INTRODUCTION TO

PHYSICS OF CONTINUA

by

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for

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PREFACE

These notes are designed to provide the student at late sophomore or early junior standing, some basic understanding of the thermomechanics of deformable bodies, i.e. gases, liquids, and solids. As prerequisites some knowledge of the calculus of several variables, partial differentiation and multiple integration, commonly taught to undergraduates in science and engineering early in their sophomore year, is required. In addition, it is assumed that the student has had some exposure to elementary particle mechanics, commonly given in introductory physics courses in high school or during the first year of college. Nevertheless, the attempt is made to make the notes self-sufficient by assuming minimal prior knowledge, both in calculus and in mechanics.

The notes contain material for more than a one-quarter course. Therefore, only a selected portion will be covered, which essentially relates to one-dimensional and some special two-dimensional states of motion. The additional material could serve as future reference, or for coverage in a second one-quarter course. However, the notes are written in such a manner that such selective coverage in a one-quarter course will maintain complete continuity.

I would like to express my thanks to my wife Eva who took dictation and helped me in developing these notes. I am also grateful to Erika Ivansons, Judy Kozlov, and Brenda I. Wilson, who typed the first version of these notes last year, and to Marlene A. Papp, who retyped almost the entire manuscript which included major additions.

S. Nemat-Nasser
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CHAPTER 1

INTRODUCTION

1.1 WHAT IS CONTINUUM MECHANICS

In our everyday life we are constantly exposed to objects which to our immediate senses appear as continuous bodies that are in motion or that deform in the course of their interaction with each other. From our understanding of physics we know that matter is not continuously distributed within the space that it occupies, that at the microscopic level, matter consists of small elements known as atoms which themselves are composed of subatomic constituents. Notwithstanding this, there are many aspects of natural phenomena that can be described to within a satisfactory degree of accuracy under the assumption that matter is continuously distributed: i.e., without taking into account its discrete atomic structure. These phenomena, by necessity, must be macroscopic. The formation of waves in a large body of water due to wind, the deformation and motion of a tall building or a slender bridge in a gust of wind, and the vibration of the surface of the earth due to earthquakes, are examples of such macroscopic phenomena.

Continuum mechanics, therefore, is an abstract mathematical theory which deals with certain abstract objects called continua, that can interact with each other in a well-defined manner under a set of explicitly stated laws.

While this definition by its very nature is so general that it does not do justice to the student who for the first time is exposed to the
subject, it does bring out a very basic point that is shared by all mathematical theories which relate to nature, i.e. physical theories. This point is that a physical theory is, in general, developed in response to a need for understanding, describing, and predicting in a quantitative manner, certain manifestations of nature. In this sense, the abstract objects and the laws which are laid down to formulate the theory, are necessarily related to nature. The theory itself, however, constitutes a conceptual exercise that is created by man to enable him to approach the involved natural phenomena in a systematic and quantitative manner. In this sense the physical theory can be thought of as a mathematical model of that aspect of nature which is being studied.\footnote{1}{If such a theory does not contain in its inner structure contradictions and logical fallacies, it then constitutes an acceptable theory. Whether such an acceptable theory is a useful one or a useless one, then depends on the degree of its success in describing the actual occurrences in nature. For example, classical Newtonian mechanics deals with particles that are thought to be mathematical points endowed with a certain measure called mass, and with rigid bodies which are dense collections of particles that do not change their positions relative to each other in each such collection. These mathema-

\footnote{1}{Note that a consistent mathematical theory can be developed without any reference or relation to any physical situation. The usefulness of such a theory then would be accidental.}

