\gamma}}{\partial t} + curl (\mathbf{\gamma} \times \mathbf{\gamma})
\]

\[
= \frac{d \mathbf{\gamma}}{dt} - \mathbf{\gamma} : \text{grad} \mathbf{\gamma} + \mathbf{\gamma} \cdot \text{div} \mathbf{\gamma}
\]

Hence, we deduce

\[
\frac{d}{dt} \left( \frac{\mathbf{\gamma}}{\rho} \right) = \frac{\mathbf{\gamma}}{\rho} \cdot \text{grad} \mathbf{\gamma} + \frac{1}{\rho} \text{curl} \mathbf{\gamma}
\]

which expresses the rate of change of vorticity for a general continuum.
To obtain (3.7), we have also employed the following continuity equation
which will be discussed in Section 5 of this chapter:

\[
\frac{d}{dt} (\log \rho) + \text{div} \mathbf{\gamma} = 0
\]
3.4 Instantaneous Plane Motion

Let us consider a special instantaneous motion called plane.

An instantaneous motion is said to be plane if, with a suitable choice of the rectangular Cartesian coordinate system, the instantaneous velocity field can be written in the form

\[ v_1 = v_1(x_1, x_2) \quad , \quad v_2 = v_2(x_1, x_2) \quad , \quad v_3 = 0 \quad . \quad (4.1) \]

The vorticity vector \( \omega \), with the components

\[ w_1 = w_2 = 0 \quad , \quad w_3 = \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \quad , \quad (4.2) \]

is instantaneously parallel to the \( x_3 \)-axis. A positive value of \( w_3 \) represents a counterclockwise rotation-rate of the considered material neighborhood about an axis that is parallel to the \( x_3 \)-axis.

The stretch-rate in the direction of the unit vector \( \mu \) with the components

\[ \mu_1 = \cos \phi \quad , \quad \mu_2 = \sin \phi \quad , \quad \mu_3 = 0 \quad , \]

is given by

\[ D(\mu) = D_{11} \cos^2 \phi + D_{22} \sin^2 \phi + D_{12} \sin 2\phi \]

\[ = \frac{1}{2} (D_{11} + D_{22}) + \frac{1}{2} (D_{11} - D_{22}) \cos 2\phi + D_{12} \sin 2\phi \quad , \quad (4.3) \]

where \( \phi \) is the angle formed by the direction of \( \mu \) and the positive \( x_1 \)-direction. If \( \mu \) is a unit vector in the \( x_1, x_2 \)-plane, forming a right-handed orthogonal triad with \( \mu \) and the positive \( x_3 \)-direction, we have

\[ v_1 = -\sin \phi \quad , \quad v_2 = \cos \phi \quad , \quad v_3 = 0 \quad . \]
The shear-rate for the material elements instantaneously along the \( \mu \) and \( \nu \)-directions then is

\[
- \frac{1}{2} \ddot{\theta} (\mu \nu) = - \frac{1}{2} (D_{11} - D_{22}) \sin 2 \varphi + D_{12} \cos 2 \varphi \quad .
\] (4.4)

In the plane \( x_3 = 0 \), and along the principal directions of the stretch-rate, we now choose a new system of rectangular coordinates \( x'_1 \), \( x'_2 \), and \( x'_3 \), and label them in such a manner that the principal stretch-rates \( D_1 \) of the \( x'_1 \)-direction and \( D_\Pi \) of the \( x'_2 \)-direction, satisfy the condition \( D_1 \geq D_\Pi \). Denoting by \( \varphi' \) the angle formed by the direction of \( \mu \) and the positive \( x'_1 \)-direction, we reduce (4.3) and (4.4) to

\[
D(\mu) = \frac{1}{2} (D_1 + D_\Pi) + \frac{1}{2} (D_1 - D_\Pi) \cos 2 \varphi' \quad (4.5)
\]

\[
- \frac{1}{2} \ddot{\theta}(\mu \nu) = - \frac{1}{2} (D_1 - D_\Pi) \sin 2 \varphi' \quad .
\] (4.6)

A line element instantaneously parallel to the \( x'_1 \), \( x'_2 \)-plane has an angular velocity \( \Omega \) parallel to the \( x_3 \)-axis and equal in magnitude to the sum of \( \frac{1}{2} w_3 \) and \( - \frac{1}{2} \ddot{\theta}(\mu \nu) \), i.e.,

\[
\Omega = \frac{1}{2} w_3 - \frac{1}{2} (D_1 - D_\Pi) \sin 2 \varphi' \quad .
\] (4.7)

Equations (4.5) and (4.7) completely define the instantaneous motion of the material neighborhood of a given particle. They may be given the following graphical interpretation which has been developed by Prager.

In a plane, called the plane of relative velocities, we represent angular velocity and stretch-rate of an element instantaneously along the unit vector \( \mu \) by the abscissa \( \Omega \) and the ordinate \( D \) of a point
As the angle $\phi'$ of this direction varies, the representative point $M$ describes a circle, called the circle of relative velocities, whose center $O$ has the coordinates $\Omega = \frac{1}{2} w_3$ and $D = \frac{1}{2} (D_I + D_{II})$, and whose radius is $\frac{1}{2} (D_I - D_{II})$, see Fig. 4.1. On this circle the highest point $A_I$ and the lowest point $A_{II}$ have the maximum $D_I$ and the minimum $D_{II}$ stretch-rates, and they correspond, respectively, to the principal $x'_1$ and $x'_2$-directions. The abscissa of the point $O$ represents the angular velocity $\frac{1}{2} w_3$ of the rigid-body rotation. If the $D$-axis is translated parallel to itself so that it passes through the point $O$, we obtain a construction known as Mohr's circle. This gives only the stretch-rates of line elements and shear-rates for pairs of orthogonal line elements, but no angular velocities.

To locate on the circle the point $M$ that corresponds to the direction $\mu$, we measure the angle $A_IOM$ equal to $2 \phi'$. A line drawn through $M$ in the direction of $\mu$ intersects the circle at a point $P^*$ which is the pole of this circle. Since the angle $A_IP^*M$ is equal to $\phi'$, the line $A_I P^*$ is in the principal $x'_1$-direction. The stretch-rate and the angular velocity of a given line element is now defined, respectively, by the ordinate and the abscissa of the second intersection of the circle with a line through $P^*$ that has the direction of the considered line element. To obtain the angular distortion-rate of the two line elements emanating from the considered
particle, we first draw lines through $P^*$ in the respective directions of these elements to intersect the circle. The difference between the abscissae of these intersection points now is the desired value of the angular distortion-rate. Since this quantity is a maximum for points $B_1$ and $B_2$ in Fig. 4.1, we conclude that the orthogonal directions represented by these points correspond to the directions of the maximum shear-rate, that is, they are the principal directions of the shear rate.

Fig. 4.1

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3.5 Strain-Rate, Kinematics of Surface and Volume Elements

Since all strain measures that are introduced in Chapter II are functions of the deformation gradient \( x_{i,\alpha} \), we first calculate the material derivative of this quantity as follows:

\[
\frac{d}{dt} (x_{i,\alpha}) = \frac{\partial}{\partial t} \left( \frac{\partial x_i(X, t)}{\partial X_{\alpha}} \right)
\]

\[
= \frac{\partial x_i(X, t)}{\partial X_{\alpha}} = \dot{x}_{i,\alpha}
\]

\[
= \frac{\partial x_i}{\partial x_j} \frac{\partial x_j}{\partial X_{\alpha}} = v_{i,j} x_{j,\alpha} \quad . \tag{5.1a}
\]

From the identity \( x_{i,\alpha} x_{\alpha,j} = \delta_{ij} \), we now obtain the material derivative of \( X_{\alpha,j} \):

\[
\frac{d}{dt} (x_{i,\alpha} X_{\alpha,j}) = \dot{x}_{i,\alpha} X_{\alpha,j} + x_{i,\alpha} \frac{d}{dt} (X_{\alpha,j}) = 0
\]

or

\[
x_{i,\alpha} \frac{d}{dt} (X_{\alpha,j}) = -\dot{x}_{i,\alpha} X_{\alpha,j}
\]

which yields

\[
\frac{d}{dt} (X_{\beta,j}) = -\dot{x}_{i,\beta} x_{\beta,i} \quad . \tag{5.1b}
\]
To obtain the material derivative of the Lagrangian strain tensor \( \tilde{\varepsilon} \), we form the material derivative of both sides of equation (II-5.9c) and, using (2.3a), arrive at

\[
\frac{d}{dt} (ds)^2 = 2 \dot{E}_{\alpha \beta} \frac{dX_\alpha}{\alpha} \frac{dX_\beta}{\beta} = 2 D_{ij} x_i, \alpha \ x_j, \beta \ dx_\alpha \ dx_\beta. \tag{5.2a}
\]

Hence

\[
\dot{E}_{\alpha \beta} = D_{ij} x_i, \alpha \ x_j, \beta. \tag{5.2c}
\]

Now, noting equation (II-5.9b), we obtain

\[
\dot{C}_{\alpha \beta} = 2 \dot{E}_{\alpha \beta}. \tag{5.2d}
\]

for the material derivative of Green's deformation tensor \( \tilde{\varepsilon} \). The material derivative of Almansi's strain tensor \( \tilde{\varepsilon} \) can, similarly, be obtained using equations (II-5.6c), (2.3a). Hence we obtain

\[
\frac{d}{dt} (ds)^2 = 2 \dot{D}_{ij} dx_i \ dx_j = 2 \dot{e}_{ij} dx_i \ dx_j
\]

\[
+ 2 e_{ij} (\dot{x}_i, k \ dx_k \ dx_j + \dot{x}_j, k \ dx_k \ dx_i)
\]

\[
= 2 [\dot{e}_{ij} + e_{kj} v_{k, i} + e_{ki} v_{k, j}] dx_i \ dx_j. \tag{5.3a}
\]
or
\[
\dot{e}_{ij} = D_{ij} - (e_{kj} v_{k,i} + e_{ki} v_{k,j})
\]  
(5.3b)

The material rate of change of Cauchy's deformation tensor \( \zeta \) is now given by substitution from (II-5.12a) into (5.3b), yielding
\[
\dot{\zeta}_{ij} = -2 \dot{\zeta}_{ij} = - (c_{kj} v_{k,i} + c_{ki} v_{k,j})
\]  
(5.3c)

From Eqs. (5.2c) and (5.2d) we see that the vanishing of the deformation-rate tensor \( \dot{\zeta} \) is a sufficient condition to assure the vanishing of the components of the rate tensors \( \dot{x} \) and \( \zeta \). This implies that, in an instantaneous rigid-body motion, the components of the Lagrangian strain tensor \( \zeta \) and Green's deformation tensor \( \xi \) stay constant. Equations (5.3b) and (5.3c), on the other hand, reveal that the components of Almansi's strain tensor \( x \) and Cauchy's deformation tensor \( \zeta \), in general, appear variable in such a motion to an observer moving with the material elements.

An element of area \( da \) in an instantaneous configuration \( C_t \) is related to the corresponding element \( d\zeta \) in the reference state at the time \( t = 0 \) by the following equation (Prob. II-11):
\[
da_i = J \chi_{i} \nu_{\alpha} \varepsilon_{\alpha} \]
(5.4a)
Taking the material derivative of both sides of this equation, we obtain

\[
(da_i)' = \dot{J} \left( X_{\alpha, i} \right) \; dA_\alpha + J (X_{\alpha, i})' \; dA_\alpha .
\]  

(5.4b)

The material derivative of the Jacobian determinant is

\[
\dot{J} = \frac{d}{dt} \left[ \det | x_i, \alpha' \right] = \left( \frac{\partial J}{\partial x_i, \alpha} \right) \; (x_i, \alpha') .
\]

\[
= J \left( X_{\alpha, i} \right) \; \dot{x}_i, \alpha
\]

\[
= J \; v_{i, i} = J I_D ,
\]  

(5.5)

where, in addition to the result of Prob. II-1a, Equations (2.5c) and (5.1a) are employed. Substitution from (5.5) and (5.1b) into (5.4b) now yields

\[
(da_i)' = I_D \; da_i - v_{j, i} \; da_j .
\]  

(5.4c)

The material derivative of an element of volume \(dv\) may be easily calculated using equations (II-4.9b) and (5.5). One obtains

\[
(dv)' = \frac{d}{dt} (Jdv) = v_{i, i} \; dv = I_D dv .
\]  

(5.6)
Any elementary material volume \( dV \) of a continuum which consists of an incompressible material must remain constant for any continuous motion of the continuum. The material derivative of \( dV \) must, therefore, vanish and we must have

\[
I_D = v_i, i = \text{div} \nabla = 0 \quad .
\] (5.7b)

Consider now the conservation of mass expressed by Eq. (II-4.10).

This may also be written as

\[
\frac{d}{dt} (\rho J) = \frac{d}{dt} (\rho_0) = 0
\]

or

\[
\dot{J} + \rho I_D J = J [\frac{\partial \rho}{\partial t} + (v_i \rho), i] = 0 \quad .
\] (5.8a)

Since \( J \) is finite, (5.8a) becomes

\[
\frac{\partial \rho}{\partial t} + (v_i \rho), i = 0
\] (5.8b)

which may also be written as

\[
\frac{d}{dt} (\log \rho) + I_D = 0 \quad .
\] (5.8c)

Equation (5.8b) [or (5.8c)] is known as the spatial continuity equation.

According to Helmholtz's theorem, any sufficiently smooth vector field in a finite region may be represented as the sum of two fields: one an irrotational field whose curl is zero, and the other a solenoidal field whose divergence vanishes. Using this theorem, we may write

\[
\nabla = \nabla \phi + \nabla \times A \quad , \quad \nabla \cdot A = 0
\] (5.9a)

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Now, a motion with zero vorticity is called irrotational motion. The velocity field for irrotational motion is derivable from a scalar potential \( \phi \) called velocity potential, and (5.9a) reduces to

\[
\psi = \text{grad} \phi \\ (5.10a)
\]

Substitution from (5.9) into (5.8b) yields

\[
\dot{\rho} + \rho \nabla^2 \phi = 0 \\ (5.9b)
\]

If the continuum is incompressible, then \( \dot{\rho} = 0 \) and we obtain Laplace's equation

\[
\nabla^2 \phi = 0 \\ (5.10b)
\]

for the velocity potential \( \phi \). Hence, (5.10b) together with suitable boundary conditions define the potential flow of an incompressible continuum.
3.6 **Kinematics of Volume, Surface, and Line Integrals**

Let $\mathbf{J}(\mathbf{x}, t)$ denote a tensor-valued function of spatial variables $\mathbf{x}$ and $t$ that is defined on a material volume $\mathcal{V}$ at each instant $t$.

The material derivative of the integral

$$ I = \int_{\mathcal{V}} \mathbf{J}(\mathbf{x}, t) \, d\mathbf{v} \quad (6.1) $$

is

$$ \dot{I} = \frac{d}{dt} \int_{\mathcal{V}} \mathbf{J} \, d\mathbf{v} = \int_{\mathcal{V}} \left[ \frac{\partial \mathbf{J}}{\partial t} + \mathbf{J} \nabla \cdot \mathbf{v} \right] \, d\mathbf{v} $$

$$ = \int_{\mathcal{V}} \left[ \frac{\partial \mathbf{J}}{\partial t} + \mathbf{v} \times \mathbf{J} \right] \, d\mathbf{v} $$

$$ = \int_{\mathcal{V}} \left[ \frac{\partial \mathbf{J}}{\partial t} + (\mathbf{v} \times \mathbf{J}) \right] \, d\mathbf{v} $$

where equation (5.6) is also employed. Using Gauss' theorem (I-7.2b), we reduce (6.2a) to

$$ \frac{d}{dt} \int_{\mathcal{V}} \mathbf{J} \, d\mathbf{v} = \int_{\mathcal{V}} \frac{\partial \mathbf{J}}{\partial t} \, d\mathbf{v} + \int_{\partial \mathcal{V}} \mathbf{J} \cdot \mathbf{n} \, d\mathbf{a} $$

$$ \quad (6.2b) $$

where $\partial \mathcal{V}$ is the surface bounding $\mathcal{V}$, and $\mathbf{v}_{(\mathbf{n})} = \mathbf{v} \cdot \mathbf{n}$ is the projection of the velocity $\mathbf{v}$ on the direction of the exterior unit normal $\mathbf{n}$ to $\partial \mathcal{V}$. The balance equation (6.2b) is known as the transport theorem. It was first stated by Reynolds in 1903.
Consider now the flux of the vector quantity \( \mathbf{q}(x,t) \) across a material surface \( \mathcal{S} \):

\[
\int_{\mathcal{S}} \mathbf{q} \cdot d\mathbf{\mathcal{S}},
\]

(6.3)

where \( d\mathbf{\mathcal{S}} = \mathbf{\nu} \, d\mathbf{\mathcal{A}} \) is the element of vector area of \( \mathcal{S} \), and \( \mathbf{\nu} \) denotes the exterior unit normal to \( \mathcal{S} \). The material derivative of (6.3) is

\[
\frac{d}{dt} \int_{\mathcal{S}} \mathbf{q} \cdot d\mathbf{\mathcal{S}} = \int_{\mathcal{S}} d\mathbf{\mathcal{S}} \cdot \left[ \dot{\mathbf{q}} + \mathbf{q} \, (\text{div} \, \mathbf{\nu}) - \mathbf{q} \cdot \text{grad} \, \mathbf{\nu} \right].
\]

(6.4)

The flux of \( \mathbf{q} \) across any material surface \( \mathcal{S} \) is, therefore, constant if the quantity in the brackets vanishes. Conversely, if this flux is constant, then the right-hand side of (6.4) must vanish for every \( \mathcal{S} \).

Therefore, the condition

\[
\dot{\mathbf{q}} + \mathbf{q} \, (\text{div} \, \mathbf{\nu}) - \mathbf{q} \cdot \text{grad} \, \mathbf{\nu} = 0
\]

(6.5a)

is both necessary and sufficient to ensure that the flux of \( \mathbf{q} \) is constant across every material surface \( \mathcal{S} \). This result is known as Zorawski's criterion. It may also be expressed as

\[
\frac{\partial \mathbf{q}}{\partial t} + \nabla \times (\mathbf{q} \times \mathbf{\nu}) + \mathbf{\nu} \, (\nabla \cdot \mathbf{q}) = 0.
\]

(6.5b)
We now consider a material line \( C \) and let the vector-valued function \( \mathbf{p} = \mathbf{p}(\xi, t) \) be defined on \( C \). The material derivative of the integral

\[
\int_C \mathbf{p} \cdot d\xi,
\]

where \( d\xi = d\xi_i e_i \) is an element of \( C \), is

\[
\frac{d}{dt} \int_C \mathbf{p} \cdot d\xi = \int_C [\dot{\mathbf{p}} \cdot d\xi + \mathbf{p} \cdot (d\xi)']
\]

\[
= \int_C (\dot{p}_j + p_i v_{i,j}) \, d\xi_j.
\]

The integrand in the right side of (6.7) may be written as

\[
\dot{p}_j + p_i v_{i,j} = \dot{p}_j + (p_i v_i)_j - p_i,_{j} v_i,
\]

and since \( \oint_C (p_i v_i)_{,j} \, d\xi_j = 0 \) when \( C \) is a closed curve, we obtain

\[
\dot{J} = \oint_C [\dot{p}_j - p_i,_{j} v_i] \, d\xi_j.
\]

(6.8a)
The integral $J$ is called the circulation when the vector-valued function $p$ is replaced by the velocity vector $v$. In this case, (6.8a) reduces to

$$\frac{d}{dt} \oint_{C} v \, dl = \oint_{C} \dot{v} \, dl,$$  \hspace{1cm} (6.8b)

where $\dot{v} = a$ is the acceleration. Note that, in general, the material derivative and the integration around a closed material curve are not commutative; equation (6.8b) is a very special case.

From the definition of circulation and Stokes' theorem, equation (I-7.6), we obtain

$$\oint_{C} v \, dl = \int_{S} w \, da,$$ \hspace{1cm} (6.9)

where $w$ is the vorticity vector, and $S$ is any surface bounded by the closed curve $C$. This equation states that the circulation around a circuit $C$ is the same as the vortex-flux through a surface bounded by $C$. 

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3.7 Integrability Conditions

Having studied certain general properties of the deformation-rate tensor, we shall now consider the question of the compatibility of a given field of deformation-rate, that is, we shall seek conditions under which an arbitrarily prescribed field \( \mathbf{\omega} \) in a simply connected region\(^\dagger \) corresponds to a continuous single-valued velocity field. At the outset, it should be noted that such a velocity field can be unique only to within a rigid body motion, because rigid-body translations and rotations do not affect the deformation-rates. Thus, whenever we speak of a "unique velocity field" we shall mean that it is unique to within a rigid-body velocity.

For a prescribed velocity field, the deformation-rate tensor \( \mathbf{\omega} \) may be calculated by simple differentiation, see Eq. (3.2). Given a field \( \mathbf{\omega} \), on the other hand, six differential equations

\[
\frac{1}{2} (v_i, j + v_j, i) = D_{ij}
\]  

(7.1)

yield unique single-valued solutions for the three functions \( v_i \) only when certain restrictions are met by the six components of the tensor \( D_{ij} \). These restrictions are known as integrability or compatibility conditions.

\(^\dagger \) A region is called simply connected if, by a continuous deformation, every closed curve in this region can be reduced to a point without crossing the boundaries of the region.
Let the instantaneous velocity of a point $\mathbf{x}^o$ be denoted by $\mathbf{v}^o$. If to a given sufficiently smooth deformation-rate tensor $\mathbf{\dot{\gamma}}$, there corresponds a unique, continuous, single-valued velocity field $\mathbf{v}$, then the velocity $\mathbf{v}'$ of a point $\mathbf{x}'$ can be expressed as

$$
\mathbf{v}'_i - \mathbf{v}_i^o = \int_{\mathbf{x}^o}^{\mathbf{x}'} d\mathbf{v}_i = \int_{\mathbf{x}^o}^{\mathbf{x}'} \mathbf{v}_{i,j} \, d\mathbf{x}_j
$$

$$
= \int_{\mathbf{x}^o}^{\mathbf{x}'} (\mathbf{D}_{ji} + \mathbf{W}_{ji}) \, d\mathbf{x}_j , \quad (7.2a)
$$

where the integration path, which we assume to be a rectifiable curve, may be selected arbitrarily in the considered simply connected region $\mathcal{V}$. Note that the assumption of $\mathcal{V}$ being simply connected is rather essential here, since for a multiply connected region the velocity field may turn out to be multiple-valued.

Equation (7.2a) may be written as

$$
\mathbf{v}'_i = \mathbf{v}_i^o + \mathbf{W}_{ji}^o (x'_j - x_j^o) + \int_{\mathbf{x}^o}^{\mathbf{x}'} \left[ \mathbf{D}_{ji} + (x'_k - x_k^o) \mathbf{W}_{ki,j} \right] \, d\mathbf{x}_j , \quad (7.2b)
$$

where integration by parts is used, and where $\mathbf{W}_{ji}^o$ is the value of the spin tensor at point $\mathbf{x}^o$. Noting that

$$
\mathbf{W}_{ki,j} = \frac{1}{2} \{ \mathbf{v}_{i,k} - \mathbf{v}_{k,i} \}, \quad j = D_{ji}, \quad k - 1, j, i
$$

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we reduce (7.2b) to

\[ v'_i = v_i^0 + W_{ji}^0 (x'_j - x_j^0) + \int_{x^0}^{x'} [D_{ji} + (x'_k - x_k)(D_{ij}, k - D_{jk}, i)] \, dx_j \quad \text{(7.2c)} \]

For a smooth, single-valued velocity field, \( \nabla' (x', t) \) is uniquely defined by (7.2c) independently of a particular path of integration from \( x^0 \) to \( x' \). Therefore, the integrand

\[ U_{ij} = D_{ij} + (x'_k - x_k)(D_{ij}, k - D_{kj}, i) \]

in (7.2c) must be an exact differential, that is, we must have

\[ U_{i1} \, dx_1 + U_{i2} \, dx_2 + U_{i3} \, dx_3 = d\Phi_i \]

A necessary and sufficient condition for this is

\[ \Phi_i, jk = \Phi_i, kj \]

\[ = U_{ij}, k = U_{ik}, j \]

from which we obtain

\[ (x'_k - x_k)(D_{ij}, k\ell + D_{k\ell}, ij - D_{ik}, j\ell - D_{j\ell}, ik) = 0 \]

This is satisfied for all \( x_k \) in \( \nu \) if

\[ D_{ij}, k\ell + D_{k\ell}, ij - D_{ik}, j\ell - D_{j\ell}, ik = 0 \quad \text{(7.3)} \]

which are the desired integrability conditions. We note that if Eq. (7.3) is satisfied, the integrand in (7.2c) becomes an exact differential, rendering the velocity \( v'_i \) independent of the considered path of integration. This implies that (7.3) are sufficient conditions for integrability of (7.1). To show that they are also necessary, we assume the existence of a single-valued continuous velocity field \( v_i(x) \) of class \( C^3 \) and by successive differentiation obtain

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\[ D_{ij, k\ell} = \frac{1}{3} (v_i, jk\ell + v_j, ik\ell) \]

Now, interchanging the subscripts, we obtain

\[ D_{k\ell, ij} = \frac{1}{3} (v_k, \ell ij + v_\ell, kij) \]

Interchanging \( i \) and \( k \) in the first equation, and \( k \) and \( i \) in the second equation, we may now combine the results to arrive at (7.3).

Therefore, in a simply connected region, conditions (7.3) are both necessary and sufficient to ensure the existence of a single-valued continuous velocity field. There are 81 equations expressed by (7.3), out of which only 6 are independent; the others are either identities or repetitions of these six equations. The six independent equations are

\[
\begin{align*}
D_{11, 23} &= (D_{12, 3} + D_{31, 2} - D_{23, 1})_1, \\
D_{22, 31} &= (D_{23, 1} + D_{12, 3} - D_{31, 2})_2, \\
D_{33, 12} &= (D_{31, 2} + D_{23, 1} - D_{12, 3})_3,
\end{align*}
\]

\[
\begin{align*}
2D_{12, 12} &= D_{11, 22} + D_{22, 11}, \\
2D_{23, 23} &= D_{22, 33} + D_{33, 22}, \\
2D_{31, 31} &= D_{33, 11} + D_{11, 33}.
\end{align*}
\]

(7.4)

When the region \( \mathcal{U} \) is a multiply connected one, conditions (7.4) while still necessary, are no longer sufficient for the existence of a continuous single-valued velocity field. Since a multiply connected region can be reduced to a simply connected one by the introduction of suitable cuts, Eq. (7.2c) yields a unique velocity \( v_i' \) if the integration path does not cross any one of those cuts and (7.4) is also satisfied. Additional conditions are now obtained by the requirement that the velocity should be continuous across the cuts. In general, we need
m - 1 cuts to render an m-multiply connected region simply connected, and for each cut we need three conditions to ensure the continuity of three velocity components. Therefore, there are, in general, 3(m-1) additional conditions which must be met if the velocity field in an m-multiply connected region is to be single-valued and continuous.
3.8 Mass, Linear and Angular Momenta, and Kinetic Energy

Intuitively, the mass of a body is a non-negative quantity which measures the amount of the matter that is contained in the body. Experience shows that mass is additive, that is, the mass of a body is the sum of the masses of its parts.

In mechanics of continua, mass is assumed to be an absolutely continuous function of volume. Hence, a non-negative measure, called mass-density, may be defined at each particle such that, at an instant $t$, the mass of the body is given by the following integral:

$$
m = \int_{\mathcal{V}} \rho \, d\mathcal{V} = \int_{V} \rho_0 \, dV \quad , \quad (3.1)$$

where $\mathcal{V}$ is the spatial volume instantaneously occupied by the body, $V$ is the initial volume, $\rho = \rho(\mathbf{x}, t)$ is the mass-density distribution at the instant $t$, and $\rho_0$ is that at the initial time $t = 0$. Equation (3.1) states that the mass $m$ of the body, or any part of it, is conserved under all possible motions of the body. Any subset of the body with the instantaneous spatial volume $\mathcal{V}'$ has a mass $m'$ which goes to zero as $\mathcal{V}' \rightarrow 0$. Thus the mass-density can never approach infinity, and we must have

$$0 \leq \rho < \infty \quad . \quad (8.2)$$
Clearly enough, taking the material derivative of (8.1) one arrives at
one of the forms (5.8) which state the assumption of conservation
of mass.

The linear momentum or simply momentum of a body with
instantaneous spatial volume v, velocity field \( \mathbf{v} = \mathbf{v}(\mathbf{x}, t) \), and
mass-density distribution \( \rho = \rho(\mathbf{x}, t) \) is defined by

\[
\mathbf{P} = \int_{\mathbf{V}} \mathbf{v} \rho \, d\mathbf{v} 
\]  
(8.3a)

or

\[
P_i = \int_{\mathbf{V}} v_i(\mathbf{x}, t) \rho(\mathbf{x}, t) \, d\mathbf{v} .
\]  
(8.3b)

The moment of momentum or angular momentum about the
origin of the coordinate system is defined by

\[
\mathbf{H} = \int_{\mathbf{V}} (\mathbf{x} \times \mathbf{v}) \rho \, d\mathbf{v} 
\]  
(8.4a)

or

\[
H_i = \int_{\mathbf{V}} e_{ijk} \mathbf{x}_j v_k(\mathbf{x}, t) \rho(\mathbf{x}, t) \, d\mathbf{v} .
\]  
(8.4b)

Note that, using the mean-value theorem, \( \int_{\mathbf{V}} \mathbf{v} \rho \, d\mathbf{v} \) may be viewed as
the linear momentum of particles instantaneously contained in a volume.
element $d\mathbf{u}$. Equations (8.3) and (8.4) are, therefore, in accord with the corresponding definitions in particle dynamics. Following this line of thought, we may define the kinetic energy of the particles in $d\mathbf{u}$ by

$$\frac{1}{2} |\mathbf{v}|^2 \rho \, d\mathbf{u},$$

and, thus, the kinetic energy of the body at the instant $t$ can be expressed as

$$\dot{K} = \frac{1}{2} \int_{\mathbf{u}} |\mathbf{v}|^2 \rho \, d\mathbf{u} = \frac{1}{2} \int_{\mathbf{u}} \mathbf{v}^2 \rho \, d\mathbf{u} = \frac{1}{2} \int_{\mathbf{u}} \mathbf{v}_1 \cdot \mathbf{v}_1 \rho \, d\mathbf{u}. \quad (8.5)$$
References


PROBLEM III

1. Consider a plane motion with the following velocity field:
   \[ v_1 = x_1 (1 + t)^{-1} \]
   \[ v_2 = -K x_2^2 \]
   \[ v_3 = 0. \]

   Find the stream lines, the streak line at point \( x \) at \( t = t' \), and the path of a particle that at \( t = t' \) is at \( x' \).

2. Prove that a necessary and sufficient condition for a moving surface \( g(x, t) = 0 \) to be a material surface is the vanishing of the material derivative of \( g \), i.e., \( \dot{g}(x, t) = 0 \).

3. Show that a necessary and sufficient condition for the vanishing of the shear-rate of any two directions is the vanishing of the distortion-rate tensor \( \mathbf{Q}' \).

4. Show that the quadric of the distortion-rate tensor is never an ellipsoid.

5. Show that in an instantaneous motion of a continuum, there exists at each point at least one instantaneously rotationless direction. Discuss cases in which more than one direction is rotationless.

6. For a steady plane motion with the following velocity field:
   \[ v_1 = Kx_2 \]
   \[ v_2 = v_3 = 0, \]

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a) describe the components of $\mathbf{q}$ and $\mathbf{v}$,

b) show that even though each particle has a straight path and uniform speed, the motion is not irrotational.

Show that for an instantaneous volume $\mathcal{V}$ bounded by a regular closed surface $\mathcal{S}$ we have

$$\int_{\mathcal{V}} \mathbf{v} \, d\mathcal{V} = \int_{\mathcal{S}} \mathbf{x} \cdot \mathbf{v}(\mathcal{V}) \, d\mathcal{A} - \int_{\mathcal{V}} \mathbf{x} \cdot \mathbf{I}_D \, d\mathcal{V},$$

where $\mathbf{v}(\mathcal{V}) = \mathbf{v} \cdot \mathcal{V}$, and $\mathcal{V}$ is the exterior unit normal to $\mathcal{S}$.

8. Show that in a purely dilatational motion, we have

$$\left( \frac{1}{3} \mathbf{I}_D \right)^3 = \left( \frac{1}{3} \mathbf{I}_D^2 \right)^{3/2} = \mathbf{III}_D.$$

9. Prove the theorem of Kelvin and Tait which states that a necessary and sufficient condition for a motion to be instantaneously irrotational at a point is the existence of three mutually orthogonal instantaneously rotationless directions at that point (compare with rigid-body motion). Discuss the possibility of the existence of more than three such directions.

10. Verify the validity of the following expression for acceleration:

$$a_i = \frac{\partial v_i}{\partial t} - \mathbf{I}_D v_i + (v_i v_j) j.$$

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11. Consider a motion whose velocity field can be expressed as

$$\mathbf{v} = \mathbf{v} \mathbf{H} + \mathbf{F} \mathbf{G}$$

where $\mathbf{H}$, $\mathbf{F}$, and $\mathbf{G}$ are scalar-valued functions of spatial variables $\mathbf{x}$ and $t$. Express vorticity, and acceleration fields of this motion in terms of $\mathbf{H}$, $\mathbf{F}$, $\mathbf{G}$, and their derivatives. In particular, verify Duhem's formula, i.e.,

$$a_i = \left( \frac{1}{2} v^2 + \frac{\partial H}{\partial t} + F \frac{\partial G}{\partial t} \right)_i + \dot{F} G_i - \dot{G} F_i$$

12. Let a continuous motion of a continuum be defined by

$$\mathbf{x} = \mathbf{x} (\mathbf{x}, t)$$

relative to a fixed rectangular Cartesian frame. Consider an observer situated in another rectangular Cartesian frame, called starred, which moves relative to the fixed frame such that

$$\mathbf{x}^* (t) = \mathbf{c} (t) + \mathbf{q} (t) \cdot \mathbf{x} (t)$$

where $\mathbf{x} (t)$ is the position of a moving particle in the fixed frame, $\mathbf{x}^* (t)$ is the position of the same particle relative to the starred frame, and $\mathbf{q} = O_{ij} \mathbf{e}_i \mathbf{e}_j$ is a time-dependent, proper orthogonal tensor. Show that

$$\mathbf{q}^* = \mathbf{q} \cdot \mathbf{q} \cdot \mathbf{q}^T$$
\[ \mathbf{\dot{w}}^* = 2 \cdot \mathbf{\dot{w}} \cdot \mathbf{\omega}^T + \mathbf{\ddot{\omega}} \cdot \mathbf{\omega}^T, \]

where \( \mathbf{\omega}^* \) and \( \mathbf{\dot{w}}^* \) are the deformation-rate and spin tensors as observed by the observer in the starred frame.
CHAPTER IV

EULER'S LAWS - STRESS - ENERGY - ENTROPY

The mechanics of continua is concerned with the response of certain bodies to applied forces and torques (loads) and imposed surface displacements.

The applied forces may act over a part or the entire surface of the body in its contact with other bodies. These forces are usually specified per unit area of the surface element upon which they act, and are called surface tractions (or traction vectors). The hydrostatic pressure on submerged bodies is an example of a surface traction. We shall use the symbol $\mathbf{T}$ to denote the applied surface tractions which are measured per unit instantaneous area of the element on which they act. The physical dimension of $\mathbf{T}$ is force divided by squared length.

In a gravitational field, particles of a body are subjected to forces which are proportional to their mass. Since the mass of a continuum is regarded as continuously distributed throughout the space occupied by the body, the forces which relate to mass are likewise continuously distributed. These types of forces are defined per unit mass of the body, and are called body forces. We shall use the symbol $\mathbf{f}$ to designate body forces. The physical dimension of $\mathbf{f}$ is force divided by mass. We note that body forces may also stem from the
interaction of pairs of particles forming the continuum, as in the case of a body under its own gravitation. Since forces of this kind, called mutual loads, may be prescribed a priori, we assume that they are accounted for in the specification of $\mathcal{f}$.

In addition to surface tractions and body forces, one may consider surface and body couples acting on a body. For example, material points of a polarized continuum may be subjected to body couples, in addition to possible body forces, when this continuum is placed in an electromagnetic field. We note that surface and body couples should not be viewed as moments of surface and body forces. The existence of these couples may be postulated independently of, and in addition to the other types of forces and their corresponding moments. We shall use the symbol $\mathcal{\lambda}$ to denote body couples, measured as couple per unit mass, and employ the symbol $\mathcal{\tau}$ to denote surface couples which are measured as couple per unit area of the element upon which they act.

The forces and couples defined above represent the mechanical environment of the body. They are external to the body and for this reason may be termed extrinsic loads. Therefore, at each instant $t$, a considered body that occupies the region $\mathcal{V}$ with a surface $\mathcal{S}$ may be subjected to the following extrinsic loads: A resultant force $\mathcal{F}$ given by

\[
\mathcal{F} = \int_{\mathcal{S}} \mathcal{\lambda} \, da + \int_{\mathcal{V}} \mathcal{\tau} \, \rho \, d\mathcal{V}.
\]
and a resultant moment (torque) $\mathbf{L}$, taken with respect to the origin $O$ of the coordinate system, defined as

$$\mathbf{L} = \int_A (\mathbf{m} + \mathbf{z} \times \mathbf{T}) \, d\mathbf{a} + \int_L (\mathbf{f} + \mathbf{z} \times \mathbf{f}) \, \rho \, d\mathbf{v} \ ,$$

where $\rho$ is the instantaneous mass-density distribution of the continuum.

Note that these loads can act on bodies of all kinds.
4.1 Euler's Laws of Motion

The motion of a continuum is assumed to be governed by Euler's laws which are assertions regarding the manner by which extrinsic loads affect linear and angular momenta of bodies. The first law is concerned with linear momentum. It states that the instantaneous rate of change of the linear momentum $\mathbf{P}$ of a body $B$ is equal to the resultant external force $\mathbf{F}$ that acts on the body at the considered instant.

Let $\mathbf{v}(\mathbf{x}, t)$ be the velocity field of the body at the instant $t$, and denote by $\rho$ its mass-density. Euler's first law may then be written as

$$
\frac{\mathbf{P}}{\mathbf{P}} = \int_S \mathbf{T} \, d\mathbf{a} + \int_U f \, \rho \, d\mathbf{v},
$$

where

$$
\frac{\mathbf{P}}{\mathbf{P}} = \frac{d}{dt} \mathbf{P} = \frac{d}{dt} \int_U \mathbf{v} \, \rho \, d\mathbf{v} \quad (1.1)
$$

Here a superposed dot $(\dot{} = \frac{d}{dt})$ is to be interpreted as material-time-derivative.

Euler's second law is concerned with angular momentum or moment of momentum. It states that the instantaneous rate of change of angular momentum $\mathbf{H}$ of a body $B$ is equal to the resultant external torque $\mathbf{L}$ that acts on the body at the considered instant. Since the angular momentum relative to the origin of coordinates is defined by

$$
\mathbf{H} = \int_U (\mathbf{x} \times \mathbf{v}) \, \rho \, d\mathbf{v} \quad (1.2a)
$$

the second law becomes

$$
\dot{\mathbf{H}} = \int_S (\mathbf{m} + \mathbf{x} \times \mathbf{T}) \, d\mathbf{a} + \int_U (\mathbf{f} + \mathbf{x} \times \mathbf{f}) \, \rho \, d\mathbf{v} \quad (1.2b)
$$

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4.2 Cauchy's Stress Tensor

For the sake of clarity in presentation, let us begin by considering a non-polar continuum, excluding surface and body couples and including only surface and body forces. In Section 7 of this chapter, however, the effect of body and surface couples will be examined.

We shall now introduce a basic concept which characterizes the continuum theory, that is, the concept of internal surface tractions. This concept may be stated as follows: the mechanical action of the material points, which are situated on one side of an arbitrary material surface within a body, upon those on the other side can be completely accounted for by prescribing a suitable set of traction vectors on this surface. With this concept (assumption), a part of a body can be removed and in its place suitable surface tractions prescribed on the newly formed boundaries without affecting the motion and deformation of the remaining part of the body; such a removal of matter must not, of course, alter body forces which may include the mutual loads acting on the remaining part of the continuum.