\footnote{2}{Newton, Sir Isaac (1642-1727), English physicist and mathematician. He is considered one of the greatest scientists in the entire human history. His early work was in mathematics: binomial theorem, innovation of differential and integral calculus. Among his many contributions to the physical sciences are his experiment on the composition of white light and the nature of color, his laws of motion, and his discovery of the law of gravitation.}
tical objects are then assumed to move in a mathematical frame called 1 inertial, under a set of rules known as Newton's laws. Whether or not the motion of the moon, for example, can be described within such a context, then indicates the extent of the applicability and usefulness of Newtonian mechanics. Indeed, the fact that Newtonian mechanics has survived the scrutiny of many outstanding thinkers and is being taught in almost all schools to undergraduates, should be the indication of its usefulness. Classical mechanics, however, has limited applicability. For example, when motion at speeds near that of light and when the mechanics at atomic scales are considered, it fails to yield results which are even approximately borne out by experiment. Indeed, precisely these failures have given rise to the creation of new theories (i.e. relativity and quantum mechanics) based on new sets of rules and assumptions, which give results compatible with actual observation. This does not imply that classical mechanics as a theory is an unacceptable one, or should be thrown out. It simply indicates that classical mechanics cannot be used as a good mathematical model for certain physical phenomena.

Continuum mechanics similarly to classical mechanics, is a mathematical theory. It seeks to describe on a macroscopic scale, the motion and deformation of matter. To this end it uses the assumption that matter fills, in a continuous manner, the space that it occupies. This is an abstraction which needs further justification. However, whether or not one provides physical justification for this assumption, the usefulness

1 This is an imaginary coordinate system whose three axes are fixed with respect to the so-called "fixed stars."
of the theory will be dictated by the range of its applicability to actual situations. Although experience has revealed the importance of continuum theory, we shall briefly discuss in the sequel the justification commonly presented to support the basic assumption of continuum theory stated above.

Roughly speaking, matter may be divided into solids, liquids, and gases. Solids are commonly defined to consist of those material bodies which retain their shape, or stated more specifically, those bodies which deform by a small amount when subjected to small forces. Fluids which include both liquids and gases, are defined to consist of those material bodies which under suitable small forces undergo large changes of shape. Moreover, liquids are characterized by the property that, while they can "flow," their volume does not change considerably. Gases, on the other hand, are those material bodies which fill all the space within their containers.

Clearly enough, these definitions are not rigorous. They are intuitive, and therefore meet with the common everyday understanding of the corresponding terms. Closer examination reveals that, for example, there is no clear-cut distinction between solids and fluids under the above definitions. It is known that some solids can flow, although very slowly, under sustained loads. For example, silicone putty has all the appearance of a solid when subjected to short-time loading, while it flows under a sustained force of small magnitude. A ball of this substance bounces just like a rubber ball, while the same ball placed on a flat surface

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1 It should be stressed that these are all intuitive arguments, since what is a small force in one context may very well be a substantial force in another context.
slowly spreads flat under its own weight. Nevertheless, under most circumstances, a distinction can be made, although not quite precisely, between solids and fluids.

On the microscopic scale, for solids and liquids, the intermolecular forces are substantive; two adjacent molecules exert considerable attractive or repulsive forces upon each other, depending on whether their average equilibrium distance is slightly increased or decreased. In solids these forces are much more important, for example, in crystals, imposing a definite arrangement to the atoms. These atoms vibrate about their equilibrium position. The kinetic energy of these vibrations measures the hotness of the material and increases with the temperature. In the case of liquids, the kinetic energy of the molecular vibrations is sufficiently large so as to permit easy motion of molecules relative to each other within the body of liquid, while they are not sufficiently large to permit the molecules to escape the attractive forces provided collectively by the molecules as a group. As the temperature is increased, the kinetic energy of the vibration of the molecules becomes large, and the molecules become able to liberate themselves, forming gases. Hence the molecules of a gas have so much kinetic energy that they completely overcome the intermolecular attraction. The average distance between molecules in solids and liquids is an order of magnitude less than that in gases. For example, for simple molecules, this average distance in solids is about $3 \times 10^{-8}$ cm which is of the order of the size of the atoms, while in gases at $0^\circ C$ and 1 atmosphere pressure, this distance is about $3 \times 10^{-7}$ cm. At this latter distance the attractive intermolecular forces are almost negligible.  

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1 See Fig. }, page .
From the above remarks it is evident that most of the space occupied by the gas is void of gas molecules, since the distance between these molecules is large. Moreover, it is known that most of the mass on an atom is concentrated at its nucleus, indicating that the mass of an atom is not distributed evenly within the space occupied by that atom.