At a generic point \( \mathbf{x} \), consider a surface element \( da \) and let \( \mathbf{y} \) denote a unit vector normal to this element. The mechanical action of the material points, situated on the side of \( da \) toward which \( \mathbf{y} \) is pointing, upon those on the other side is represented by surface tractions \( \mathbf{T}^{(v)} \) acting on \( da \), Fig. 2.1. Clearly enough, one expects that these surface tractions should, in general, depend on the orientation of the element \( da \), specified by \( \mathbf{y} \), as well as on its position \( \mathbf{x} \) but not on its shape or on the shape of a particular surface which has \( da \) as an element. Since infinitely many directions can be identified at a given point, it is clear that at each point infinitely many traction vectors can also be identified, each
Fig. 2.1

Fig. 2.2
acting on an element with a given unit normal. These traction vectors are not, however, all independent, and, according to Cauchy's theorem, may all be expressed in terms of traction vectors on three distinct planes that pass through the considered point. Let us choose, at point \( \mathbf{x} \), three orthogonal planes which are parallel to the coordinate planes and thus have the unit base vectors \( \mathbf{e}_i \), \( i = 1, 2, 3 \), as unit normals. We denote by \( -\mathbf{T}_i \), \( i = 1, 2, 3 \), the traction vector on the plane whose unit normal is \( -\mathbf{e}_i \). We then consider a small tetrahedron with vertex at \( \mathbf{x} \), and three faces that pass through \( \mathbf{x} \) parallel to the coordinate planes having \( -\mathbf{e}_i \), \( i = 1, 2, 3 \), as exterior unit normals. We let the fourth face of this tetrahedron be at a distance \( h \) from \( \mathbf{x} \), and denote its area by \( da \), its exterior unit normal by \( \mathbf{n} \), and the traction vector acting on it by \( T^{(v)} \) which represents the action exerted by the material outside of the tetrahedron upon those inside. We can now isolate this tetrahedron from the rest of the body and study its motion. We apply Euler's first law (1.1), and, as the height \( h \) shrinks to zero, obtain (see Fig. 2.2)

\[
- (T_1 \nu_1 + T_2 \nu_2 + T_3 \nu_3) + T^{(v)} = 0
\]

(2.1)

where inertia and body forces are neglected because they are proportional to the volume which, in comparison with the surface area of the element, constitutes an infinitesimal of a higher order. The above equation may be written as

\[
T^{(v)} = T_i \nu_i, \quad i = 1, 2, 3,
\]

(2.2)
which states that, at a typical point \( \mathbf{x} \), the traction vector on an arbitrary plane with the unit normal \( \mathbf{\nu} \) is given as a linear and homogeneous vector-valued function of the direction cosines \( \mathbf{\nu}_i \); the coefficients in this linear relation are the traction vectors \( \mathbf{T}_i \) of the three orthogonal planes which pass through the considered point and are parallel to the coordinate planes. This is Cauchy's theorem.

If we choose a plane with the unit normal \( \mathbf{\mu} = -\mathbf{\nu} \), we obtain

\[
\mathbf{T}(\mathbf{\mu}) = \mathbf{T}(-\mathbf{\nu}) = -\mathbf{T}_i \mathbf{\nu}_i \\
= -\mathbf{T}(\mathbf{\nu})
\]

which states that traction vectors acting on opposite sides of the same surface element are equal in magnitude but opposite in direction.

Let \( T_{ij} \) denote the \( j \)th component of the traction vector on the surface with the unit normal \( \mathbf{\nu}_i \), where positive \( T_{ij} \) represents a component in the positive \( x_j \)-direction. We have

\[
T_{ij} = \mathbf{T}_i \cdot \mathbf{\nu}_j
\]

The traction vector \( \mathbf{T}(\mathbf{\nu}) \) can also be expressed in terms of its components along the coordinate axes, i.e.,

\[
\mathbf{T}(\mathbf{\nu}) = T_{ij} \mathbf{\nu}_i \mathbf{\nu}_j
\]

and thus (2.2) yields

\[
T_{ij} = T_{ji} \mathbf{\nu}_j
\]

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The quantity \( \mathbf{T} = T_{ij} \mathbf{e}_i \mathbf{e}_j \) is called Cauchy's or the **true stress tensor** at point \( x \). The traction vector on an element with the unit normal \( \nu \) at this point is given by

\[
T^{(\nu)} = \nu \cdot \mathbf{T} = (\nu_j \mathbf{e}_j) \cdot (T_{ki} \mathbf{e}_k \mathbf{e}_i)
\]

\[
= T_{ji} \nu_j \mathbf{e}_i
\]

\[
= T_{ij} \nu_j
\]

(2.6)

which is the dot product of \( \nu \) with \( \mathbf{T} \). The component \( T_{ij} \) of the stress tensor represents the orthogonal projection along the \( x_j \)-axis of the traction vector that acts on the plane normal to the \( x_i \)-direction. For the positive face\(^\dagger\) of this plane, positive \( T_{ij} \) denotes a component that is directed toward the positive \( x_j \)-direction. On the negative face of the plane normal to the \( x_i \)-axis, on the other hand, negative \( T_{ij} \) points toward the positive \( x_j \)-direction. This sign convention, which is commonly employed, is illustrated in Fig. 2.3, where it is assumed that the components of \( \mathbf{T} \) are all positive; they are shown for all the positive faces, and also for the negative face that is normal to the \( x_2 \)-direction. The components \( T_{11}, T_{22}, \) and \( T_{33} \), which are normal to the corresponding coordinate planes, are called **normal stresses**. The tangential components \( T_{12}, T_{23}, \) and \( T_{31} \) are called **shearing stresses**. We note that the normal stress \( T_{11} \), for

\[^\dagger\] The positive face of the plane normal to the \( x_i \)-direction is the one whose unit normal points in the positive \( x_i \)-direction.
example, is the normal component of the stress tensor along the $x_1$-direction, while the shearing stress $T_{12}$ is the tangential component of this tensor for the $x_1$, $x_2$-directions.

The normal stress on an element of area $da$ with the unit normal $\nu$ is the normal component of the corresponding traction vector $T^{(\nu)}$ in the direction of $\nu$. We denote this quantity by $N^{(\nu)}$, and obtain

$$N^{(\nu)} = T^{(\nu)} \cdot \nu$$

$$= T_{ij} \nu_i \nu_j \quad (2.7)$$

The shearing stress on $da$ is the tangential component $S^{(\nu)}$ of $T^{(\nu)}$, i.e., it is given by

$$S^{(\nu)} = T^{(\nu)} \cdot T^{(\nu)} - N^{(\nu)}^2 \quad (2.8)$$

Note that $S^{(\nu)}$ is defined as an unconditionally positive quantity.

Consider now the motion of a part $B_1$ of the body. Denote by $\nu_1$ the instantaneous region occupied by $B_1$, and let $\partial_1$, with the exterior unit normal $\nu$, be its boundary surface. According to our basic assumption, the effect of the material points outside of this region upon those within the region is completely defined by the specification of surface tractions $T^{(\nu)}$ on $\partial_1$. Euler's laws of motion then yield

$$\int_{\nu_1} (\dot{T}_{ij} - \dot{\nu}_i \nu_j) \rho \, d\nu + \int_{\partial_1} T_{ji} \nu_j \, da = 0 \quad (2.9)$$

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\[ \int_{V_1} e_{ijk} x_j (f_k - \dot{v}_k) \rho \, dv + \int_{\partial V_1} e_{ijk} x_j T_{jk} \nu_k \, d\alpha = 0, \quad (2.10) \]

where (2.5) is used, and \( f_i \) is the component of the body force. Using the Gauss theorem, we reduce these equations to

\[ \int_{V_1} [ T_{ji,j} + \rho (f_i - \dot{v}_i) ] \, dv = 0, \]

\[ \int_{V_1} e_{ijk} T_{jk} \, dv = 0, \]

where the first equation is used for deriving the second. Since these equations are to be valid for an arbitrary part \( B_1 \) of the body, the integrands must vanish, yielding Cauchy's laws

\[ T_{ji,j} + \rho f_i = \rho \dot{v}_i, \quad (2.11) \]

\[ e_{ijk} T_{jk} = 0, \quad (2.12) \]

Equation (2.11) expresses the conservation of linear momentum, while (2.12) is the statement of the conservation of angular momentum when surface and body couples are absent; (2.12) shows that the stress tensor \( \mathbf{T} \) is symmetric.

For a surface element with the exterior unit normal \( \nu \), the traction vector \( T_i(\nu) = T_{ji} \nu_j \) must equal the externally applied surface traction \( T_i \). We thus have

\[ T_{ji} \nu_j = T_i \text{ on } \partial V \quad (2.13) \]

IV-11
where \( \partial_T \) denotes that part of the boundary \( \partial \), upon which the surface tractions are prescribed. If \( \partial_T \) is traction-free, we obtain \( T_{ji} \nu_j = 0 \) on \( \partial_T \).

Since \( \mathbf{J} \) is a real, symmetric, second order tensor, it has all the properties of such a tensor. Therefore, at each point \( x \) at time \( t \), there exist three orthogonal planes upon which the traction vectors \( \mathbf{T}^{(\nu)} \) are normal; the corresponding shearing stresses are zero, and the normal stresses are extrema. The directions of these planes are given by the principal directions of \( \mathbf{J} \). These orthogonal directions are called the principal directions of stress. The corresponding principal values \( T_J \), \( J = I, II, III \), are the principal stresses which are the roots of the equation

\[
\det | T_{ij} - T \delta_{ij} | = 0 \tag{2.14}
\]

or, equivalently, the equation

\[
T^3 - I_T T^2 + II_T T - III_T = 0 \tag{2.15}
\]

where

\[
I_T = T_{ii} = T_{11} + T_{22} + T_{33} = T_I + T_{II} + T_{III} \]

\[
II_T = \frac{1}{2} e_{ijk} e_{ilm} T_{jl} T_{km} = \frac{1}{2} (l_T^2 - T_{ij} T_{ji}) = T_I T_{II} + T_{II} T_{III} + T_{III} T_I
\]

IV-12
\[ \Pi_T = \det | T_{ij} | = \frac{1}{6} e_{ijk} e_{lmn} T_{ijl} T_{jm} T_{kn} \]

\[ = \frac{1}{6} \left( 2 T_{ij} T_{jk} T_{ki} - 3 I_T T_{ij} T_{ji} + I_T^3 \right) \]

\[ = T_I T_{\Pi} T_{\text{III}} \]  \hspace{1cm} (2.16)

are the basic invariants of the stress tensor \( \sim \). The principal stresses may be positive, negative, or zero; a positive value denotes tension, and a negative value, compression. The principal directions of stress are defined by the equations

\[ T_{ji} \nu_j^J = T_J \nu_i^J \hspace{0.5cm} , \hspace{0.5cm} J = I, \Pi, \text{III}, \text{(no sum on } J) \]  \hspace{1cm} (2.17)

which show that the corresponding shearing stresses are zero, and that the principal triad \( \nu_j^J \), \( J = I, \Pi, \text{III} \), is orthogonal, i.e.,

\[ \nu_j^J \cdot \nu_j^K = \delta_{JK} = \begin{cases} 1 & \text{if } J = K \\ 0 & \text{if } J \neq K \end{cases} \]

With respect to the principal triad, the matrix of the stress tensor is diagonal, and this tensor can be expressed as

\[ T_{ij} = \sum_{J=1}^{\text{III}} T_J \nu_i^J \nu_j^J \]  \hspace{1cm} (2.18)

Employing an analysis similar to that which led to Eq. (III.2.13), we conclude that

\[ S = \frac{1}{2} | T_J - T_K | \hspace{0.5cm} , \hspace{0.5cm} J \neq K \]  \hspace{1cm} (2.19)
define the extreme value of the shearing stress, acting on the orthogonal planes with unit vectors

\[ \mu = \frac{\sqrt{2}}{2} \left( \nu^J \pm \nu^K \right), \quad J \neq K \]  

(2.20)

At a point \( \mathbf{x} \) in the body, we introduce a right-handed rectangular Cartesian coordinate system with axes \( y_i, \ i = 1, 2, 3 \), parallel to the corresponding \( x_i \)-axes. The quadric surface

\[ \pm \varphi = T_{ij} y_i y_j \]  

(2.21)

is called Cauchy's stress quadric, where the sign of the left-hand side is to be chosen so that (2.21) represent a real surface. The normal stress, transmitted at this point across an element with the unit vector \( \nu_i \), is

\[ N^v = T_{ij} \nu_i \nu_j \]

\[ = \frac{1}{r^2} T_{ij} y_i y_j = \pm \frac{\varphi}{r^2} \]  

(2.22)

where \( y_i = r \nu_i \), and \( r \) is the length of the radius vector of the quadric surface measured along the unit normal \( \nu \), see Fig. 2.4.

The traction vector acting on this element is
Fig. 2.3

Fig. 2.4
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\[ T_i^{(\nu)} = T_{ji} \nu_j \]

\[ = \frac{1}{r} T_{ji} \nu_j = \frac{1}{2} \frac{\partial}{\partial y_i} (\pm \phi) \quad (2.23) \]

Since the right side of this equation is proportional to the gradient of \( \phi \), we conclude that the direction of the traction vector on the considered element is perpendicular to a plane tangent to the quadric surface at the terminus of the radius vector normal to that element. The magnitude of the traction vector is inversely proportional to the length of this radius vector.

A quadric has three principal axes which are orthogonal, and correspond to radius vectors with stationary lengths. These directions coincide with the principal directions of stress, and, as is clear from (2.22), define the orientation of surface elements with stationary normal stresses.

The stress quadric may be an ellipsoid, a hyperboloid, or any one of their degenerate forms. The quadric is ellipsoid if the principal stresses are distinct and of the same sign, and it is a hyperboloid if these stresses are distinct but one has a sign different from the other two. When two of the principal stresses are equal at a point, there is only one unique principal axis for the stress quadric which, in this case, reduces to a surface of revolution. The axis of revolution is defined by the direction of the distinct principal stress. Normal to this axis, any two orthogonal directions may be taken as the principal axes. If all three principal stresse
are equal at a point, then the stress quadric reduces to a sphere. In this case, the state of stress is isotropic, i.e., \( T_{ij} = p \delta_{ij} \), where \( p \) denotes the normal stress common to all elements of area passing through the considered point. A state of stress of this type is called \textit{spherical}. Note that, in general, the stress tensor \( \mathbb{J} \) may be decomposed into a spherical part and a deviator \( \mathbb{J}' \) whose trace is zero, i.e.,

\[
\mathbb{J} = \mathbb{J}' + \frac{1}{3} I_T \mathbb{J}
\]

where \( I_T = T_{ii} \) is the first stress-invariant. The spherical part of the stress tensor is also called the \textit{mean normal stress}.
4.3 Mohr's Geometric Representation of Stress-State

Let the directions of the coordinate axes \( x_1 \), \( x_2 \), and \( x_3 \) coincide with the principal directions of stress, and label them in such a manner that \( T_I > T_{II} > T_{III} \). The normal stress \( N(\nu) \) transmitted across an element with the unit vector \( \nu \) then is

\[
N(\nu) = T_I \nu_1^2 + T_{II} \nu_2^2 + T_{III} \nu_3^2
\]  

(3.1)

and the corresponding shearing stress is given by

\[
S(\nu)^2 = T_I^2 \nu_1^2 + T_{II}^2 \nu_2^2 + T_{III}^2 \nu_3^2 - N(\nu)^2
\]  

(3.2)

To simplify the notation, we set \( N = N(\nu) \) and \( S = S(\nu) \) and, using the condition

\[
\nu_1^2 + \nu_2^2 + \nu_3^2 = 1
\]

solve Eqs. (3.1) and (3.2) for the direction cosines \( \nu_1 \). Thus,

\[
\nu_1^2 = \frac{(T_{II} - N)(T_{III} - N) + S^2}{(T_{II} - T_I)(T_{III} - T_I)}
\]

\[
\nu_2^2 = \frac{(T_{III} - N)(T_I - N) + S^2}{(T_{III} - T_{II})(T_I - T_{II})}
\]
\[ \nu_3^2 = \frac{(T_I - N)(T_{II} - N) + S^2}{(T_I - T_{III})(T_{II} - T_{III})} \]  

(3.3)

Since the denominators of the first and last equations in (3.3) are positive and that of the second equation is negative, the admissible values of \( N \) and \( S \) must satisfy the following inequalities:

\[ S^2 + [N - \frac{1}{3}(T_{II} + T_{III})]^2 - \frac{1}{3}(T_{II} - T_{III})^2 > 0 \],

\[ S^2 + [N - \frac{1}{3}(T_{III} + T_{II})]^2 - \frac{1}{3}(T_{III} - T_{II})^2 < 0 \],

\[ S^2 + [N - \frac{1}{3}(T_I + T_{II})]^2 - \frac{1}{3}(T_I - T_{II})^2 > 0 \].

(3.4)

In the \( N, S \)-plane, a stress point with the abscissa \( N \) and ordinate \( S \) corresponds to a real direction \( \gamma \) if (3.4) is satisfied. Setting the left side of (3.4) equal to zero, we obtain the equation of three circles which we label as I, II, and III, respectively. The centers of these circles lie on the \( N \)-axis at points \( O_I = \frac{1}{3}(T_{II} + T_{III}) \), \( O_{II} = \frac{1}{3}(T_{III} + T_I) \), and \( O_{III} = \frac{1}{3}(T_I + T_{II}) \), respectively, and their respective radii are \( R_I = \frac{1}{3}(T_{II} - T_{III}) \), \( R_{II} = \frac{1}{3}(T_I - T_{III}) \), and \( R_{III} = \frac{1}{3}(T_I - T_{II}) \). All admissible stress points are in the upper half-plane (since \( S \) is by definition non-negative), inside or on circle II and outside or on circles I and III, see Fig. 3.1.

The stress points on circle I correspond to the directions that lie in the \( x_2, x_3 \)-plane, since for these directions \( \nu_1 = 0 \). Similarly, stress points on circles II and III correspond to the directions for which \( \nu_2 = 0 \) and \( \nu_3 = 0 \), respectively.
Consider now a direction in the $x_1$, $x_2$-plane that forms an angle $\varphi$ with the positive $x_1$-direction. From (2.7) and (2.8) we obtain, for this direction,

$$N = \frac{1}{2}(T_I + T_{II}) + \frac{1}{2}(T_I - T_{II}) \cos 2\varphi,$$

$$S = \frac{1}{2}(T_I - T_{II}) \sin 2\varphi.$$  \hspace{1cm} (3.5)

Thus, the associated stress point $Q$ on semicircle III is obtained by measuring the angle $A_{I O_{III} Q}$ equal to $2\varphi$. From the first equation in (3.3), we have

$$S^2 + [N - \frac{1}{2}(T_{II} + T_{III})]^2 = \frac{1}{2}(T_{II} - T_{III})^2 + \cos^2 \varphi(T_{II} - T_I)(T_{III} - T_I),$$

which states that the locus of stress points for directions that form a constant angle $\varphi$ with the $x_1$-axis is a circular arc $QQ'$ with center at $O_I$ and radius equal to the length of the line $O_I Q$.

Next, consider a direction in the $x_2$, $x_3$-plane that forms an angle $\theta$ with the positive $x_3$-direction. Following the same line of reasoning as above, we conclude that the corresponding stress point $P$, on semicircle I, is obtained by measuring the angle $A_{III O_I P}$ equal to $2\theta$. The locus of stress points of the directions which form a constant angle $\theta$ with the positive $x_3$-direction is the circular arc $PP'$ with center at $O_{III}$ and radius equal to the length of the line $O_{III} P$.

The stress point $R$ corresponding to a direction which forms the angles $\varphi$ and $\theta$ with the $x_1$- and $x_3$-directions, respectively, is given by the intersection of the circular arcs $QQ'$ and $PP'$. Note that

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the stress points $P'$ and $Q'$, on semicircle $\Pi$, are associated with the
directions which lie in the $x_1$, $x_3$-plane. To obtain $Q'$, for example,
we measure the angle $A_Q Q'$ equal to $2\phi$. Note also that the highest
point $M$ on circle $\Pi$ corresponds to the maximum shearing stress of
magnitude $\frac{1}{2}(T_I - T_{III})$. Since $\phi = \theta = 45^\circ$ for this point, the associated
direction bisects the angle formed by the $x_1$- and $x_3$-directions. The
normal stress across this element is equal to $\frac{1}{2}(T_I + T_{III})$ which is the
abscissa of $M$. The geometric interpretation of the state of stress pre-
sented above is commonly named after O. Mohr who first conceived it.
It can readily be adapted to the representation of the state of deformation-
rate at a point.

When two of the principal stresses vanish at a point, the state of
stress is said to be locally uniaxial with the axis of stress defined by the
principal axis that is associated with the non-zero principal stress. The
stress field is said to be uniaxial if the state of stress is uniaxial every-
where in $\nu$, and the direction of the principal axis, corresponding to the
non-zero principal stress, is constant.