Notwithstanding these facts, in the continuum theory it is assumed that material fills all the space that a body occupies, and, as mentioned before, the theory yields very reasonable results regarding the macroscopic behavior of matter. The reason for this is that there is a very large number of molecules in every macroscopically accessible region occupied, say, by gases. With the usual instrumentation the sensitivity of the measurements in most macroscopic experimentation does not exceed $\frac{1}{100}$ mm, and within a cube with sides $\frac{1}{100}$ mm there are approximately $2.7 \times 10^{10}$ molecules of a gas at $0^\circ$C and 760 mm of mercury. Hence, as far as measurement is concerned, the continuum assumption in most usual cases provides a good approximation to the macroscopic behavior of material bodies.

Continuum mechanics can rightly be considered a branch of general field theories. It has become customary, however, to deal with only the thermo-mechanical aspects of material bodies in continuum mechanics. Moreover, while relativistic continuum mechanics has been developed and can be discussed, most fundamental developments which lend themselves to useful application have been created within the context of classical mechanics, and that is how the subject will be presented in this book.
1.2 BASIC INGREDIENTS OF CONTINUUM MECHANICS

As discussed above, continuum mechanics deals with matter which is viewed as continuously distributed throughout the space which it occupies. Thus, the fact that matter has, indeed, atomic structure, does not enter into the theory. The elements of a continuum are called particles. A particle similarly to that in classical particle mechanics, is a mathematical point which moves with respect to the inertial frame. But, in contrast with classical particle mechanics, each particle is not endowed with a finite mass, rather, the particles are viewed as densely (continuously) populating the continuum, and, therefore, endowing the continuum with a mass-density which is a point-function that measures the amount of matter per unit volume of the continuum. In this manner the mass of a particle is zero, since a particle does not have any volume, but any finite portion of the continuum has a finite mass. Loosely speaking, the mass-density at a particle can be thought of as the ratio between the mass contained within a small volume which contains within it the particle, and that volume. Actually, this is an average density.

MASS-DENSITY

A more precise definition of mass-density is the following. Consider particle P of a continuum, and imagine a closed surface S with volume V about this particle. Assume that V is sufficiently small so that it is completely within the continuum; P is an interior particle, i.e., it is not on the boundary of the continuum. Let the mass of all the matter within V be m. Consider in V now a nested sequence of volumes V_1, V_2, ..., V_n, ..., each containing within itself particle P, and each being totally contained
within the preceding volume; this means that $V_1$ contains $V_2$, $V_2$ contains $V_3$, and so on, see Fig. 2.1. Let the mass of the continuum contained within each of these volumes be $m_1$, $m_2$, ..., $m_n$, ..., respectively, and let the largest dimension of $V_n$ become smaller, approaching zero together with $V_n$ as $n$ becomes very large. At the same time the corresponding mass $m_n$ approaches zero. Assume that the ratio $\frac{m_n}{V_n}$ remains finite, admitting a limit as $n$ becomes large. This limit defines the mass-density of the continuum at particle P, and is commonly denoted by $\rho(P)$,

$$\rho(P) = \lim_{n \to \infty} \frac{m_n}{V_n}.$$  

(2.1)

**TEMPERATURE**

When one deals with the thermo-mechanical aspects of a continuum, one endows each particle with another measure called its *temperature.* In classical thermodynamics temperature is defined operationally in the same manner as mass, time, and length.\(^1\) To this end one employs the *thermometer* and assumes that two bodies in thermal equilibrium (the net exchange of heat energy between them is zero) have equal temperatures.\(^2\) In this manner temperature at a particle can be regarded as the measure of

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1 See Sec. 1.3 for a discussion of the physical units.

2 This assumption is a consequence of the so-called zeroeth law of thermodynamics which states that two bodies in thermal equilibrium with a third body, are themselves in thermal equilibrium. For a discussion see Joseph Kestin, *A Course in Thermodynamics*, Blaisdell Publishing Co. 1966, Ch. 2.
To define the mass-density at particle $P$, consider a nested sequence of volumes, $V_n$, each containing $P$, and each being totally contained within the preceding. As $n$ increases, let the largest dimension of $V_n$ go to zero together with the mass $m_n$ contained in $V_n$. The mass-density at $P$ is then defined by Eq. (2.1).
the hotness of the continuum there. Temperature is a scalar-valued function of the particle, which, in general, changes from one particle to another (usually in a continuous manner).