The state of stress is said to be plane at a point if one and only
one principal stress vanishes there. The plane of the stress is specified
by the principal axes which correspond to the non-zero principal stresses.
The stress field is said to be plane if the state of stress is plane every-
where in $\nu$, and the direction of the zero principal stress is constant.

Consider a state of plane stress with $T_{III} = 0$ and $T_I > T_{II}$
$\neq 0$, and let the $x_1$- and $x_2$-axes be along the principal directions,
respectively. The normal stress $N$ and the shearing stress $S$ transmitted across an element whose unit normal $\upsilon$ lies in the $x_1$, $x_2$-plane and forms an angle $\phi$ with the positive $x_1$-direction are given by Eqs. (3.5). In the $N$, $S$-plane, these equations represent a circle with center at $O = \frac{1}{2} (T_I + T_{I\Pi})$ and radius $R = \frac{1}{2} (T_I - T_{I\Pi})$; this is called Mohr's circle or the circle of stress, see Fig. 3.2b. To discuss Mohr's circle, we adopt the following sign convention: the normal stress $N$ is regarded positive if it points in the direction of $\upsilon$; the shearing stress $S$ is regarded positive if it has the same relation with $\upsilon$ as the positive $x_1$- and $x_2$-directions have with respect to each other.

To locate on the stress circle the stress point $M$ that corresponds to the direction $\upsilon$, we measure the angle $\angle AOM$ equal to $2\phi$.

Suppose now that the state of plane stress at a point is specified by three stress-components $T_{11}$, $T_{22}$, and $T_{12} = T_{21}$, see Fig. 3.2a, where $T_{11}$ and $T_{22}$ are not the principal stresses. To obtain the corresponding stress circle, we locate in the $N$, $S$-plane two points $P$ and $P'$ with coordinates $(T_{11}, -T_{12})$ and $(T_{22}, T_{12})$, respectively. As is clear from Fig. 3.2b, the line $PP'$ intersects the $N$-axis at the center $O = \frac{1}{2} (T_{11} + T_{22})$ of the stress circle whose radius is defined by $R^2 = T_{12}^2 + \frac{1}{4} (T_{11} - T_{22})^2$. The principal stresses then are

$$T_{I, \Pi} = \frac{1}{2} (T_{11} + T_{22}) \pm \sqrt{T_{12}^2 + \frac{1}{4} (T_{11} - T_{22})^2}.$$
and the angle $\text{POA}_I = 2\Psi$ is defined by

$$\tan 2\Psi = \frac{-2T_{12}}{T_{11} - T_{22}}$$
4.4 Other Stress Tensors

Cauchy's laws (2.11) and (2.12) are expressed in terms of current variables $\mathbf{X}$ and $t$. Kinematical quantities, on the other hand, are most conveniently described using Lagrangian variables $\mathbf{X}$ and $t$. This is especially useful when the reference configuration $C_0$ constitutes a natural state of stress-free configuration of the continuum. Thus, it may often be required to consider other forms of the stress tensor that are expressed in reference to an initial configuration of the continuum.

To this end, let us introduce the Lagrangian variables $\mathbf{X}$ and $t$, and for a part $B_1$ of the body obtain, from Eqs. (1.1) and (2.5),

$$
\frac{d}{dt} \int_{V_1} v_i(\mathbf{X}, t) \rho_0(\mathbf{X}) \, dV = \int_{S_1} \left[ T_{ji} X^j_\beta, \beta \right] \, dA_\beta + \int_{V_1} F_i(\mathbf{X}, t) \rho_0(\mathbf{X}) \, dV
$$

(4.1)

where $v_i(\mathbf{X}, t) = \dot{x}_i(\mathbf{X}, t)$ is the velocity vector, $V_1$ and $S_1$ are the initial material volume and surface of $B_1$, respectively, and $F_i(\mathbf{X}, t) = f_i(\mathbf{X}(t), t)$ is body force per unit mass in material description. Now, using the Gauss theorem and the fact that (4.1) is to be valid for any volume $V_1$, we obtain

$$
\rho_0 \dot{v}_i = (T_{ji} J X^j_\beta, \beta) + \rho_0 \mathbf{F}_i
$$

(4.2a)

which is Cauchy's first law in Lagrangian description.

The quantities
\[
T_{\alpha i}^R = J \frac{\partial X_{\alpha}}{\partial x_j} T_{ji}
\]

\[
= \frac{\rho_0}{\rho} X_{\alpha,j} T_{ji}
\]

\[i, j, \alpha = 1, 2, 3\]

are the components of the stress tensor at \( \mathcal{X} \) measured per unit of corresponding area at \( \mathcal{X} \). The tensor \( \mathcal{E}^R = T_{\alpha i}^R e_{\alpha} e_{i} \) is called Lagrange's stress tensor or sometimes the first Piola-Kirchhoff stress tensor. Note that \( \mathcal{E}^R \) is a nonsymmetric tensor.

Solving (4.3a) for \( T_{ji} \), we obtain

\[
T_{ji} = \frac{\rho}{\rho_0} x_{j,\alpha} T_{\alpha i}^R
\]

and hence Cauchy's second law takes on the following complicated form:

\[
x_{j,\alpha} T_{\alpha i}^R = x_{i,\alpha} T_{\alpha j}^R
\]

while the first law becomes

\[
\rho_0 v_i = T_{\alpha i}^R e_{\alpha} e_{i} + \rho_0 F_i
\]

Let us define a second order, symmetric tensor \( \mathfrak{S} = S_{\alpha \beta} e_{\alpha} e_{\beta} \) as follows:

\[
S_{\alpha \beta} = X_{\alpha,i} T_{\beta i}^R
\]

\[
= \frac{\rho_0}{\rho} \frac{\partial X_{\alpha}}{\partial x_i} \frac{\partial X_{\beta}}{\partial x_j} T_{ji}
\]

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which is commonly called **Kirchhoff's stress tensor**, and also the **second Piola-Kirchhoff stress tensor**; it apparently has been first introduced by Piola in 1833. In terms of $\bar{S}$, the second law becomes

$$ S_{\alpha\beta} = S_{\beta\alpha} \quad , $$

(4.6)

while the first law takes on the following complicated form:

$$ \rho_0 \dot{v}_i = (x, S_{\alpha\beta})\dot{\beta} + \rho_0 F_i ; \quad \alpha, \beta, \gamma, i = 1, 2, 3 \quad . $$

(4.7)

Let us note that the stress tensors $\bar{S}^R$ and $\bar{S}$, which are introduced consistently with the natural measure of stress $\bar{S}$, may be given certain mechanical interpretations as follows: Consider an element of area $dA$ with its image $dA$ in the reference configuration. From Eq. (4.1), it is clear that the tensor $T^R_{\alpha i}$ is introduced in such a manner as to leave the resultant force $T^{(\nu)} dA$ on the element $dA$ unchanged, i.e.,

$$ T^{(\nu)} dA = T_{ji} x\alpha, j dA \alpha \sim_i $$

or

$$ T_{ji} da_j = T^R_{\alpha i} dA \alpha \quad . $$

(4.8)

The stress tensor $\bar{S} = S_{\alpha\beta} \sim\alpha \sim_\beta$, on the other hand, is defined as follows: The resultant force $T^{(\nu)} dA$, instantaneously acting on the element $dA$, is first subjected to the following transformation:

$$ dp\alpha = T_{ji} da_j \frac{\partial X\alpha}{\partial x_i} \quad , $$

(4.9a)
where $d\mathbf{p} = dp_\alpha e_\alpha$ is the transformed resultant force on $da$.

Now, at the image point $X$ in the reference configuration $C_0$, a stress tensor $\bar{\Sigma}$ is introduced in such a manner that its dot product with the element of area-vector $d\bar{A}$ at $X$ yields a vector equal to $d\bar{p}$, i.e.,

$$\bar{\Sigma} \cdot d\bar{A} = d\bar{p}$$

or

$$S_{\alpha\beta} dA_\beta = X_\alpha, i \ T_{ji} \ da_j$$

which is equivalent to (4.5).

We close this section by defining another stress tensor

$$\bar{\tau} = T_{ij} \varepsilon_i \varepsilon_j$$

called convected stress tensor, which is useful for the discussion of constitutive equations. The convected stress tensor is defined by

$$\bar{\tau} = \bar{\tau}^T \cdot \bar{\tau} = \bar{T}_{\alpha\beta} \varepsilon_\alpha \varepsilon_\beta$$

$$= T_{ij} X_\alpha, i \ X_\beta, j \varepsilon_\alpha \varepsilon_\beta$$

Solving (4.10a) for $T_{ij}$, we obtain

$$T_{ij} = \bar{T}_{\alpha\beta} \ X_\alpha, i \ X_\beta, j$$

Let us now summarize various stress tensors and their relations to each other as follows:

$$\bar{\Sigma} = T_{ij} \varepsilon_i \varepsilon_j$$

Cauchy's or the true stress tensor,
\( \mathcal{S}_R = \mathcal{T}^R_{\alpha i} \mathcal{e}_\alpha \mathcal{e}_i \): Lagrange's or the first Piola-Kirchhoff stress tensor

\( \mathcal{S} = S_{\alpha \beta} \mathcal{e}_\alpha \mathcal{e}_\beta \): Kirchhoff's or the second Piola-Kirchhoff stress tensor,

\( \mathcal{\tilde{S}} = \mathcal{T}^R_{\alpha \beta} \mathcal{e}_\alpha \mathcal{e}_\beta \): convected stress tensor.

These tensors are related to each other by the following relations:

\[
\mathcal{S}_R = J (\mathcal{S}^{-1})^T \mathcal{S} \quad \text{or} \quad \mathcal{T}^R_{\alpha i} = J X_{\alpha, j} T_{ji}
\]

\[
\mathcal{S} = \mathcal{S} \cdot \mathcal{S}^{-1} \quad \text{or} \quad S_{\alpha \beta} = J X_{\alpha, i} T_{ij} X_{\beta, j}
\]

\[
\mathcal{\tilde{S}} = \mathcal{S} \cdot \mathcal{S} \cdot \mathcal{S}^T \quad \text{or} \quad \mathcal{T}^R_{\alpha \beta} = x_{i, \alpha} T_{ij} x_{j, \beta} \quad \text{(4.11a)}
\]

where in the left-hand column dyadic notation is used. For example, the first relation may be expressed as

\[
\mathcal{S}_R = J (X_{\alpha, i} \mathcal{e}_i \mathcal{e}_\alpha)^T \cdot (T_{j \ell} \mathcal{e}_j \mathcal{e}_\ell)
\]

\[
= (J X_{\alpha, i} T_{j \ell}) \mathcal{e}_\alpha (\mathcal{e}_i \cdot \mathcal{e}_j) \mathcal{e}_\ell
\]

\[
= (J X_{\alpha, i} T_{j \ell} \delta_{ij}) \mathcal{e}_\alpha \mathcal{e}_\ell
\]

\[
= J X_{\alpha, j} T_{j \ell} \mathcal{e}_\alpha \mathcal{e}_\ell
\]

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It is often more convenient to identify a second order tensor with the matrix of its components and use the matrix multiplication rule. For instance, \( \underline{\underline{\mathbf{a}}}^R \) may be written as \( \underline{\underline{\mathbf{a}}}^R = J \underline{\underline{\mathbf{a}}}^{-1} \underline{\underline{\mathbf{a}}} \). In this notation, (4.11a) reduces to

\[
\underline{\underline{\mathbf{a}}}^R = J \underline{\underline{\mathbf{a}}}^{-1} \underline{\underline{\mathbf{a}}} = \underline{\underline{\mathbf{a}}} \underline{\underline{\mathbf{a}}}^T
\]

\[
\underline{\underline{\mathbf{a}}} = J \underline{\underline{\mathbf{a}}}^{-1} \underline{\underline{\mathbf{a}}} (\underline{\underline{\mathbf{a}}}^{-1})^T = \underline{\underline{\mathbf{a}}}^{-1} (\underline{\underline{\mathbf{a}}}^R)^T
\]

\[
\underline{\underline{\mathbf{a}}}^H = \underline{\underline{\mathbf{a}}}^T \underline{\underline{\mathbf{a}}} \underline{\underline{\mathbf{a}}}
\]

(4.11b)
4.5 **Work and Energy**

In a purely thermo-mechanical theory, there are four conservation laws governing the motion and deformation of a continuum: 
a) conservation of mass, b) conservation of linear momentum, c) conservation of angular momentum, and d) conservation of energy. The first conservation law states that mass is neither destroyed nor created, i.e.,

\[ m = \int_V \rho \, dV = \int_V \rho_0 \, dV \quad (5.1) \]

where \( \rho_0 \) is a reference and \( \rho \) the instantaneous mass-density, respectively. The conservation laws (b) and (c), on the other hand, yield Cauchy's laws of motion (2.11) and (2.12), where the latter must be modified for polar continua or if surface and body couples also exist (see Section 4.6). We shall now consider the last conservation law, namely, that of energy which, in a purely thermo-mechanical system, asserts the equivalence of heat energy and mechanical work. We shall, however, consider only a non-polar continuum and do not include body and surface couples in our discussion.

Let \( \epsilon \) denote the instantaneous **specific internal energy** per unit mass, and \( \mathbf{v}_i(x, t) \) the instantaneous velocity field of the continuum occupying a region \( V \) with a regular surface \( \partial V \). The internal and kinetic energies contained within \( V \) then are...
\[ \mathcal{E} = \int u \varepsilon \rho \, du \quad , \quad \chi = \int u \frac{1}{2} v_i v_i \rho \, du \quad . \tag{5.2} \]

If the rate of heat supply is \( \tilde{Q} \), the conservation of energy asserts
that
\[ \chi + \dot{\mathcal{E}} = \mathcal{F} + \tilde{Q} \quad , \tag{5.3} \]

where
\[ \mathcal{F} = \int \mathcal{g} \cdot \mathbf{T}_i v_i \, ds \quad + \quad \int u f_i v_i \rho \, du \quad . \tag{5.4} \]

is the rate at which the prescribed surface tractions \( \mathbf{T}_i \) and body
forces \( f_i \) do work on the continuum. Note that, since the heat
supply \( \tilde{Q} \) is supposed to be known, (5.3) may be viewed as defining
the internal energy \( \mathcal{E} \).

Let \( \mathcal{g} \) denote the heat flux through the surface \( \mathcal{g} \) and \( h \),
the heat created per unit mass in the body; \( h \) may be, for example,
created by radiation-absorption. The heat supply \( \tilde{Q} \) then is
\[ \tilde{Q} = - \int \mathcal{g} q_i v_i \, ds \quad + \quad \int u h \rho \, du \]
\[ = \int u (-q_i + \rho h) \, du \quad . \tag{5.5} \]

where \( y \) denotes the exterior unit normal to \( \mathcal{g} \). Substitution from
(2.5) into (5.4) and then from (5.2), (5.4), and (5.5) into (5.3) now
yields
\[
\int_{V} \left\{ \left( \frac{1}{2} v_i v_i + e \right) \left[ \frac{\partial}{\partial t} + (\rho v_i), i \right] + \left[ \rho \dot{\varepsilon} - (T_{ij} D_{ji} - q_{i,i} + \rho h) \right] \right. \\
+ T_{ij} W_{ji} - v_i \left[ T_{ji,j} + \rho f_i - \rho \dot{v}_i \right] \right\} \, dv = 0 \tag{5.6a}
\]

which must hold for any volume \( V \). We thus obtain

\[
(\frac{1}{2} v_i v_i + e) \left[ \frac{\partial}{\partial t} + (\rho v_i), i \right] + \left[ \rho \dot{\varepsilon} - (T_{ij} D_{ji} - q_{i,i} + \rho h) \right] - \\
v_i \left[ T_{ji,j} + \rho f_i - \rho \dot{v}_i \right] + [T_{ij} W_{ji}] = 0 . \tag{5.6b}
\]

The quantity inside of the first set of brackets in (5.6) is zero because of conservation of mass, while the last two sets of brackets vanish due to conservation of linear and angular momenta (Cauchy's laws). We, therefore, obtain

\[
\rho \dot{\varepsilon} = T_{ij} D_{ji} - q_{i,i} + \rho h \tag{5.7}
\]

as the statement of local conservation of energy. This equation states that the local rate of change of internal energy is due to heat energy plus the rate of deformation-work \( T_{ij} D_{ji} \) which is also called stress power.
It is of interest to note that equation (5.6) includes, in addition to the local conservation of mass and energy, statements of local conservation of linear and angular momenta. Indeed, it is argued by Green and Rivlin\(^1\) that Cauchy's laws can be deduced from the equation of energy by making use of invariance conditions under superposed rigid-body motions. To this end, one writes (5.3) for the body that is in the same configuration at time \( t \) but has a velocity field \( \vec{v} + \vec{v}_0 \), where \( \vec{v}_0 \) is a constant vector. Now assuming that under this superposed rigid-body translational velocity the quantities \( \varepsilon, T^{(v)}, \vec{f}, q, \) and \( h \) stay unaltered, and using conservation of mass, one obtains

\[
\left[ \rho \dot{\varepsilon} - (T_{ij} D_{ji} - q_{i,1} + \rho h) \right] - (v_i + v_i^0) \left[ T_{ji,j} + \rho f_i - \rho \dot{v}_i \right] + T_{ij} W_{ji} = 0 \tag{5.8}
\]

Comparing this equation with (5.6), and noting conservation of mass, we obtain

\[
v_i^0 \left[ T_{ji,j} + \rho f_i - \rho \dot{v}_i \right] = 0 \tag{5.9}
\]

which must hold for all constant vectors \( \chi^0 \). Thus the quantity in the brackets must vanish identically; this is Cauchy's first law. Equation (5.6a), therefore, reduces to

\[
\int_U \left\{ \left[ \rho \dot{\varepsilon} - (T_{ij} D_{ji} - q_{i,1} + \rho h) \right] + T_{ij} W_{ji} \right\} dU = 0 \tag{5.10}
\]

Now consider a motion of the body that differs from the given motion only by a superposed uniform rigid body angular velocity \( \Omega^0_{ij} \), the body being in the same configuration at the time \( t \). Assuming that the quantities \( \varepsilon \), \( T_{ij} \), \( q \), and \( h \) are unaltered by such motion, we may write Eq. (5.10) using the velocity gradient \( v_{i,j} + \Omega^0_{ij} \), where \( \Omega^0_{ij} \) is a constant antisymmetric tensor. Comparison of the results so obtained with (5.10) yields

\[
\Omega^0_{ij} T_{ji} = 0 \tag{5.11}
\]

which must hold for an arbitrary antisymmetric tensor \( \Omega^0_{ij} \), yielding Cauchy's second law. Using these results, local conservation of energy is now readily obtained from (5.6b). Note that the approach presented by Green and Rivlin postulates that the quantities \( T^{(\nu)} \), \( T_{ij} \), and \( f \) are unaltered under superposed translational and uniform angular rigid body velocities; this, of course, is also implied by conservation of linear and angular momenta which yielded Cauchy's laws.