As mentioned before, temperature is the macroscopic manifestation of the average random kinetic energy of molecules that compose matter. For example, in the classical kinetic theory of monoatomic gases such as helium, it is shown that temperature \( \theta \) is directly proportional to the average random translational kinetic energy of the atoms, \( \theta = \frac{1}{3} \frac{2 m_o v^2}{k} \), where \( k \) is the Boltzmann constant equal to \( 1.38 \times 10^{-23} \) Joule/molecule °K, \( v^2 \) is the average squared speed of the gas particles, and \( m_o \) is the mass of one particle.

**CONFIGURATION**

The particles of a continuum move in the inertial frame, each tracing a curve which defines the trajectory of that particle. Henceforth the inertial frame will be referred to as the frame of reference.

At a given instant of time, the particles of a continuum occupy places in space. The collection of all the points in space occupied by these particles forms the configuration of the continuum at that time.

Imagine, for example, a raindrop falling toward the earth. If the raindrop is viewed as a continuum consisting of particles of water, then as it falls toward the earth, its configuration changes because of interaction with its environment. At each instant of time the particles of

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1 Boltzmann, Ludwig (1844-1906), Austrian physicist, among whose many contributions is the development of statistical mechanics and the statistical account of the second law of thermodynamics.
water can be put into one-to-one correspondence with points in space that they occupy. The collection of these points defines the configuration of the droplet at the considered instant. A snapshot of the droplet gives a view of its configuration at the time of exposure.

**BASIC LAWS**

The particles of a given continuum interact with each other and with particles of other continua. In continuum mechanics such interactions are assumed to occur so that certain basic laws are not violated. Here these laws are mentioned, but the student will find a more detailed account of each law and its various aspects later on. These laws are:

1. Conservation of mass,
2. Balance Law for linear momentum,
3. Balance Law for angular momentum,

**BASIC INGREDIENTS**

The deformation and motion of the continuum can be studied without considering the causes (forces) which induce them. This constitutes the kinemathical ingredient of the theory. The relationship between the forces that represent the action of other continua upon the considered continuum and the forces of reaction that are induced within the continuum, constitute the dynamical ingredient of the theory. Continua consisting of different materials (for example, wood versus steel), but having the same geometry (similar geometrical shape with equal dimensions) and being subjected to the same external action (subjected to the same set of forces),
behave differently. This relates to the constitution of the matter composing the continuum. These differences are accounted for by a set of relations called constitutive equations, which is laid down under a set of basic rules. These aspects of continuum mechanics will be studied more explicitly in various contexts later. Therefore, if the student finds some of the terms used somewhat vague, he should be patient for the time being. The following example, however, may help to clarify a few points. This example deals with perfect gases viewed as continua.

**EXAMPLE 2.1**

Consider a cylindrical container made of "rigid" walls, one end of which is closed, to the other end of which a piston is fitted which can move within the cylinder. Consider a body of gas in thermal equilibrium within the cylinder, having a constant temperature, $T$, a volume, $V$, and exerting on the walls of the cylinder and on the piston, a constant pressure, $p$. The behavior of this body of gas can be described either in the context of the classical kinetic theory of gases, or in the macroscopic manner in the context of continuum theory. Each of these theories constitutes a mathematical model yielding results which can be used to approximate the actual behavior of certain gases.

In the first theory the gas is assumed to consist of very small elastic balls which bounce around within the container, hitting each other and the walls of the container. In the second theory, which will be discussed now, one assumes that the container is filled all through its space by a continuously distributed gas-continuum with uniform density $\rho = \frac{M}{V}$, where $M$ is the total mass of the gas. In this case the kinematics
of the gas-continuum is such that it tends to occupy all the space within a container. Hence, the volume of a given mass of gas is determined by the volume of its container. The gas-continuum exerts at each point on the boundary of the container a force which is normal to the wall of the container, and has a uniform intensity (i.e. force per unit area of the container) all over the walls of the container. This constitutes the pressure of the gas-continuum. At each particle of the gas-continuum a temperature is defined, and when thermal equilibrium is attained, this temperature is uniform throughout the container.