Let \( \mathcal{Q} = \mathcal{Q}_{\alpha \beta} \alpha \beta \) denote the heat flux at time \( t \), measured per unit of time and per unit of area in the reference configuration \( C_0 \). Equation (5.5) may be written in terms of the Lagrangian variables \( X \) and \( t \) as

\[
\mathcal{Q} = - \int_S \mathcal{Q} \cdot dA + \int_V h \rho_0 dV
= \int_V (- \mathcal{Q}_{\alpha \beta} \alpha \beta + \rho_0 h) dV. \tag{5.12}
\]
Thus the local conservation of energy, equation (5.7), takes on the following form when it is expressed in terms of the Lagrangian variables:

$$\rho_0 \dot{\varepsilon} = T^R_{\alpha i} v_i,\alpha - Q_{\alpha,\alpha} + \rho_0 \dot{h}$$  \hspace{1cm} (5.13a)

The first term in the right side of (5.13a) is the stress power which may also be written as

$$T^R_{\alpha i} v_i,\alpha = x_{\alpha,\beta} v_i,\alpha \dot{s}_{\alpha\beta} = s_{\alpha\beta} \dot{e}_{\alpha\beta}$$  \hspace{1cm} (5.13b)

Equation (5.13a) thus becomes

$$\rho_0 \dot{\varepsilon} = s_{\alpha\beta} \dot{e}_{\alpha\beta} - Q_{\alpha,\alpha} + \rho_0 \dot{h}.$$  \hspace{1cm} (5.13c)
4.6 Entropy and Clausius-Duhem Inequality

To discuss the thermodynamics of a deformable medium we introduce two new time-dependent functions which describe the effects of heat energy in a continuum. These are: 1) the temperature \( \Theta = \Theta (\vec{x}, t) \) which is a scalar field with positive values at every particle \( \vec{x} \) for all time \( t \), and which serves as a measure of the relative hotness of various material neighborhoods of a body or various bodies; 2) the entropy \( \mathcal{K} \) which is a time-dependent, continuous, additive function of mass, admitting a density \( \eta = \eta (\vec{x}, t) \) so that the entropy \( \mathcal{K} \) contained within a material volume \( V \) can be written as

\[
\mathcal{K} = \int_V \eta (\vec{x}, t) \, \rho (\vec{x}, t) \, dV
\]  

(6.1)

Using the spatial variables, we write for the temperature

\[
\Theta = \Theta (\vec{x}(\vec{x}, t), t)
\]

\[
= \Theta (\vec{x}, t) > 0
\]  

(6.2)

In a thermo-mechanical system, the change in entropy consists essentially of two parts, one part being irreversible and the other part reversible. The irreversible part is usually associated with the generation of entropy because of internal energy dissipation or the existence of a temperature gradient in the body. The second law of thermodynamics states that the rate of internal production of entropy...
\( \Gamma \) is non-negative. The reversible part of the entropy is merely the flow of entropy across the boundary of the system, and may be positive, negative, or zero. If we denote by \( \gamma = \gamma(x, t) \) the specific entropy production per unit mass per unit time, i.e.,

\[
\Gamma = \int_{\mathcal{U}} \gamma(x, t) \, \rho(x, t) \, dv \quad , \quad (6.3)
\]

then the following balance equation is, in general, valid:

\[
\frac{d}{dt} \kappa = \Gamma + \left\{ - \int_{\mathcal{S}} \frac{1}{\partial} q \cdot \nu \, da + \int_{\mathcal{U}} \frac{1}{\theta} h \, \rho \, dv \right\} \quad , \quad (6.4)
\]

where the terms inside of the braces denote the rate of entropy flow due to heat supply. Since (6.4) is valid for any part \( B_1 \) of a body \( B \), using (6.1) and (6.2), and Gauss' theorem, we obtain

\[
\gamma = \{ \dot{\eta} - (\frac{1}{\rho \theta} \, \text{div} \, q + \frac{h}{\theta}) \} - \frac{1}{\rho \theta^2} q \cdot \text{grad} \, \theta \quad . \quad (6.5)
\]

The last term in the right-hand side of (6.5) is the rate at which entropy is produced per unit mass due to heat conduction, and the terms inside of the braces denote the rate of entropy generation per unit mass, by energy dissipation. We set

\[
\gamma_1 = - \frac{1}{\rho \theta^2} q \cdot \text{grad} \, \theta \quad , \quad (6.6a)
\]
\[
\gamma_2 = \dot{\eta} - (\frac{1}{\rho \theta} \, \text{div} \, q + \frac{h}{\theta}) \quad , \quad (6.6b)
\]

and reduce (6.5) to

\[
\gamma = \gamma_1 + \gamma_2 \geq 0 \quad , \quad (6.7)
\]

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where the inequality follows from the fact that, for any part \(B_1\) of a body \(B\), the entropy generation \(\Gamma\) is non-negative.

The inequality

\[
\Gamma = \int \gamma \rho \, dv \geq 0
\]

or equivalently (6.7), does not necessarily imply that \(\gamma_1\) and \(\gamma_2\) are each individually non-negative. We shall adopt the commonly held views that heat does not flow against a temperature gradient, and that a body at a uniform temperature cannot absorb heat energy and automatically release an equivalent amount of mechanical energy. Hence, we shall write

\[
\gamma_1 \geq 0, \quad \gamma_2 \geq 0
\]

These inequalities may be regarded as special cases of (6.7).

As will be discussed in Chapter VI, the Clausius-Duhem inequality (6.8) places severe restrictions on the forms of the constitutive equations which are used to account for the thermo-mechanical constitution of a particular material that may comprise a body.

Eliminating \(\theta\) between (5.7) and (6.5), and noting that \(\theta\) is non-negative, we obtain

\[
\theta \gamma = \theta \dot{\eta} - \dot{\varepsilon} + \frac{1}{\rho} \text{tr} \left( \mathcal{F} \cdot \mathcal{L} \right) + \frac{1}{\rho \theta} \mathcal{Q} \cdot \text{grad} \theta \geq 0
\]

where \(\text{tr} \left( \mathcal{F} \cdot \mathcal{L} \right) = T_{ij} D_{ij}\).

It is often convenient to introduce the Helmholtz free energy, or the specific free energy.
\[ \psi = e - \eta \theta \quad , \quad (6.11) \]

and rewrite (6.10) as

\[ \theta \gamma = - \dot{\psi} - \eta \dot{\theta} + \frac{1}{\rho} \text{tr} (\bar{J} \cdot \bar{q}) - \frac{1}{\rho} q \cdot \nabla \theta \geq 0 \quad . \quad (6.12) \]

Consider now a material volume \( \mathcal{V} \) bounded by a regular surface \( \partial \mathcal{V} \). Suppose that the field quantities \( \bar{x}(\bar{X}, t), \bar{J}, e, q, \eta \) and \( \bar{q} \) are given in \( \mathcal{V} \) and on \( \partial \mathcal{V} \). If \( \bar{J} \) is a symmetric tensor, (2.12) is satisfied, and substitution into (2.11) and (5.7) yields, respectively, body forces \( \bar{f} \) and heat sources \( \bar{h} \) which are required to render the above field quantities compatible with these conservation laws. A set of field quantities \( \bar{x}(\bar{X}, t), \bar{J}, e, q, \eta \) which satisfy the conservation laws (2.11), (2.12), and (5.7), for suitable body forces \( \bar{f} \) and heat sources \( \bar{h} \), is said to constitute a thermodynamic process. We note that body forces and heat sources involved in the definition of a thermodynamic process need not correspond to a real situation.
4.7 **Couple-Stress Theory**

In continuum mechanics, the term "internal forces" does not refer to forces at the atomic, molecular, or crystalline levels, but rather to forces between adjacent macroscopic elements which, while small in comparison with typical dimensions of the considered body, are large in comparison with typical dimensions of crystallites. Similarly, the terms "local deformation" and "particle displacement" also refer to macroscopic elements. Hence each of these elements may be regarded as itself being a deformable medium, resulting in a continuum theory with "microstructure." If we assume that each micromedium undergoes a homogeneous deformation, then, in a three-dimensional case, the microdeformation of an element is completely defined by the specification of the deformation of three distinct directors. The points of such a continuum enjoy more degrees of freedom than three, and across its material surfaces not only surface tractions but also surface couples are transmitted.

A simple couple-stress theory is obtained if we assume that each point of the medium has the degrees of freedom of a rigid body. If it is then assumed that the (micro) rotation of the points of such a continuum is equal to the local (macro) rotation of the medium, and that couple stresses are transmitted across material surfaces, we obtain the simplest couple-stress theory which is founded on only a

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1 This is the point of view adopted by Mindlin.
modified concept of internal forces. In the following, we shall briefly examine this theory, referring the interested students to references, which are cited at the end of this chapter, for further study.

A simple couple-stress theory is obtained if we suppose that the mechanical action of the material points, which are situated on one side of an elementary material surface \( da \) within a continuum upon those on the other side, is completely accounted for by prescribing suitable sets of surface tractions and surface couples on this surface. The continuum may be subjected to externally applied surface and body forces, as well as surface and body couples; the latter is denoted by \( \mathbf{N} \) and is measured as couple per unit mass.

Consider, at a point \( x \), a surface element \( da \) with the unit vector \( \mathbf{y} \), and let \( T^{(\nu)} \) and \( m^{(\nu)} \) denote, respectively, the surface traction and surface couple transmitted across this element.

Denote by \( \mathbf{m}_i \) the surface couple that, at this point, acts on that face of a surface element that has the unit vector \( \mathbf{e}_i \). Consider now a small tetrahedron whose three faces are perpendicular to the directions of the coordinate axes, whose height is \( h \), and upon whose fourth face with the exterior unit normal \( \mathbf{y} \), surface tractions \( T^{(\nu)} \) and surface couples \( m^{(\nu)} \) are acting. Let \( m_{ij} \) denote the jth component of the surface couple on the face that is normal to the \( x_i \)-direction, i.e.,

\[
m_i = m_{ij} \mathbf{e}_j.
\]  

(7.1)
Apply Euler's second law (1.2) to this tetrahedron, let the height $h$ shrink to zero, and obtain

$$m_i^{(\nu)} = m_{ji} \nu_j$$  \hspace{1cm} (7.2)$$

where the left side defines the components of the surface couple $\mathbf{m}^{(\nu)}$.

The quantity $\mathbf{m} = m_{ij} \mathbf{e}_i \mathbf{e}_j$ is called the couple-stress tensor.

The surface couple acting on any element with the unit vector $\mathbf{y}$ is now given by the dot product of $\mathbf{y}$ with $\mathbf{m}$, i.e.,

$$\mathbf{m}^{(\nu)} = \mathbf{y} \cdot \mathbf{m} = m_{i} \nu_i$$  \hspace{1cm} (7.3)$$

Set $\mathbf{y} = -\mathbf{y}$, and obtain

$$\mathbf{m}^{(\mu)} = \mathbf{m}^{(-\nu)} = -m_{i} \nu_i$$

$$= -\mathbf{m}^{(\nu)}$$  \hspace{1cm} (7.4)$$

which states that surface couples acting on opposite sides of the same surface element are equal in magnitude but opposite in direction.

Consider now the conservation of linear and angular momenta.

Since linear momentum is unaffected by surface and body couples, Cauchy's first law (2.11) retains its validity also in the couple-stress theory. Apply Euler's second law (1.2) to an arbitrary part $B_1$ of the body, and using Eq. (7.2) obtain

$$m_{ji,j} + \rho \mathbf{E}_i = -\epsilon_{ijk} T_{jk}$$  \hspace{1cm} (7.5)$$
which replaces Cauchy's second law (2.12). Note that (7.5) reduces to (2.12) if surface and body couples are absent. Therefore, Cauchy's stress tensor \( \mathfrak{M} \) is symmetric if and only if the left side of (7.5) vanishes.

Let us now introduce a third order tensor \( \mathfrak{M} = M_{ijk} \tilde{e}_i \tilde{e}_j \tilde{e}_k \), which is dual to the couple-stress tensor \( m_{i\ell} \) that acts on a plane normal to the \( x_\ell \)-axis, i.e.,

\[
M_{ijk} = \frac{1}{2} e_{jkl} m_{i\ell} \quad . \tag{7.6a}
\]

This can be solved for \( m_{i\ell} \), yielding

\[
m_{i\ell} = M_{ijk} e_{jkl} \quad . \tag{7.6b}
\]

If we also introduce the dual tensor \( \mathfrak{L} = L_{ij} \tilde{e}_i \tilde{e}_j \) of the body couple \( \ell_k \), i.e.,

\[
L_{ij} = \frac{1}{2} e_{ijk} \ell_k \quad \tag{7.7a}
\]

or

\[
\ell_k = e_{ijk} L_{ij} \quad . \tag{7.7b}
\]

Cauchy's second law (6.5) reduces to

\[
M_{ipq, i} + \rho L_{pq} + T_{[pq]} = 0 \quad , \tag{7.8}
\]

where \( T_{[pq]} = \frac{1}{2} (T_{pq} - T_{qp}) \) is the antisymmetric part of Cauchy's stress tensor \( \mathfrak{M} \). Both \( \mathfrak{M} = M_{ijk} \tilde{e}_i \tilde{e}_j \tilde{e}_k \) and \( \mathfrak{M} = m_{ij} \tilde{e}_i \tilde{e}_j \) are commonly referred to as couple-stress tensors. Similarly, \( \ell = \ell_k \tilde{e}_k \) and \( \mathfrak{L} = L_{ij} \tilde{e}_i \tilde{e}_j \) are called body couples.

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Consider now the conservation of energy. Equation (5.3) remains the same provided that we modify the rate at which the prescribed body and surface loads, which must include surface and body couples, do work on the continuum, that is, provided that we define \( \mathcal{F} \) as

\[
\mathcal{F} = \int_{\delta} (T_{ij} v_i + \frac{1}{2} m_i w_i) \, da + \int_{\nu} (f_i v_i + \frac{1}{2} \ell_i w_i) \, d\nu,
\]

(7.9)

where \( w_i = \epsilon_{ijk} v_k, j \) is the vorticity vector whose magnitude at a point is twice the angular speed of the rigid-body rotation of a small material neighborhood of that point. Substitution from (7.9), (5.2), and (5.5) into (5.3) now yields

\[
\frac{d}{dt} \int_{\nu} (\varepsilon + \frac{1}{2} v_i v_i) \rho \, d\nu = \int_{\nu} \left( T_{ji} v_i + \frac{1}{2} m_{ji} w_i \right) v_j \, da
\]

\[
+ \int_{\nu} \left( f_i v_i + \frac{1}{2} \ell_i w_i - \frac{q_{i,i}}{\rho} + h \right) \rho \, d\nu,
\]

where (2.5) and (7.2) are also used. This equation may now be written as

\[
\int_{\nu} \left\{ \left( \frac{1}{2} v_i v_i + \varepsilon \right) \left[ \rho \frac{\partial}{\partial t} \left( \rho v_i \right), i \right] + \rho \ddot{v} - \left( T_{ij} D_{ij} + \frac{1}{2} m_{ij} w_j, i - q_{i,i} + h \right) \right. \\
- \frac{1}{2} w_i \left[ m_{ji,j} + \rho \ell_i + \varepsilon_{ijk} T_{jk} \right] v_i \left[ T_{ji,j} + \rho f_i - \rho \dot{v}_i \right] \} \, d\nu = 0
\]

(7.10)

which must hold for any material volume \( \nu \), yielding, in addition to the local conservation of mass (5.8b), and conservation of momentum.
(7.5), the following expression for the local energy-conservation:

\[ \rho \dot{e} = T_{ij} \dot{D}_{ij} + \frac{1}{2} M_{ij} w_{j,i} - q_{i,i} + \rho h \]

\[ = T_{ij} \dot{D}_{ij} + M_{ijk} \dot{W}_{jk,i} + q_{i,i} + \rho h \]  \hspace{1cm} (7.11)

We shall not pursue further the study of the couple-stress theory here and, unless otherwise stated explicitly, throughout the rest of these notes. Interested students will find a number of relevant references cited at the end of this chapter.
4.8 Summary of Basic Equations

Before proceeding to the study of certain constitutive equations of some ideal materials, we summarize in this section the basic equations of motion of a continuum.

a) Conservation of mass is stated as

$$\frac{\partial \rho}{\partial t} + (\rho v_i)_i = 0$$  \hspace{1cm} (8.1a)

where $\rho$ and $v_i$ are, respectively, the instantaneous mass-density and velocity field of the continuum. In terms of the motion

$$x = x(X, t) \quad , \quad 0 \leq t \leq \infty$$  \hspace{1cm} (8.2a)

and its inverse

$$X = X(x, t) \quad , \quad 0 \leq t \leq \infty$$  \hspace{1cm} (8.2b)

the velocity field is defined as

$$v(X, t) = \frac{\partial x}{\partial t} = \dot{x}(X, t)$$  \hspace{1cm} (8.3a)

which may be written as

$$v(x, t) = \dot{x} (x(x), t)$$  \hspace{1cm} (8.3b)

by substitution from (8.2b) into (8.3a).

Equation (8.1a) may also be put into the following form:

$$(\log \rho)' + v \cdot v = 0$$  \hspace{1cm} (8.1b)
This equation, or Eq. (8.1a), is called the spatial equation of continuity. For incompressible (also for isochoric motions) bodies \( \dot{\rho} = 0 \) and we obtain

\[
\nabla \cdot \vec{\mathbf{v}} = v_i, \quad i = \frac{I_D}{D} = 0
\]

(8.1c)

In this case, \( \vec{\mathbf{v}} \) is divergence-free and it may be solenoidal, in which case we have

\[
\vec{\mathbf{v}}(\vec{x}, t) = \nabla \times \vec{A}(\vec{x}, t)
\]

(8.1d)

where the vector potential \( \vec{A} \) is such that

\[
\nabla \cdot \vec{A} = 0
\]

(8.1e)

or if it is irrotational, then the velocity field \( \vec{v} \) is derivable from a scalar potential \( \varphi(\vec{x}, t) \), i.e.,

\[
\vec{v} = \text{grad} \varphi
\]

(8.1f)

and hence

\[
\nabla \cdot \vec{v} = \text{div} \text{grad} \varphi = \nabla^2 \varphi = 0
\]

(8.1g)

b) Conservation of linear momentum has the form

\[
a_i = f_i + \frac{1}{\rho} \sum_{j} T_{ji,j}
\]

(8.4a)

or

\[
a = f + \frac{1}{\rho} \nabla \cdot \vec{v}
\]

(8.4b)
where

\[ \ddot{\mathbf{x}}(\mathbf{\xi}, t) = \frac{\partial \mathbf{v}(\mathbf{\xi}, t)}{\partial t} + \mathbf{v}(\mathbf{\xi}, t) \cdot \nabla \mathbf{v}(\mathbf{\xi}, t) \quad (8.5a) \]

is the acceleration field, \( \mathbf{f} = \mathbf{f}(\mathbf{x}, t) \) is the body force per unit mass, and \( \mathbf{S} = T_{ij} \mathbf{e}_i \mathbf{e}_j \) is Cauchy's stress tensor. Using the expression for vorticity

\[ \mathbf{\omega} = \nabla \times \mathbf{v} \quad , \quad (8.5b) \]

we express the acceleration \( \ddot{\mathbf{x}} \) as

\[ \ddot{\mathbf{x}} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega} \times \mathbf{v} + \frac{1}{2} \text{grad} (\mathbf{v}^2) \quad , \quad (8.5c) \]

where \( \mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v} \). Equation (8.4b) now becomes

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega} \times \mathbf{v} + \nabla (\frac{1}{2} \mathbf{v}^2) = \mathbf{\tau} + \frac{1}{\rho} \mathbf{\omega} \cdot \mathbf{S} \quad . \quad (8.4c) \]

When the body force \( \mathbf{f} \) is derivable from a scalar potential \( U \), i.e.,

\[ \mathbf{f} = - \text{grad} U \quad , \quad (8.6) \]

then the local conservation of linear momentum, Eq. (8.4c), reduces to

\[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{\omega} \times \mathbf{v} = - \text{grad} (U + \frac{1}{2} \mathbf{v}^2) + \frac{1}{\rho} \text{div} \mathbf{S} \quad . \quad (8.4d) \]

Equations (8.4) and (8.5) are expressed in terms of the Eulerian variables \( \mathbf{\xi} \) and \( t \). Introducing the stress tensor

\[ \mathbf{S}^R = T_{\alpha i}^R \mathbf{e}_\alpha \mathbf{e}_i \] defined by

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\[ T^R_{\alpha i} = \frac{\dot{\rho}}{\rho} X_{\alpha, j} T_{ji} \quad (8.7a) \]

and using Lagrangian variables \( \dot{\omega}, t \), we reduce (8.4b) to

\[ \rho_0 \dot{v}_i (\dot{\omega}, t) = \rho_0 F_i (\dot{\omega}, t) + T^R_{\alpha i, \alpha} (\dot{\omega}, t) \quad (8.8a) \]

where \( F_i (\dot{\omega}, t) = f_i (\dot{\omega}, t) \) is body force per unit mass in the initial configuration \( C_0 \), and \( T^R \) is the nonsymmetric, first Piola-Kirchhoff stress tensor. Introducing Kirchhoff's stress tensor

\[ \tilde{\tau} = S_{\alpha \beta} e_\alpha e_\beta = \frac{\rho}{\rho} X_{i, \alpha} X_{j, \beta} T_{ij} e_\alpha e_\beta \quad (8.7b) \]

we obtain, from (8.8a),

\[ \rho_0 \dot{v}_i = \rho_0 F_i + (x_{i, \alpha} S_{\alpha \beta}, \beta) \quad (8.8b) \]

\section{c) Conservation of angular momentum for a non-polar body,}

and excluding couple stresses, asserts the symmetry of Cauchy's stress tensor

\[ \tilde{\tau} = \tilde{\tau}^T \quad (8.9a) \]

or

\[ T_{ij} = T_{ji} \quad (8.9b) \]

In terms of the first Piola-Kirchhoff stress tensor \( T^R \), we have

\[ x_{j, \alpha} T^R_{\alpha i} = x_{i, \alpha} T^R_{\alpha j} \quad (8.10) \]

For a polar continuum with couple stresses, (8.9) must be replaced by

\[ m_{ji, j} + \rho \ell_i + \epsilon_{ijk} T_{jk} = 0 \quad (8.11a) \]
where \( \tilde{\sigma} = m_{ij} \tilde{e}_i \tilde{e}_j \) is the couple-stress tensor, and \( \tilde{\tau} = \ell_i \tilde{e}_i \) is body couple per unit mass. Introducing the tensors
\[
\tilde{\sigma} = M_{ijk} \tilde{e}_i \tilde{e}_j \tilde{e}_k \quad \text{and} \quad \tilde{\tau} = L_{ij} \tilde{e}_i \tilde{e}_j \quad \text{as}
\]
\[
M_{ijk} = \frac{1}{2} \varepsilon_{ijk} \, m_{ij} \quad ,
\]
\[
L_{ij} = \frac{1}{2} \varepsilon_{ijk} \, \ell_k \quad ,
\]
which are dual tensors to couple stress \( \tilde{\sigma} \) and body couple \( \tilde{\tau} \), respectively, we express (8.11a) as
\[
M_{ijk} + \rho L_{jk} + T_{[jk]} = 0 \quad .
\]

\( d) \) Conservation of energy for a non-polar continuum, and excluding couple stresses, has the following local form in terms of the Eulerian variables:
\[
\rho \ddot{\varepsilon} = T_{ij} D_{ij} - q_{i,j} + \rho h \quad ,
\]
where \( \varepsilon \) is the internal energy per unit mass, \( D_{ij} = v_{(i,n)} \) is the deformation-rate tensor, \( q = q_i \tilde{e}_i \) is the heat flux vector, and \( h \) is the heat source per unit mass. The stress power may be written as
\[
\frac{1}{\rho} T_{ij} D_{ij} = \frac{1}{\rho} T_{ij} \dot{E}_{\alpha\beta} X_{\alpha}, i X_{\beta}, j
\]
\[
= \frac{1}{\rho_0} (\frac{\rho_0}{\rho} X_{\alpha}, i X_{\beta}, j T_{ij}) \dot{E}_{\alpha\beta}
\]
\[
- \frac{1}{\rho_0} S_{\alpha\beta} \dot{E}_{\alpha\beta} = \frac{1}{\rho_0} T_{\alpha\beta} X_{\alpha} \dot{r}_{\beta} \quad .
\]

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where $\varepsilon = E_{\alpha\beta} \sim_{\alpha} \sim_{\beta}$ is the Lagrangian strain tensor. Denoting by $\mathbf{Q} = Q_{\alpha} \sim_{\alpha}$ the heat flux vector per unit initial area, Eq. (8.13a) becomes

$$\rho \dot{\varepsilon} : S_{\alpha\beta} \dot{E}_{\alpha\beta} - Q_{\alpha} + p_{0} h = 0$$  \hspace{1cm} (8.13b)$$

which is the local conservation of energy in the Lagrangian description.

When couple stresses are included, Eq. (8.13a) must be replaced by

$$\rho \dot{\varepsilon} = T_{ij} D_{ij} + \frac{1}{2} m_{ij} w_{j,i} - q_{i,i} + \rho h$$

$$= T_{ij} D_{ij} + M_{ijk} W_{jk,i} - q_{i,i} + \rho h \hspace{1cm} (8.15)$$

where Cauchy's spin tensor $\mathbf{w} = W_{ij} \sim_{i} \sim_{j}$ is given by

$$W_{ij} = \frac{1}{2} \varepsilon_{ijk} w_{k} \hspace{1cm} (8.16)$$

e) The Clausius-Duhem inequality has the following local form:

$$\gamma = \gamma_{1} + \gamma_{2} \geq 0 \hspace{1cm} (8.17a)$$

where

$$\gamma_{1} = -\frac{1}{\rho \theta^{2}} q \cdot \text{grad} \theta \hspace{1cm} (8.17b)$$

is the rate of entropy production per unit mass due to heat conduction, and

$$\gamma_{2} = \dot{\eta} - (\frac{1}{\rho \theta} \text{div} q + \frac{h}{\theta}) \hspace{1cm} (8.17c)$$
is the rate of entropy production per unit mass due to energy dissipation. Here $\eta$ is the specific entropy (per unit mass), and $\theta$ is the temperature field.

If we eliminate the heat source $h$ between (8.13a) and (8.17a), we obtain

$$\theta \gamma = \theta \eta - \dot{\varepsilon} + \frac{1}{\rho} \text{tr} (\mathcal{Y} : \dot{\mathcal{E}}) - \frac{1}{\rho \theta} q \cdot \text{grad} \theta \geq 0$$

(8.18)

Introducing the specific free energy, or Helmholtz free energy,

$$\psi = e - \eta \theta$$

(8.19a)

we express (8.18) as

$$\theta \gamma = - \dot{\psi} - \eta \dot{\theta} + \frac{1}{\rho} \text{tr} (\mathcal{Y} : \dot{\mathcal{E}}) - \frac{1}{\rho \theta} q \cdot \text{grad} \theta \geq 0$$

(8.19b)

For a given material volume $V$ with surface $\partial$, a set of field quantities $x(X,t), \mathcal{Y}, e, q, \eta$ constitutes a thermodynamic process if $\mathcal{Y}$ is symmetric (for a non-polar continuum, excluding couple stresses) and the balance equations (8.4a), and (8.13a) are satisfied for some suitable body forces $f$ and heat sources $h$.

The body forces and heat sources obtained in this manner need not correspond to a real situation.

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References


Green, A. E. and Zerna, W., Theoretical Elasticity, Oxford University Press, 1960.


Relevant References on Couple-Stress Theories


PROBLEM IV

1. Let \( \mathbf{u} \) and \( \mathbf{u}' \) denote the unit normals to two elementary areas \( da \) and \( da' \) at a generic material point \( X \) of a deformed body \( B \). If \( \mathcal{T}(\mathbf{u}) \) and \( \mathcal{T}(\mathbf{u}') \) are the respective traction vectors transmitted across \( da \) and \( da' \), show that

\[
\mathcal{T}(\mathbf{u}) \cdot \mathbf{u}' = \mathcal{T}(\mathbf{u}') \cdot \mathbf{u}.
\]

2. For two distinct stress states \( \mathbb{S} \) and \( \mathbb{S}' \) with distinct principal values, show that there exists a common system of principal axes if and only if \( \mathbb{S} \cdot \mathbb{S}' = \mathbb{S}' \cdot \mathbb{S} \).

3. Show that the following quantity

\[
\mathcal{J} = \mathcal{T}_i \cdot \mathcal{T}_i
\]

is a stress-invariant.

4. Show that in the absence of body and inertial forces, the equilibrium equations (2.11) are satisfied identically by the Beltrami stress function \( \Phi_{ij} \) such that

\[
T_{ij} = e_{ipq} e_{jrs} \Phi_{qs,pr}.
\]

In particular, consider cases in which (a) \( \Phi_{ij} \) is diagonal, (b) \( \Phi_{11} = \Phi_{22} = \Phi_{33} = 0 \).

---

1 This is Cauchy's reciprocal theorem.
5. Express the second invariant \( \Pi_T \) of the stress deviator \( \mathbf{\tau}' \) in terms of the principal shearing stresses.

6. Find the principal normal and shearing stresses when the stress tensor is given by

\[
\mathbf{\tau}' = T(\mu \mathbf{\nu} + \nu \mathbf{\mu})
\]

where \( T \) is a scalar and \( \mathbf{\nu} \) and \( \mathbf{\mu} \) are unit vectors. Show that for a uniaxial state of stress in which \( \mathbf{\nu} = \mathbf{\mu} \), the maximum normal and shearing stresses are \( 2T \) and \( T \), respectively.

7. For a perfect fluid the stress tensor \( \mathbf{T} \) is assumed to be spherical for all possible motions of the fluid, that is

\[
\mathbf{T} = -p \mathbf{J}
\]

where \( p \) is regarded as a scalar-valued function of mass-density \( \rho \).

a) Show that if the body force \( \mathbf{f} \) is derivable from a scalar potential \( U \), i.e.,

\[
\mathbf{f} = -\nabla U
\]

then we have the following equation of motion:

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \times \mathbf{v} = -\nabla (U + P + \frac{1}{2} \mathbf{v}^2)
\]

where

\[
P(p) = \int_{p_0}^{p} \frac{dp}{\rho(p)}
\]
b) Under the assumptions in (a), show that the acceleration vector \( \mathbf{a} \) has a potential given by \( U + P \), that is,

\[
\mathbf{a} = -\nabla (U + P)
\]

and that the circulation of the velocity vector along a closed material curve is independent of time.\(^1\) Hence, an instantaneously irrotational motion of a perfect fluid subjected to potential body forces remains irrotational, and the strength of a vortex filament stays constant during the motion of this filament.

8. Let a motion of the perfect fluid in Problem 7 be irrotational, that is, let

\[
\mathbf{\omega} = \nabla \varphi
\]

Show that

\[
\frac{\partial \varphi}{\partial t} + U + P + \frac{1}{2} \mathbf{v}^2 = 0
\]

b) Using the result in (a), derive Bernoulli's equation

\[
U + P + \frac{1}{2} \mathbf{v}^2 = \text{const}
\]

for steady potential flows, where the equation holds with a fixed value of the constant throughout a flow-field.

9. In a state of plane stress, express the principal stresses in terms of the normal tractions acting on three planes which are perpendicular to the plane of stress and form 120° angles with each other.

---

10. Show that from the conservation of energy
\[ \frac{d}{dt} \int_U \frac{1}{2} v_i v_i \rho \, dv + \frac{d}{dt} \int_U \varepsilon \rho \, dv = \int_{\partial U} T_{ij} v_i \, da + \int_U f_i v_i \rho \, dv \]
one may deduce the conservation of mass, and linear and angular momenta if one assumes the invariance of \( \varepsilon, T^{(\nu)}, f, q, \) and \( h \) under superposed rigid-body velocities.

11. Show that the Clausius-Duhem inequality (6.8) does not imply that heat cannot flow against a temperature gradient. Can you think of a case in which heat would flow against a temperature gradient?

12. Obtain the counterpart of (6.12) for a couple-stress theory.

13. Discuss the couple-stress quadric
\[ \pm \mu = m_{ij} y_i y_j \]
where \( \mu \) is a positive, scalar-valued function of \( y_i, i = 1, 2, 3. \)

14. For a continuum with microstructure, let \( \mathcal{Q} = A_{ij} \varepsilon_i \varepsilon_j \) define the excess of angular momentum above the moment of linear momentum, so that the angular momentum per unit mass, taken with respect to origin of coordinates, is given by
\[ x_{[ij]} + A_{ij} \]
Show that the expression for the local conservation of angular momentum takes on the form
\[ M_{ipq, i} + \rho L_{pq} + T_{[pq]} = \rho \dot{A}_{ij} \]
Introduce a dual vector \( \alpha \) to \( \mathcal{Q} \) and obtain the dual of the above equation.
CHAPTER V

CONSTITUTIVE EQUATIONS

The kinematics and dynamics of deformable continua are discussed in Chapters II, III, and IV without any reference to the particular material that may comprise these continua. The motion and deformation of the bodies under extrinsic loads are supposed to be governed by basic laws of the balance of mass, linear and angular momenta, and energy which are assumed to hold for bodies of all kinds. However, bodies consisting of different materials respond differently when subjected to the same extrinsic loads. Hence this difference in the material response must be recognized by introducing constitutive equations. These are a set of equations which relate the kinematical and dynamical quantities of the motion in such a manner as to reflect the basic constitution of the material comprising the considered body. Since the dynamical and kinematical quantities in a continuum theory are defined for macroscopic elements—that is, they do not relate to elements at the atomic, molecular, or even crystalline levels—the corresponding constitutive equations, at best, describe the behavior of an ideal, rather than natural, material. Clearly enough, a constitutive equation of this kind can only be an approximate model for nature, and must be justified experimentally. Nevertheless, there exists a set of basic conditions that a constitutive equation must satisfy before it can qualify for representing any, be it an ideal or real, material. For example,
one expects that material properties, and hence constitutive equations, should not depend on whether one performs his measurements sitting down or standing up, or whether one moves or is stationary relative to the considered body.

In modern treatments of the mechanics of continua, the following principles are often cited for formulating constitutive equations:

1. **Principle of determinism** which states that the history of the motion of a body determines its present state;

2. **Principle of equipresence** which states that, unless shown otherwise, the independent variables entering one constitutive equation should be present in all constitutive equations;

3. **Principle of local action** which states that the material response of a material point $X$ is influenced only by the history of the motion of a small material neighborhood of this particle, that is, action at large distances is not permitted in formulating a material response function;

4. **Principle of the material frame-indeference, or objectivity** which states that a physically acceptable constitutive equation must be invariant under any time-dependent change of frame of reference, the new frame of reference being obtained in such a manner that time intervals, the sense of time, and lengths are preserved;

5. **Principle of entropy production** which states that constitutive equations must comply with the Clausius-Duhem inequality.

Most constitutive equations satisfy trivially the first three conditions. Conditions (4) and (5), however, impose severe restrictions on
the admissible forms of constitutive equations. Needless to say that no constitutive equation should be in violation of the basic conservation laws.

In this chapter we shall study some simple constitutive equations. If all non-mechanical effects are excluded, the constitutive equation may be regarded as a statement defining the stress tensor $\sigma$ at a typical particle $X$, at the time $t$, in terms of the history of the deformation of the body up to the present state $C_t$. Since the stress at a particle $X$ is supposed to represent the mechanical action of the material points in a small neighborhood of this particle, we expect that the stress tensor at $X$, at the time $t$, should only depend on the history of the deformation of this neighborhood, that is, the deformation history of the particles that are located at sufficiently large distances from $X$ should have negligible effects on the present state of stress at this particle. This is the principle of local action. In particular, a material is called simple material if the stress at each particle is determined from the knowledge of the history of only the deformation gradient at that particle. Thus, for a simple material, the stress at a particle at the time $t$ may be written as a functional of the history of the deformation gradient at that particle.

In this chapter, we shall consider constitutive equations of certain special kinds of simple materials, excluding all non-mechanical effects. The thermodynamics of simple materials are considered in Chapter VI.
5.1 History of Deformation. Change of Observer.

In Chapter III we agreed to identify the particles \( X \) with their initial positions \( \tilde{X} \) in the initial configuration \( C_0 \) at the time \( t=0 \), and studied the motion of the continuum using either the spatial or the material description. Unless otherwise stated explicitly, we shall adhere to this convention in the present chapter. Consider a continuum in the configuration \( C_t \) at the present instant \( t \). The positions \( \tilde{\xi} \) of the particles \( \tilde{X} \) at the time \( t - \tau \) can be expressed as

\[
\tilde{\xi} = \tilde{\xi}(X, t - \tau), \quad 0 \leq \tau \leq t,
\]

(1.1a)

where for \( \tau = 0 \) we have the present configuration, i.e.,

\[
\tilde{X} = \tilde{\xi}(X, t) \text{ for } \tau = 0,
\]

(1.1b)

and for \( \tau = t \) we obtain the reference state, i.e.,

\[
\tilde{X} = \tilde{\xi}(X, 0) \text{ for } \tau = t.
\]

(1.1c)

Thus, with \( t \) fixed and \( \tau \) varying from \( t \) to zero, equation (1.1a) defines the history of the deformation relative to the initial configuration \( C_0 \). This equation may also be used to express the past history of the motion relative to the initial state \( C_0 \) by letting the parameter \( \tau \) take on the values larger than \( t \), i.e., \( 0 \leq \tau \leq \infty \). Therefore, we may write

\[
\tilde{\xi} = \tilde{\xi}(X, t - \tau), \quad 0 \leq \tau \leq \infty
\]

(1.1d)
which defines the history of the motion relative to the configuration \( C_0 \) from the far past \( t - \tau = -\infty \) to the initial time \( \tau = t \), and finally up to the present time \( t \) for which \( \tau = 0 \).

The deformation gradient at the time \( t - \tau \) relative to the initial configuration \( C_0 \) is

\[
\nabla \xi = (\varepsilon_{,\alpha} \frac{\partial}{\partial x^\alpha})(\xi_i \ v^1_i)
\]

\[
= \xi_i, \alpha (X, t - \tau) \varepsilon_{,\alpha} v^1_i
\]

\[
= \xi (X, t - \tau)
\]

(1.2)

which, with \( t \) fixed and \( \tau \) variable, defines the history of the deformation gradient relative to the configuration \( C_0 \).

Let us now consider a change in the frame of reference which may be used for describing a motion. A frame of reference in the classical mechanics is a rigid body which carries a device for measuring the time intervals. The most general change of the observer consists of a time-dependent rigid motion and a shift in the origin of time. Such a change preserves time intervals, sense of time, and lengths. If \( \xi = \xi(t) \) defines, with respect to an observer, the position vector of a moving particle, and if \( \xi^* = \xi^*(t) \) is the position vector of the same particle as it appears to another observer, called starred, then we may write

\[
\xi^*(t^*) = \xi^0(t) + \xi(t) \xi (t) \quad , \quad t^* = t - a
\]

(1.3)

where \( \xi^0(t) \) is a time-dependent vector, \( \xi (t) \) is a time-dependent orthogonal tensor, and \( a \) is a scalar. Under the change of frame defined by
the translation \( \xi^0 \), rotation \( \mathcal{Q} \), and time-shift \( \tau \), scalars, vectors, and tensors change. These quantities are said to be objective if we have

\[
\varphi^* (t^*) = \varphi(t) \quad \text{for an objective scalar } \varphi ,
\]

\[
\chi^* (t^*) = \mathcal{Q} (t) \chi(t) \quad \text{for an objective vector } \chi ,
\]

and

\[
\mathcal{G}^* (t^*) = \mathcal{Q} (t) \mathcal{J} (t) \mathcal{Q}^T (t) \quad \text{for an objective second order tensor } \mathcal{J} ,
\]

where we have adopted the convention of matrix multiplication which was mentioned at the end of Section 4 of the last chapter. For example, the second and third equations in (1.4a) have the following forms in a fixed rectangular Cartesian coordinate system:

\[
v^*_i (t^*) = \mathcal{Q}_{ij} (t) \ n_j^i (t) ,
\]

\[
\mathcal{T}_{ij}^* (t^*) = \mathcal{Q}_{im} (t) \mathcal{Q}_{jn} (t) \ T_{mn} (t) .
\]  

(1.4b)

In particular, the motion (1.1d) and the deformation gradient (1.2) become

\[
\xi^* (X, t^* - \tau^*) = \xi^0 (t - \tau) + \mathcal{Q} (t - \tau) \ \xi (X, t - \tau) 
\]

(1.5a)

\[
\mathcal{J}^* (X, t^* - \tau^*) = \mathcal{Q} (t - \tau) \ \mathcal{J} (X, t - \tau) .
\]  

(1.5b)

Note that the change of the observer is more than just a change in the coordinate system. In Chapter I a tensor quantity is viewed as an invariant entity that exists independently of a particular fixed coordinate system.

\[\text{† Note that the right side of (1.5b) must be written as } \mathcal{J} \cdot \mathcal{J}^T \text{ if dyadic rule of multiplication is used.} \]
system which may be employed for the specification of its components. Accordingly, the velocity vector of a moving particle is an invariant quantity, since it is a vector quantity. But this velocity vector is not an objective quantity, since under the time-dependent transformation (1.3) ---that is under the change of the frame of reference---we obtain

\[ \dot{x}^* = \dot{x}^0(t) + \ddot{x}(t) \dot{x}(t) + \dddot{x}(t) x(t) \]

which does not comply with (1.4) and, therefore, is not objective.

Similarly, the acceleration vector of a moving particle is not objective. Thus, Cauchy's first law of motion, which is stated in terms of the acceleration, is not objective. Nevertheless, we shall require that all constitutive assumptions satisfy the principle of objectivity.