From the above remarks it is evident that the kinematics, dynamics, and thermal behavior of a gas-continuum in a state of thermo-mechanical equilibrium are very simple. However, when the flow of such compressible gas-continua is considered, the situation becomes much more complicated, as will be discussed later on.

To obtain the constitutive equation that relates pressure, volume, temperature, one appeals to experiment. Experiments with real gases show that, at a constant temperature, the product of pressure and volume is approximately constant; Mariotte's law.\(^1\) Moreover, that, at a constant volume, pressure changes approximately proportionately with temperature.

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\(^1\) Mariotte, Edme (c. 1620-1684), French physicist, one of the first members of the French Academy of Sciences who made, among other achievements, contributions to the mechanics of fluid motion, the nature of color, development of the barometer, and various aspects of thermodynamics. Mariott's law is also known as Boyle's law, since it was developed independently by Boyle, Robert (1627-1691), an English natural philosopher, and one of the founders of modern chemistry. Despite his scientific contributions, by his will he founded the Boyle lectures whose goal was to prove the Christian religion against "notorious infidels, viz., atheists, theists, pagans, Jews, and Mohammedans", with the proviso that no mention be made of controversies among Christians.
measured from absolute zero, and that, at constant pressure, volume increases proportionately with temperature; Charles' law.\(^1\) From these observations one may postulate that an ideal gas is such a gas-continuum which satisfies the following constitutive relation:

\[
\frac{pV}{\theta} = nR,
\]

(2.2)

where \(n\) is the number of moles of the gas in \(V\), and \(R\) is the universal gas constant, i.e. \(R = 1.986 \text{ cal/mole} \quad ^\circ\text{K}\); a mole of a gas is equal to its molecular mass in grams; for example, a mole of oxygen defines 32 g of oxygen.\(^2\)

This is a phenomenological approach. It is called phenomenological because, one, being guided by the actual phenomena, lays down relationships which do not contain logical inconsistencies, which are plausible from a purely theoretical point of view, i.e. do not violate any basic accepted laws of the considered theory, and which, at least approximately, are borne out by a wide class of experimental results on actual materials. No actual gas, for example, behaves exactly (namely, to within a very high

\(^1\) Charles, Jacques Alexander César (1746-1823), French mathematician, physicist, and inventor, who created many ingenious physical devices, among which is the thermometric hydrometer. Charles' law is also named after Gay-Lussac, Joseph Louis (1778-1850), a French chemist and physicist, whose early work was on the properties of gases. In an attempt to measure the change in terrestrial magnetism with elevation, he made meteorological observations, by personally ascending with the aid of a balloon to elevations exceeding 23,000 feet above sea level. His later works were mostly in chemistry.

degree of accuracy) in a manner postulated by (2.2). For this reason (2.2) defines the constitutive equation for an ideal rather than an actual gas. Since no logical objection can be launched against (2.2), it represents an acceptable constitutive relation. It also happens to be a relatively good constitutive equation, since gases do approximately comply with (2.2), at least under normal conditions, i.e. not under low temperatures or high pressures. A similar result can be obtained from the kinetic theory.

Although most actual gases approximately fit the above model, the model gives poor results when applied to heavy gases, for example. This implies that both the kinetic theory and the continuum theory must be modified if one wishes to obtain results which are in accord with the actual behavior of certain heavy gases. The modification in continuum theory is accomplished by introducing additional kinematical quantities, additional dynamical quantities (when the flow of such gases is considered), and most importantly, a different constitutive equation relating the basic ingredients of the theory; for example, a more complicated relation between p, V, and θ. In the case of kinetic theory, on the other hand, one modifies his assumptions regarding the gas molecules as small elastic balls, and their statistical interaction with each other and with the walls of the container.
1.3 SYSTEMS OF UNITS

In this section a brief discussion of systems of physical units is presented for the sake of completeness. For a more thorough account, other standard references should be consulted.¹

In the physical sciences two kinds of physical quantities are distinguished, (1) quantities which are defined operationally, and (2) quantities which are defined mathematically in terms of the operationally defined quantities. Some authors refer to the first class of quantities as primary, and to the second as secondary.