The stress tensor, which is introduced in order to represent any possible molecular interaction within the body, can be required to be objective. Thus, for any orthogonal, time-dependent, second order tensor \( \mathcal{Z}(t) \), we shall require that

\[ \mathcal{Z}^* = \mathcal{Z} \mathcal{Z}^T \quad (1.6a) \]

where \( \mathcal{Z} = T_{ij} \dot{x}_i \dot{x}_j \) is Cauchy's stress tensor. On the other hand, kinematical quantities may or may not be objective. For example, it may easily be shown that the stretching tensor \( \mathcal{S} \) is objective, i.e.,

\[ \mathcal{S}^* = \mathcal{S} \mathcal{S}^T \quad (1.6b) \]
while the spin tensor \( \mathbb{\omega} \) is not, i.e.,

\[
\mathbb{\omega}^* = \mathbb{\omega} \mathbb{\omega}^T + \mathbb{\omega} \mathbb{\omega}^T
\]

(1.6c)

where the last term in the right side is the spin of the starred frame of reference relative to the original observer, i.e., the unstarred frame of reference. We also note that while Cauchy’s stress tensor \( \mathfrak{J} \) is regarded as objective, its material time-derivative is not. Taking the material time-derivative of both sides of (1.6a), we obtain

\[
\ddot{\mathfrak{J}}^* = \mathbb{\omega} \dot{\mathfrak{J}} + \mathfrak{J} \mathbb{\omega}^T + \mathbb{\omega} \dot{\mathfrak{J}} + \mathfrak{J} \mathbb{\omega}^T
\]

(1.7a)

But, we have

\[
\mathbb{\omega} \mathbb{\omega}^T = \mathbb{\omega} \quad \text{or} \quad Q_{ik} Q_{jk} = \delta_{ij}
\]

(1.8a)

from which we obtain

\[
\mathfrak{J} \mathbb{\omega}^T = - \mathbb{\omega} \mathfrak{J}^T
\]

\[
= - \mathbb{\omega}^* + \mathbb{\omega} \mathbb{\omega}^T
\]

(1.8b)

where (1.6c) is used. Multiplying both sides of (1.8b) from left by \( \mathbb{\omega}^T \), we obtain

\[
\mathfrak{J}^T = - \mathbb{\omega}^T \mathbb{\omega}^* + \mathbb{\omega} \mathbb{\omega}^T
\]

(1.8c)

or

\[
\dot{Q}_{ji} = - Q_{ki} W_{kj}^* + W_{ik} Q_{jk} = Q_{ki} W_{jk}^* + Q_{jk} W_{ik}
\]

(1.8d)
Now, substitution into (1.7a) yields
\[
\frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \nabla \mathbf{V} + \frac{\partial \mathbf{V}}{\partial t} \mathbf{V}^T \mathbf{A} = \mathbf{B} \left( \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \nabla \mathbf{V} + \frac{\partial \mathbf{V}}{\partial t} \mathbf{V}^T \mathbf{A} \right) \mathbf{V}^T
\]
which shows that, while \(\frac{\partial \mathbf{A}}{\partial t}\) is not objective, the quantity
\[
\mathbf{A}^0 = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{A} \cdot \nabla \mathbf{V} + \frac{\partial \mathbf{V}}{\partial t} \mathbf{V}^T \mathbf{A}
\]
is called stress-flux or co-rotational stress-rate. In a component-form, we have
\[
T_{ij} = \dot{T}_{ij} + T_{ik} \dot{W}_{kj} - W_{ik} T_{kj}
\]
We note that, since \(\mathbf{A} \mathbf{V}\) is an objective quantity, other objective stress-rates can be generated by adding to (1.9b) terms of this kind. For example, we may consider
\[
\dot{T}_{ij} = T_{ij} + T_{ik} \dot{L}_{kj} + L_{ki} T_{kj}
\]
or
\[
\dot{L}_{ij} = L_{ij} \dot{e}_i \dot{e}_j = \dot{x}_i \dot{e}_i \dot{e}_j
\]
where
\[
\dot{L} = L_{ij} \dot{e}_i \dot{e}_j
\]
is the velocity gradient. The quantity \(\dot{T}_{ij}\) is called the convected stress-rate. It is obtained by adding to the co-rotational stress-rate the objective quantity \(\mathbf{A} \mathbf{V} + \mathbf{V}^T \mathbf{A}\).
5.2 Simple Materials

A general theory of nonlinear constitutive equations of materials with memory was developed by Green and Rivlin\(^1\) in 1957, 1961, and by Green, Rivlin and Spencer\(^1\) in 1959. A somewhat different formulation was later proposed by Noll\(^1\) in 1958, and has since been generalized and further developed in a large body of literature, which is surveyed up to 1965 by Truesdell and Noll.\(^1\) The presentation in this chapter follows that of these latter authors.

A continuum is said to consist of a **simple material** if the stress tensor \( \mathbf{\Sigma} \) at each particle \( \mathbf{X} \), at the time \( t \), can be expressed as a functional of the history of the deformation gradient \( \mathbf{G}(\mathbf{X}, t - \tau), 0 \leq \tau \leq \infty \), at that particle, i.e.,

\[
\mathbf{\Sigma}(\mathbf{X}, t) = \lim_{\tau \to 0^+} \mathcal{Q}(\mathbf{G}(\mathbf{X}, t - \tau); \mathbf{X}) \quad (2.1)
\]

Note the meaning of the right-hand side of (2.1). For a given tensor-valued function \( \mathbf{G}(\mathbf{X}, t - \tau) \), defined at a fixed particle \( \mathbf{X} \) for all values of the parameter \( \tau \) from \( \tau = 0 \) to \( \tau = \infty \), the value of the tensor-valued functional \( \mathcal{Q} \) is the symmetric stress tensor \( \mathbf{\Sigma} \) at the particle \( \mathbf{X} \) at the time \( t \). In (2.1), the explicit dependence of \( \mathcal{Q} \) on \( \mathbf{X} \) accounts for possible material inhomogeneity.

Materials of this kind are called simple because one can study their constitutive properties by subjecting them to **homogeneous deformations**, that is, deformations in which the

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\(^1\) See References at the end of this chapter.
history of the deformation gradient \( \overset{*}{\mathcal{J}} \) is independent of \( X \), and, hence, is the same for all particles of the considered body.

The constitutive equation (2.1) complies trivially with the principles of determinism, and local action, and the principle of equipresence is irrelevant in the present context. Hence, we must subject (2.1) to the requirement of the material frame-indeference, that is, we must have

\[
\overset{*}{\mathcal{J}}^{*} = \underset{\tau=0}{\mathcal{Q}} (\overset{*}{\mathcal{J}}(X, t-\tau); X), \quad (2.2a)
\]

where, without loss of generality, the time-shift \( \tau \) is taken equal to zero. Since \( \overset{*}{\mathcal{J}} \) is objective, we obtain

\[
\mathcal{Z}(t) \overset{*}{\mathcal{J}} \mathcal{Z}^{T}(t) = \underset{\tau=0}{\mathcal{Q}} (\mathcal{Z}(t-\tau) \overset{*}{\mathcal{J}}(X, t-\tau); X) \quad (2.2b)
\]

which must hold identically for all non-singular \( \overset{*}{\mathcal{J}} \) and all orthogonal \( \mathcal{Z} \).

Now, since the deformation gradient \( \overset{*}{\mathcal{J}} \), at a typical particle and time \( t-\tau \), can be decomposed as (see Eqs. (II-3.33) and (II-6.9))

\[
\overset{*}{\mathcal{J}}(X, t-\tau) = \overset{T}{\mathcal{U}}(X, t-\tau) \overset{*}{\mathcal{J}}(X, t-\tau), \quad (2.3)
\]

we may choose \( \mathcal{Z} = \overset{T}{\mathcal{U}} \) and obtain from (2.2b)

\[
\overset{T}{\mathcal{U}} \overset{*}{\mathcal{J}} \overset{T}{\mathcal{U}} = \underset{\tau=0}{\mathcal{Q}} (\overset{T}{\mathcal{U}}(X, t-\tau); X), \quad (2.2c)
\]

where the identity \( \overset{T}{\mathcal{U}} \overset{T}{\mathcal{U}} = \mathcal{J} \) is used. This equation may be written as
\[ \bar{\sigma}(\bar{x}, t) = \bar{\sigma}(\bar{x}, t) \begin{array}{c} T=\infty \\ \tau = 0 \end{array} \bar{\xi} \left( \bar{\xi}(\bar{x}, t-\tau); \bar{x} \right) \tilde{\bar{\alpha}}^T(\bar{x}, t) \] (2.2d)

which states that, for simple materials, the stress at each particle \( \bar{x} \), at the time \( t \), depends only on the history of the stretch tensor \( \bar{\xi} = \bar{\xi}^\tau \) and the instantaneous value of the rotation tensor \( \tilde{\bar{\alpha}} \) at that instant and that particle. Moreover, the rotation tensor \( \tilde{\bar{\alpha}} \) appears explicitly and does not enter in the functional \( \bar{\xi} \) at all.

In the absence of all non-mechanical effects, Eq. (2.2d) defines the most general form of the constitutive equation of simple materials. Various specializations of (2.2d) yield the constitutive equations of most ideal materials such as elastic solids, and viscous fluids which are considered in the subsequent sections.
5.3 Finite Elasticity

A body that consists of a perfectly elastic material is characterized by its preferred or natural configuration, to which it returns if the forces that maintain it in the deformed configuration are all released. Elastic materials, therefore, have a perfect memory for the preferred state and no memory for the history of the deformation from the preferred state to the final state. Taking the initial configuration $C_0$ coincident with the preferred natural state, we write the constitutive equation of a perfectly elastic material as

$$\varepsilon = \mathcal{G} (\mathcal{F}; \mathcal{X})$$

(3.1)

where $\mathcal{G}$ is a symmetric, second order tensor-valued function—and not a functional—of the deformation gradient $\mathcal{F}(X, t) = x_i^\alpha \sim_\alpha \sim_1$ at the particle $X$. Application of the material frame-indifference now yields

$$\varepsilon (X) = \hat{\mathcal{G}} \hat{\mathcal{U}} (\mathcal{U}(X, t); X) \mathcal{F}^T$$

(3.2)

where $\hat{\mathcal{G}} = \mathcal{G} \hat{\mathcal{U}}$ by the polar decomposition. This equation may also be expressed as

$$\mathcal{F}^T \varepsilon \mathcal{F} = \mathcal{U} \hat{\mathcal{G}} (\mathcal{U} ; X) \mathcal{U} = \mathcal{C}^{\mathcal{F}} \mathcal{G} (\mathcal{C}^{\mathcal{F}} ; X) \mathcal{C}^{\mathcal{F}} = \mathcal{G} (\mathcal{C} ; X)$$

or as

$$\hat{\mathcal{G}} = \mathcal{G} (\mathcal{C} ; X)$$

(3.3a)

where

$$\hat{\mathcal{G}} = \mathcal{F}^T \varepsilon \mathcal{F} = T_{ij} x_i^\alpha \sim_\alpha \sim_1 x_j^\beta \sim_\beta \sim_1$$

(3.3b)

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\[ T_{ij} = X_\alpha, i X_\beta, j \tilde{T} \alpha \beta \]
\[ = X_\alpha, i X_\beta, j \tilde{s} \alpha \beta \left( \Xi; X \right) \quad (3.3c) \]

The symmetric tensor \( \Xi \) is the convected stress tensor.

When the material is homogeneous, the response function \( \Xi \), or \( \xi \), is the same for all particles \( X \) and, therefore, independent of \( X \).

For an isotropic material, on the other hand, there exists no preferred direction in the natural state \( C_0 \). Thus, the form of the components of the tensor \( \Xi \) are the same in all rectangular Cartesian coordinate systems, and we must have

\[ \Xi \Xi^T = \Xi (\xi_0 \Xi^T; X) \quad (3.4) \]

where \( \Xi \) is an orthogonal constant tensor. Note that an isotropic material may be nonhomogeneous, that is, homogeneity and isotropy are two distinct properties which may or may not coexist.

Let us now show that, for an isotropic material, the principal axes of the convected stress tensor \( \Xi \) and Green's deformation tensor \( C \) coincide. For isotropy, we have

\[ \Xi (\Xi \Xi^T) = \Xi \Xi (\Xi) \Xi^T \quad (3.5) \]

where the dependence on \( X \) may or may not exist according to whether the material is or is not heterogeneous, and where \( \Xi \) is a constant orthogonal tensor. Let the principal directions of \( \Xi \) be \( \Xi^J \), \( J = I, II, III \), and, for a fixed \( J \), let \( \Xi \) denote a reflection about a plane with the unit normal \( \Xi^J \), i.e.,

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\[ \sim M^J = - \sim M^J \]

Equation (3.5) now becomes

\[ s (\sim \mathcal{C}) = \sim s (\mathcal{C}) \sim^T \]

or

\[ \sim \sim = \sim \sim \]

and thus

\[ \sim \sim M^J = \sim \sim M^J = - \sim \sim M^J \quad (3.6) \]

Therefore, the reflection \( \sim \) transforms the vector \( \sim M^J \) into its negative, implying that \( \sim M^J \) is parallel to \( \sim M^J \), that is,

\[ s_{\alpha \beta} M^J_B = s_J M^J_\alpha \quad , \quad J = I, \text{II, III} \quad , \quad (\text{no sum on } J) \quad (3.7) \]

This equation states that \( \sim M^J \) is the characteristic vector and \( s_J \) the characteristic number of the second order, symmetric tensor \( \sim \).

Since \( \sim M^J \) is also the principal direction of Green's deformation tensor \( \mathcal{C} \), we have proved that \( \sim \) and \( \mathcal{C} \) have the same principal axes.

Let us assume that \( \mathcal{C} \) has three distinct principal numbers \( C_J \), \( J = I, \text{II, III} \), and write

\[ s_J = \varphi_0 + \varphi_1 C_J + \varphi_2 C_J^2 \quad , \quad J = I, \text{II, III} \quad , \quad (3.8) \]

where \( \varphi_A \), \( A = 0, 1, 2 \), are functions of the invariants of the tensor \( \mathcal{C} \), i.e.,

\[ \varphi_A = \varphi_A (I_C, \Pi_C, \Pi_C) \quad , \quad A = 0, 1, 2 \quad (3.9) \]

The determinant of the coefficients of the unknown functions \( \varphi_A \), \( A = 0, 1, 2 \), is
\[
\Delta = \begin{vmatrix}
1 & C_I & C_I^2 \\
1 & C_{II} & C_{II}^2 \\
1 & C_{III} & C_{III}^2 \\
\end{vmatrix} = (C_I - C_{II})(C_I - C_{III})(C_{II} - C_{III})
\]

Since \( C_J \), \( J = I, II, III \), are distinct, \( \Delta \) is not zero and (3.8) has a unique solution for \( \varphi_A \), \( A = 0, 1, 2 \). Referred to its principal directions, \( \varphi \) may be written as

\[
\varphi = \sum_{J=1}^{III} s_J M^J M^J = \varphi_0 \varphi + \varphi_1 C + \varphi_2 C^2
\]

(3.10a)

Equation (3.10a) is in the invariant form and, thus, valid in any coordinate system. Note that the representation (3.10a) is obtained on the assumption that \( C_J \), \( J = I, II, III \), are distinct. If the tensor \( \varphi \) has multiple proper numbers, then all of the equations in (3.8) are not independent. In this case, some of \( \varphi_A \), \( A = 0, 1, 2 \), may be zero. In the component form, (3.10a) is

\[
s_{\alpha \beta} = \varphi_0 \delta_{\alpha \beta} + \varphi_1 c_{\alpha \beta} + \varphi_2 \varphi C_{\alpha \gamma} C_{\gamma \beta}
\]

(3.10b)

Substitution from (3.10b) into (3.3c) yields

\[
T_{ij} = \varphi_0 c_{ij} + \varphi_1 \delta_{ij} + \varphi_2 b_{ij}
\]

(3.10c)

where \( \varepsilon = c_{ij} \varepsilon^i \varepsilon^j \) is Cauchy's deformation tensor, and \( \varepsilon = b_{ij} \varepsilon^i \varepsilon^j \) is Finger's strain tensor. From Eq. (11-4.5a) we have

\[
\varepsilon = b^{-1}
\]

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which reduces (3.10c) to

\[ \Psi = \psi_0 \Psi + \psi_1 \Sigma + \psi_{-1} \Sigma^{-1} \quad , \quad (3.11a) \]

where

\[ \psi_0 = \varphi_1 \quad , \quad \psi_1 = \varphi_2 \quad , \quad \psi_{-1} = \varphi_0 \quad . \quad (3.11b) \]

The functions \( \psi_A \), \( A = 0, 1, -1 \), depend only on the invariants of Finger's strain tensor \( \Sigma \), i.e.,

\[ \psi_A = \psi_A (I_b, II_b, III_b) \quad , \quad A = 0, 1, -1 \quad . \quad (3.11c) \]

When the body is in the initial state \( C_0 \), the stress tensor vanishes and we must have

\[ \psi_0 (3, 3, 1) + \psi_1 (3, 3, 1) + \psi_{-1} (3, 3, 1) = 0 \quad , \quad (3.12a) \]

since, at this state, we have

\[ I_b = II_b = 3 \quad , \quad III_b = 1 \quad . \]
5.4 Hyperelasticity

A simple material is called hyperelastic if there exists a scalar function \( \sigma = \sigma(\varepsilon, X) \), called elastic potential or stored energy function, \(^*\) such that Kirchhoff's stress tensor \( \mathcal{G} \) at any particle \( X \) is given by

\[
\mathcal{G}(X) = \frac{\partial \sigma(\varepsilon, X)}{\partial \varepsilon_{\alpha\beta}} \varepsilon_{\alpha} \varepsilon_{\beta},
\]

(4.1)

that is, for hyperelastic materials, the stress tensor \( \mathcal{G} \) is derivable from a scalar potential in a same manner as conservative forces are derivable from a potential. Note that, since in the absence of non-mechanical effects Eq. (IV-5.13c) reduces to

\[
\rho_0 \dot{e} = S_{\alpha\beta} \varepsilon_{\alpha\beta},
\]

(4.2)

we may write

\[
S_{\alpha\beta} \dot{e}_{\alpha\beta} = \frac{\partial \sigma}{\partial \varepsilon_{\alpha\beta}} \dot{e}_{\alpha\beta}
\]

\[
= \ddot{\sigma} = \rho_0 \dot{e}, \quad (4.3)
\]

which states that \( \frac{1}{\rho_0} \sigma \) is the internal energy per unit mass. Hence, the work done in deforming a body that consists of a hyperelastic material is stored as an internal energy which can be recovered completely. Substitution from (4.1) into (IV-4.5) now yields

\(^*\) \( \varepsilon \) is Lagrangian strain tensor.
\[ T_{ij} = \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial \mathcal{C}_{\alpha\beta}} x_i, \alpha^j, \beta \]  \hspace{1cm} (4.4a)

Instead of expressing \( \Sigma \) in terms of the Lagrangian strain tensor \( \mathcal{C} \), we may use Green's deformation tensor and obtain

\[ T_{ij} = 2 \frac{\rho}{\rho_0} \frac{\partial \Sigma}{\partial \mathcal{C}_{\alpha\beta}} \mathcal{F}_{ij} x_i, \alpha^j, \beta \]  \hspace{1cm} (4.4b)

where

\[ \Sigma = \sigma \left( \frac{1}{2} (\mathcal{C} - \mathcal{D}) \mathcal{X} \right) = \Sigma(\mathcal{C} ; \mathcal{X}) \]

For isotropic hyperelastic materials, the scalar potential function \( \Sigma \) is a function of the strain invariants only, that is,

\[ \Sigma = \Sigma(I_1, \Pi_1, \Pi_3 ; \mathcal{X}) = \Sigma(I_1, \Pi_1, \Pi_3 ; \mathcal{X}) \]  \hspace{1cm} (4.5a)

since the basic invariants of \( \mathcal{C} \) and \( \mathcal{D} \) are the same. In the following, we shall drop the subscripts and write \( I, \Pi_1, \) and \( \Pi_3 \) which are to be interpreted as the basic invariants of \( \mathcal{C} \) or \( \mathcal{D} \). Equation (4.5a) thus becomes

\[ \Sigma = \Sigma(I, \Pi_1, \Pi_3 ; \mathcal{X}) \]  \hspace{1cm} (4.5b)

which yields

\[ \frac{\partial \Sigma}{\partial \mathcal{C}_{\alpha\beta}} = \frac{\partial I}{\partial \mathcal{C}_{\alpha\beta}} + \frac{\partial \Pi_1}{\partial \mathcal{C}_{\alpha\beta}} + \frac{\partial \Pi_3}{\partial \mathcal{C}_{\alpha\beta}} + \frac{\partial \Pi_3}{\partial \mathcal{C}_{\alpha\beta}} \]  \hspace{1cm} (4.6)

Using Eqs. (II-3.15), we obtain

\[ \frac{\partial I}{\partial \mathcal{C}_{\alpha\beta}} = \delta_{\alpha\beta} \]

\[ \frac{\partial \Pi_1}{\partial \mathcal{C}_{\alpha\beta}} = C_{\gamma\gamma} \delta_{\alpha\beta} - C_{\alpha\beta} \]

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\[
\frac{\partial \Pi}{\partial C_{\alpha\beta}} = C_{\alpha\gamma} C_{\gamma\beta} - I C_{\alpha\beta} + II \delta_{\alpha\beta} \\
= III B_{\alpha\beta},
\] (4.7)

where \( B = C^{-1} \) by equation (II-4.5b). Substitution from (4.7) into (4.6) gives

\[
\frac{\partial \Sigma}{\partial C_{\alpha\beta}} = \frac{\partial \Sigma}{\partial I} \delta_{\alpha\beta} + \frac{\partial \Sigma}{\partial III} (I \delta_{\alpha\beta} - C_{\alpha\beta}) + \frac{\partial \Sigma}{\partial III} III B_{\alpha\beta}.
\] (4.8)

From (4.8) and (4.4b) we now arrive at

\[
T_{ij} = 2 \frac{\rho}{\rho_0} \left\{ III \frac{\partial \Sigma}{\partial III} \delta_{ij} + \left( \frac{\partial \Sigma}{\partial I} + I \frac{\partial \Sigma}{\partial III} \right) b_{ij} - \frac{\partial \Sigma}{\partial III} b_{ik} b_{kj} \right\}.
\] (4.9a)

But we also have

\[ b_{ik} b_{kj} = I b_{ij} - II \delta_{ij} + III c_{ij} \]

which reduces (4.9a) to the following form:

\[
T_{ij} = 2 \frac{\rho}{\rho_0} \left\{ \left( II \frac{\partial \Sigma}{\partial III} + III \frac{\partial \Sigma}{\partial III} \right) \delta_{ij} + \frac{\partial \Sigma}{\partial III} b_{ij} - III \frac{\partial \Sigma}{\partial III} c_{ij} \right\}.
\] (4.9b)

This equation may be written as

\[
\bar{\Sigma} = 2 \frac{\rho}{\rho_0} \left[ \left( II \frac{\partial \Sigma}{\partial III} + III \frac{\partial \Sigma}{\partial III} \right) \bar{g} + \frac{\partial \Sigma}{\partial I} \bar{b} - III \frac{\partial \Sigma}{\partial III} \bar{b}^{-1} \right]
\] (4.10)

which, after comparison with (3.11a), yields the following expressions for the functions \( \psi_i \):

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\[ \psi_0 = \frac{2}{\sqrt{III}} \left( II \frac{\partial \Sigma}{\partial II} + III \frac{\partial \Sigma}{\partial III} \right), \]
\[ \psi_1 = \frac{2}{\sqrt{III}} \frac{\partial \Sigma}{\partial I}, \]
\[ \psi_{-1} = -2 \sqrt{III} \frac{\partial \Sigma}{\partial II}. \]  

(4.11)

The stress-strain relation (4.10) is due to Finger.

When the material is incompressible, we have \( III = 1 \), and the deformation-work is only due to deviatoric part of the stress tensor, since trace \( \Sigma \cdot \Sigma = 0 \). From equations (4.3) and (IV-5.7) we have

\[ \dot{\sigma} = \rho_0 \dot{\varepsilon} = \frac{\rho_0}{\rho} T_{ij} D_{ij} \]

which may also be written as

\[ \frac{\partial \Sigma}{\partial \Sigma_{\alpha \beta}} \dot{\Sigma}_{\alpha \beta} = 2 \frac{\partial \Sigma}{\partial \Sigma_{\alpha \beta}} x_i, \alpha \cdot x_j, \beta D_{ij} = T_{ij} D_{ij}, \]  

(4.12)

where equations (III-5.2c and d) are also used. Now, for compressible hyperelastic materials, equation (4.12) immediately reduces to (4.4b). But, for incompressible materials, (4.12) must be solved subject to the constraint

\[ D_{ij} \delta_{ij} = 0 \]  

(4.13)

Introducing Lagrange's multiplier \( p \), we obtain, from (4.12) and (4.13),
\[ D_{ij} \left\{ 2 \frac{\rho}{\rho_0} \frac{\delta \Sigma}{\delta \alpha \beta} x_i, \alpha x_j, \beta - T_{ij} - p \delta_{ij} \right\} = 0 \]

which yields

\[ T_{ij} = - p \delta_{ij} + 2 \frac{\delta \Sigma}{\delta \alpha \beta} x_i, \alpha x_j, \beta \] \quad (4.14)

where \( p \) is an arbitrary hydrostatic pressure. For isotropic, incompressible, hyperelastic bodies, (4.14) reduces to

\[ \Sigma = - p \beta + 2 \frac{\delta \Sigma}{\delta \alpha I} \beta - 2 \frac{\delta \Sigma}{\delta \alpha \Pi} \beta^{-1} \] \quad (4.15)

where \( \Sigma = \Sigma(I, \Pi, \beta) \). Equation (4.15) is due to Rivlin.

Ideal materials for which the constitutive Eq. (4.15) reduces to

\[ \Sigma = - p \beta + \alpha \beta + \beta \beta^{-1} \] \quad (4.16a)

where \( \alpha = 2 \frac{\delta \Sigma}{\delta \alpha \Pi} \), and \( \beta = - 2 \frac{\delta \Sigma}{\delta \alpha \Pi} \) are assumed to be constants characterizing the material, were considered by Mooney for representing the elastic behavior of rubber-like materials. With \( \beta = 0 \), we obtain the stress-strain law for the so-called neo-Hookean materials considered by Rivlin. Note that the elastic potential corresponding to Eq. (4.16a) is

\[ \Sigma = \frac{1}{2} \alpha (I - 3) - \frac{1}{2} \beta (\Pi - 3) \] \quad (4.16b)

From Eq. (4.15), the principal stresses \( T_J, J = I, \Pi, \Pi \), can be related to the principal stretches \( \Lambda_{(\beta)} \) as follows:

\[ T_J = - p + 2 \frac{\delta \Sigma}{\delta \alpha I} \Lambda_{(J)}^2 - 2 \frac{\delta \Sigma}{\delta \alpha \Pi} (\Lambda_{(J)}^2) \beta^{-1} \] \quad (4.17a)
where, because of incompressibility, we also have

\[ \Lambda_{(I)} \Lambda_{(II)} \Lambda_{(III)} = 1 \quad . \quad (4.17b) \]

We eliminate \( p \) in Eqs. (4.17a), and obtain the following expressions for the differences of the principal stresses:

\[
T_1 - T_2 = 2(\Lambda_{(I)}^2 - \Lambda_{(II)}^2)[\frac{\partial \sigma}{\partial I} + \Lambda_{(III)}^2 \frac{\partial \sigma}{\partial II}] \quad ,
\]

\[
T_2 - T_3 = 2(\Lambda_{(II)}^2 - \Lambda_{(III)}^2)[\frac{\partial \sigma}{\partial I} + \Lambda_{(I)}^2 \frac{\partial \sigma}{\partial II}] \quad ,
\]

\[
T_3 - T_1 = 2(\Lambda_{(III)}^2 - \Lambda_{(I)}^2)[\frac{\partial \sigma}{\partial I} + \Lambda_{(II)}^2 \frac{\partial \sigma}{\partial II}] \quad ,
\]

which are useful for application to special problems.
5.5 Simple Fluids

A simple fluid is a simple material for which the stress tensor \( \mathcal{F} \) at each particle \( X \) at time \( t \) is defined as a functional of the history of the deformation gradient at \( X \) taken relative to a configuration of the body, that is, there exists no preferred state and any configuration may be used as a reference. Thus, we may use the instantaneous configuration \( C_t \) for the reference configuration and write

\[
\mathcal{F} = \mathcal{F}(\mathcal{X}, t - \tau), \quad 0 \leq \tau \leq \infty
\] (5.1a)

which defines at time \( t - \tau \) the position \( \mathcal{X} \) of the particle \( X \) that instantaneously—that is, at the present time \( t \)—is at point \( \mathcal{X} \). For \( \tau = 0 \), we have

\[
\mathcal{X} = \mathcal{F}(\mathcal{X}, t); \quad \tau = 0
\] (5.1b)

The deformation gradient at time \( t - \tau \) may now be defined relative to the instantaneous configuration \( C_t \) as

\[
\mathcal{F}_t(\mathcal{X}, t - \tau) = \left( \frac{\partial}{\partial x_i} e_i \right)(\mathcal{F}_j(\mathcal{X}, t - \tau) e_j)
\]

\[
= \mathcal{F}_{j,i}(\mathcal{X}, t - \tau) e_i e_j
\] (5.2)

where the subscript \( t \) in the left side of (5.2) is to stress the fact that the configuration \( C_t \) at the present time \( t \) is used as the reference. For \( 0 \leq \tau \leq \infty \), this equation defines the history of the deformation gradient relative to the configuration at the present time \( t \). The history of Green's deformation tensor may be written as
\[ C_{\tau t}(\tilde{\chi}, t - \tau) = \tilde{\tilde{\chi}} T_{\tau t} \tilde{\tilde{\chi}} \]  
\[ (5.3a) \]

and that of Finger's strain tensor as

\[ b_{\tau t}(\tilde{\chi}, t - \tau) = \tilde{\tilde{\chi}}_{\tau t} \tilde{\tilde{\chi}} T \]  
\[ (5.3b) \]

where the subscript \( t \) is again to denote the fact that present configuration \( C_t \) is used as the reference. Note that, from (5.3a, b), we have

\[ C_t(\tilde{\chi}, t) = b_{\tau t}(\tilde{\chi}, t) = \tilde{\tilde{\chi}} \]  
\[ (5.3c) \]

The time-derivatives \( G_n \) of the tensor \( C_{\tau t} \) at \( \tau = 0 \), i.e.,

\[ (-1)^n \frac{d^n}{d\tau^n} C_{\tau t}(\tilde{\chi}, t - \tau) \bigg|_{\tau=0} = G_n \]  
\[ (5.4a) \]

are called the Rivlin-Ericksen tensors. For \( n = 1 \), we have

\[ G_1 = - \frac{d}{d\tau} \left\{ \tilde{\tilde{\chi}} T_{\tau t}(\tilde{\chi}, t - \tau) \tilde{\tilde{\chi}}_{\tau t}(\tilde{\chi}, t - \tau) \right\} \bigg|_{\tau=0} = 2 \tilde{\tilde{\chi}} \]  
\[ (5.4b) \]

and for \( n = 2 \), we obtain

\[ G_2 = \tilde{\tilde{\chi}}_2 + \tilde{\chi}_2 T + 2 \tilde{\chi}_1 T \tilde{\chi}_1 \]  
\[ (5.5a) \]

where

\[ \tilde{\tilde{\chi}}_n = (-1)^n \frac{d^n}{d\tau^n} \tilde{\tilde{\chi}}_{\tau t}(\tilde{\chi}, t - \tau) \bigg|_{\tau=0}, \quad n = 1, 2, \ldots, \]  
\[ (5.5b) \]

and

\[ \tilde{\tilde{\chi}}_0 = \tilde{\tilde{\chi}} \]  
\[ (5.5c) \]
Also (5.4a) may be written as

\[ \mathcal{G}_n = \sum_{k=0}^{n} \binom{n}{k} \mathcal{Z}_k T \mathcal{Z}_{n-k} \]  

(5.4d)

Let us now consider the constitutive equation of simple fluids.

In the general case, this constitutive equation defines the stress tensor \( \mathcal{F} \) at a particle \( X \) at time \( t \) as a functional of the history of Green's deformation tensor \( \mathcal{C}_t (\mathcal{X}, t - \tau) \), i.e.,

\[ \mathcal{F}(\mathcal{X}, t) = \frac{\tau = \infty}{\tau = 0} \left( \mathcal{C}_t (\mathcal{X}, t - \tau); \mathcal{P}(t), X, t \right), \]  

(5.6a)

where the dependence on the particle \( X \) that at time \( t \) is at point \( \mathcal{X} \) is denoted in order to include possible hetrogeneities. This equation must satisfy the following identity:

\[
\mathcal{Z} \left\{ \frac{\mathcal{G}}{\tau = 0} \left( \mathcal{C}_t (\mathcal{X}, t - \tau); \mathcal{P}(t), X, t \right) \right\} \rightarrow \mathcal{T}_t
\]

\[
= \mathcal{G} \left( \mathcal{Z} \left\{ \frac{\mathcal{C}_t (\mathcal{X}, t - \tau) \rightarrow \mathcal{P}(t), X, t}{} \right\} \right) \]  

(5.6b)

for all constant orthogonal tensors \( \mathcal{Q} \). This condition states the fact that simple fluids are isotropic. If the fluid has been at rest for all \( 0 \leq \tau \leq \infty \), we would have

\[ \mathcal{C}_t (\mathcal{X}, t - \tau) = \mathcal{J} \]  

(5.7a)

and (5.6a) would become

\[
\mathcal{F} = \frac{\mathcal{G}}{\tau = 0} \left( \mathcal{J} \cdot \mathcal{P}(t), X, t \right) \]

\[
= \mathcal{K} (\mathcal{P}(t), X, t) \]  

(5.7b)

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Using (5.6b), (5.7b) becomes

\[ \tilde{\mathbf{j}} \tilde{\mathbf{j}} = \tilde{\mathbf{j}} \tilde{\mathbf{j}}^T = \tilde{\mathbf{j}} \]  

which implies that \( \tilde{\mathbf{j}} \) is proportional to the identity tensor \( \mathbf{j} \), i.e.,

\[ \tilde{\mathbf{j}} = -p(\rho, X) \mathbf{j} \]  

(5.8)

where \( p \) is the hydrostatic pressure which depends on the mass density \( \rho \). Thus, if a simple fluid has been at rest for all times, then the present state of stress is a hydrostatic pressure \( p \) depending on the mass density \( \rho \). When the fluid is heterogeneous, then \( p \) also depends on \( X \).

If the fluid is incompressible, the stress tensor \( \tilde{\mathbf{j}} \) is defined only to within a hydrostatic pressure \( -p \), and (5.6a) becomes

\[ \tilde{\mathbf{j}} = -p \tilde{\mathbf{j}} + \sum_{\tau=0}^{\infty} \tilde{Q} (\tilde{\mathbf{c}}_t (\bar{x}, t - \tau); X, t) \]  

(5.9a)

where

\[ \text{tr} \sum_{\tau=0}^{\infty} \tilde{Q} (\tilde{\mathbf{c}}_t (\bar{x}, t - \tau); X, t) = 0 \]  

(5.9b)

The simplest form of the constitutive equation (5.9a) is obtained if we set \( \tilde{Q} \equiv 0 \). This yields the constitutive equation of perfect fluids, i.e.,

\[ \tilde{\mathbf{j}} = -p(\rho, X) \mathbf{j} \]  

(5.10a)

Taking \( \tilde{Q} \) as a linear function of stretching tensor \( \tilde{\mathbf{g}} \), we obtain

\[ \tilde{\mathbf{j}} = (-p + \lambda \text{tr} \tilde{\mathbf{g}}) \tilde{\mathbf{g}} + 2\mu \tilde{\mathbf{g}} \]  

(5.10b)

which defines the constitutive equation of compressible viscous fluids. \( \lambda \) and \( \mu \) in (5.10b) are assumed constant. For incompressible fluids, we have \( \text{tr} \tilde{\mathbf{g}} = 0 \) which yields
\[
\dot{\mathbf{\Sigma}} = -\mathbf{p} \mathbf{\Sigma} + 2\mu \mathbf{\dot{\Sigma}} \quad .
\] (5.10c)

The coefficient \( \mu \) is called the coefficient of viscosity. Note that equations (5.10) are in accord with the principle of material frame-independence, since \( \mathbf{\Sigma} \) is objective. A generalization of (5.10c) is

\[
\dot{\mathbf{\Sigma}} = -\mathbf{p} \mathbf{\Sigma} + \mathbf{\zeta} \left( \mathbf{\Sigma} ; \mathbf{X} \right) \quad ,
\] (5.10d)

where \( \mathbf{\zeta} \) is an isotropic-tensor-valued function of \( \mathbf{\Sigma} \);

\[
\dot{\mathbf{\Sigma}} \sim \left( \mathbf{\Sigma} ; \mathbf{X} \right) \dot{\mathbf{\Sigma}}^T = \mathbf{\zeta} \left( \mathbf{\Sigma} \mathbf{\Sigma} \mathbf{\Sigma}^T ; \mathbf{X} \right) \quad (5.10e)
\]

for all orthogonal \( \mathbf{\Sigma} \). Using the results of the last section, we may write

\[
\dot{\mathbf{\Sigma}} = -\mathbf{p} \mathbf{\Sigma} + \varphi_1 \mathbf{\dot{\Sigma}} + \varphi_2 \mathbf{\Sigma} \quad .
\] (5.10f)

where \( \varphi_A = \varphi_A \left( \mathbf{I}_D, \mathbf{II}_D, \mathbf{III}_D \right), A = 1, 2, \) and \( \mathbf{I}_D = \text{tr} \mathbf{\Sigma} = 0 \). Equation (5.10f) defines the incompressible Reiner-Rivlin fluids. When the fluid is compressible, we obtain

\[
\dot{\mathbf{\Sigma}} = \left( -\mathbf{p} + \varphi_0 \right) \mathbf{\dot{\Sigma}} + \varphi_1 \mathbf{\dot{\Sigma}} + \varphi_2 \mathbf{\Sigma} \quad ,
\] (5.10g)

where

\[
\varphi_A = \varphi_A \left( \mathbf{I}_D, \mathbf{II}_D, \mathbf{III}_D \right); A = 0, 1, 2 \quad .
\]

A more general simple fluid is obtained by allowing \( \mathbf{\zeta} \) to depend on all \( \mathbf{\Sigma}_n \) rather than on only \( \mathbf{\Sigma}_1 = 2 \mathbf{\Sigma} \). This way we obtain

\[
\dot{\mathbf{\Sigma}} = -\mathbf{p} \mathbf{\Sigma} + \mathbf{\zeta} \left( \mathbf{\Sigma}_1, \mathbf{\Sigma}_2, \ldots, \mathbf{\Sigma}_n \right) \quad ,
\] (5.11)

where \( \mathbf{\Sigma}_n \) is the Rivlin-Ericksen tensor (Eq. (5.4a)).
References


Green, A. E. and Zerna, W., Theoretical Elasticity, Oxford University Press, 1960.


Principles of Continuum Mechanics, Socony Mobil Oil Company, Inc., Field Research Laboratory, Dallas, Texas, 1960.
Continuum Mechanics: a collection of reprints of some major publications from 1945 to 1961 in Continuum Mechanics is edited by Truesdell, C., Published by Gordon and Breach Science Publishers, Inc., 1965-66, and consists of the following parts

Part 1 The Mechanical Foundations of Elasticity and Fluid Dynamics; author: C. Truesdell.


Part 4 Problems of Nonlinear Elasticity; authors: Bernstein, Ericksen, Green, Knowles, Reissner, Rivlin, Shield, Toupin, Truesdell.

Original Contributions in Nonlinear Constitutive Equations of Materials with Memory:


PROBLEMS V

1. If a change of the frame of reference should preserve time intervals, the sense of time, and lengths, show that it must consist of a time-dependent rigid-body motion and a shift in the origin of time.

2. Show that, if an objective vector must change according to the rule \( (1.4)_2 \) under a change of an observer, and if \( \mathcal{J} \) is a second order tensor which transfers objective vectors into objective vectors such that

\[
\mathbf{v}^* = \mathcal{J} \mathbf{v} , \quad \mathbf{w}^* = \mathcal{J} \mathbf{w} ,
\]

then \( \mathcal{J} \) is an objective tensor of second order.

3. Show that the kinematical quantities \( \mathbf{v}, \mathbf{u} \), and \( \mathbf{C} \) are objective, while \( \mathbf{v}^* \), \( \mathbf{u}^* \), and \( \mathbf{C}^* \) change according to

\[
\mathbf{v}^* = \mathcal{J} \mathbf{v} , \quad \mathbf{u}^* = \mathbf{u} , \quad \mathbf{C}^* = \mathbf{C}
\]

under a change of the observer. Verify \((1.6b)\) and \((1.6c)\).

4. Using the expression for the co-rotational stress-rate \( \mathbf{q}^* \) and Eq. \((1.9a)\) show that the following stress-rate tensor is an objective quantity:

\[
\mathbf{q}^* = \mathbf{q} - \mathbf{q} \mathbf{L}^T - \mathbf{L} \mathbf{q}
\]