PRIMARY QUANTITIES

A physical quantity is said to be primary (or defined operationally), if a standard unit is first constructed and then a description is given for a procedure by means of which that quantity can be measured in terms of the standard unit. Examples of such quantities are length, time, mass, and temperature.

SECONDARY QUANTITIES

A secondary physical quantity is defined with the aid of a mathematical statement which involves in its physical meaning the primary quantities. For example, area is defined in terms of length, so that the area $A$ of a square with sides having length $a$ is given by $A = a \cdot a = a^2$. Having defined operationally length, one then gives the area of the square by the above mathematical definition. Another example is the speed of a particle which moves along a curve. Here the speed $s$ is defined as follows: Let

¹ See, for example, Kestin (1966), op. cit., Chapter 1.
in an interval of time from $t_1$ to $t_2$ the particle sweep a distance $\Delta l$
along the curve. The speed at the instant $t_1$ is then given by

$$\lim_{t_2 \to t_1} \frac{\Delta l}{t_2 - t_1},$$

if it exists. As is clear this mathematical definition involves the notions of length and time which are primary quantities. The speed itself, however, is defined purely mathematically, although it does have physical significance.

**PHYSICAL DIMENSIONS**

A physical quantity is characterized by its **physical dimensions**, independently of the particular set of standard units that may be used for its quantitative description. Primary quantities are characterized by independent physical dimensions. Then the physical dimensions of the secondary quantities are described in terms of those of the former.

For example, if length and time are primary quantities with physical dimensions $L$ and $T$, then the physical dimension of speed, denoted by $[s]$, is $[s] = L/T$. With this in mind it is observed that in physically meaningful equations, only terms with equal physical dimensions can be added to, or subtracted from, each other. In particular, transcendental functions which occur in physical laws or relations, must have **dimensionless** arguments, i.e., their arguments must be pure numbers without any physical dimensions.

**THE ABSOLUTE AND THE ENGINEERING SYSTEMS OF UNITS**

Depending on the nature of the primary units, different systems of physical units can be constructed. There are two commonly used such systems: (1) the **absolute**, or **physical**, and (2) the **engineering**, or gravitational.
In the absolute system of units, length, time, and mass are chosen as primary units.

In the engineering system of units, length, time, and force are chosen as primary units.

In either system, Newton's second law of motion which relates force \( \mathbf{F} \) to mass \( m \) and acceleration \( \mathbf{A} \) by \( \mathbf{F} = m\mathbf{A} \), is employed to define the fourth physical quantity, namely force in the first system, and mass in the second system mentioned above.

Both systems of units employ an additional primary unit to define temperature, and another one to define electric current.

**ABSOLUTE MKS SYSTEM OF UNITS**

The three letters MKS stand for meter (m), kilogram-mass (kg), and second (sec).

Meter (m) is the distance between two marks on a rod made of a special alloy, which distance is now standardized to represent 1,650,763.73 times the wavelength of the orange-red line of isotope 86 of krypton, exactly; originally the meter was intended to represent one-millionth of a quadrant of the terrestrial meridian.

Kilogram (kg) is the mass of a standard piece of metal kept in Paris. It was originally intended to represent the mass of one liter of distilled water at 4°C.

Second (sec) is now defined in terms of the frequency of a quantum transition in cesium in the ground state. This frequency is 9,192,631,770 cycles/sec exactly; originally, second was intended to represent the 86,400th part of the mean solar day.
In the MKS system the unit of force is called Newton (N) which is the force required to induce an acceleration of 1 m/sec\(^2\) to 1 kilogram-mass, according to Newton's law.

**CGS SYSTEM OF UNITS**

Here again the primary units are length, mass, and time, with the corresponding names of centimeter (cm), gram (g), and second (sec).

Centimeter (cm) is defined to represent one-hundredth of the standard meter.

Gram (g) is defined to represent one-thousandth of the standard kilogram.

Second (sec) is defined the same way as for the MKS system.

The standard of force is dyne (dyne) which is the force required to give 1 g an acceleration equal to 1 cm/sec\(^2\). Dyne, therefore, is one-thousandth of a Newton.

**GRAVITATIONAL MKS SYSTEM OF UNITS**

The units of length and time are the same as the corresponding ones in the absolute MKS system. The third primary unit here, however, is the standard kilogram-force which is the force exerted on 1 kilogram-mass placed at a location where the gravitational acceleration is \( g = 9.80665 \) m/sec\(^2\), exactly. Hence, one kilogram-force is 9,806.65 N. There exists no universally accepted notation which explicitly distinguishes between kilogram-mass and kilogram-force; in some European countries, kilopond, denoted by kp, is used to designate kilogram-force. Here we shall use kgm and kgf to designate kilogram-mass and kilogram-force, respectively.

In this system of units, mass is a secondary quantity defined with the aid of Newton's second law. This is the mass contained in an object
which, when subjected to 1 kgf, attains the acceleration of 1 m/sec$^2$. Since 1 kgf is 9.80665 N, and since 1 N induces 1 m/sec$^2$ acceleration to 1 kgm, the unit of mass in the gravitational MKS system of 9.80665 times 1 kgm.

**ENGLISH ABSOLUTE SYSTEM OF UNITS**

The primary units of this system are **foot** (ft) for length, **second** (sec) for time, and **pound-mass** (lbm) for mass.

Foot (ft) is defined to represent 0.3048 m, exactly; another unit of length commonly used in this system is **inch** (in) which represents 2.54 cm, exactly.

Second (sec) is the same as previously discussed.

Pound-mass (lbm) is defined to represent a mass equivalent to 0.45359237 kgm, exactly.

The unit of force in this system is called poundal (pdl), and represents a force which induces 1 ft/sec$^2$ acceleration to 1 lbm.

**ENGLISH GRAVITATIONAL SYSTEM OF UNITS**

The units of length and time are the same as those in the English absolute system of units. The third primary unit here, however, is the standard **pound-force** (lbf) which is 0.45359237 kgf, exactly.

Mass in this system of units is a secondary quantity whose unit is called **slug** (slug) and is defined to represent that mass which when subjected to 1 lbf, attains 1 ft/sec$^2$ acceleration.

Table 3.1 summarizes basic units of various systems and defines a few additional secondary units.
### TABLE 3.1

**SYSTEMS OF UNITS**

<table>
<thead>
<tr>
<th>SYSTEM</th>
<th>LENGTH</th>
<th>MASS</th>
<th>TIME</th>
<th>FORCE</th>
<th>ENERGY</th>
<th>PRESSURE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ABSOLUTE SYSTEMS</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CGS</td>
<td>cm</td>
<td>g</td>
<td>sec</td>
<td>dyne = (\frac{g \cdot \text{cm}}{\text{sec}^2})</td>
<td>erg = (\frac{g \cdot \text{cm}^2}{\text{sec}^2})</td>
<td>dyne = (\frac{g}{\text{cm} \cdot \text{sec}^2})</td>
</tr>
<tr>
<td>MKS</td>
<td>m</td>
<td>kgm</td>
<td>sec</td>
<td>(N = \frac{\text{kgm} \cdot \text{m}}{\text{sec}^2})</td>
<td>(J = N \cdot \frac{\text{m}}{\text{sec}^2})</td>
<td>(\frac{N}{\text{m}^2})</td>
</tr>
<tr>
<td>ENGLISH</td>
<td>ft</td>
<td>lbm</td>
<td>sec</td>
<td>poundal = (\frac{\text{lbm} \cdot \text{ft}}{\text{sec}^2})</td>
<td>(\frac{\text{lbm} \cdot \text{ft}^2}{\text{sec}^2})</td>
<td>(\frac{\text{lbm}}{\text{ft} \cdot \text{sec}^2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>GRAVITATIONAL SYSTEMS</strong></th>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>MKS</td>
<td>m</td>
<td>(\text{kgf sec}^2)</td>
<td>sec</td>
<td>kgf</td>
<td>kgf m</td>
<td>(\frac{\text{kgf}}{\text{m}^2})</td>
</tr>
<tr>
<td>ENGLISH</td>
<td>ft</td>
<td>slug = (\frac{\text{lbf sec}^2}{\text{ft}})</td>
<td>sec</td>
<td>lbf</td>
<td>ft lbf</td>
<td>(\frac{\text{lbf}}{\text{ft}^2})</td>
</tr>
</tbody>
</table>

**TEMPERATURE**

To characterize quantitatively the common notion of "hotness" (or coldness), standard units for temperature are introduced. Temperature, therefore, can be regarded as a primary quantity, defined operationally. To this end the *zeroth law of thermodynamics* is used, which
states that **two bodies in thermal equilibrium with a third body are themselves in thermal equilibrium**. When two bodies are in thermal equilibrium, it is then said that they have equal temperatures.

There are two commonly used standard units for temperature, (1) **Celsius** \(^1\) \(^\circ\text{C}\), previously known as centigrade, and (2) **Fahrenheit** \(^2\) \(^\circ\text{F}\). The **Celsius scale** was constructed by taking the temperature of a mixture of water and ice at normal atmospheric pressure (the ice point of water) as \(0^\circ\text{C}\), and the temperature of a mixture of water and water vapor at normal atmospheric pressure (the boiling point of water) as \(100^\circ\text{C}\). To construct a thermometer one uses a certain characteristic of a device, which changes with temperature, for example, the length of the column of mercury in the capillary tube of a mercury-in-glass thermometer; other thermometers are gas thermometer, thermocouple, resistance thermometer, and radiation thermometer or pyrometer. \(^3\) If the material of the tube is such that it is insensitive to temperature changes compared with mercury (for example, glass has this property), then one may measure the change in the length of the column from that in thermal equilibrium with ice water to that in thermal equilibrium with boiling water. This incremental length can then be divided into a hundred equal parts, each representing \(1^\circ\text{C}\).

---

\(^1\)Celsius, Anders (1701-1744), Swedish astronomer who developed the centigrade thermometer.

\(^2\)Fahrenheit, Gabriel Daniel (1686-1736), German physicist who introduced the thermometric scale named after him.

\(^3\)For a discussion of thermometry see Kestin (1966), *op. cit.*, Ch. 2; see also *Encyclopedia Britannica*, 1965, Vol. 22, p. 110.
To construct a Fahrenheit scale, the ice point temperature of water is fixed at $32{}^\circ F$, and the boiling point temperature at $212{}^\circ F$. If the temperature of a certain substance is $X{}^\circ C$, $X$ being a numerical value, for example 50, the corresponding numerical value $Y$, expressed in $^\circ F$ is given by

$$
Y = \frac{9}{5} X + 32 ;
$$

for example, a substance at $50{}^\circ C$ is at $122{}^\circ F$.

In the thermodynamics of systems in equilibrium it is argued and accepted that the ratio of the temperatures of two such given systems can be measured directly without ambiguity. To obtain a more permanent temperature scale therefore, the temperature of the triple point of water (triple point of water refers to the equilibrium state at which ice, water, and water vapor coexist\(^1\)) is fixed by convention to represent $273.16{}^\circ K$, where \(1{}^\circ K \approx 1{}^\circ C\) (very closely), and the new scale is called the **absolute Kelvin\(^2\) scale**, \({}^\circ K\) denoting the absolute degree Kelvin. The temperature at the triple point of water is set at $0.01{}^\circ C$, so that the temperature of a mixture of ice and water at one atmosphere is very closely $0{}^\circ C$. Therefore, the Celsius scale is very nearly the same as the Kelvin scale, but not identical.

---

\(^1\)See p. 4-6.

\(^2\) Kelvin, William Thomson, Baron (1824-1907), British physicist who made significant contributions to various branches of physics, including thermodynamics, electricity and magnetism, telegraphy, and oceanography, and invented a number of ingenious measuring devices. He formulated the second law of thermodynamics and, in discussing the dynamic theory of heat, formulated the law of the conservation of energy as a fundamental law of nature. He was well recognized during his lifetime, was knighted, given other honorary titles, and became president of the Royal Society.
The corresponding absolute scale relating to °F is called the Rankin scale, for which the temperature at the triple point of water is set at 491.688 °R, with °R denoting degree Rankin.

CONCLUDING REMARKS

Although it is highly desirable to have a unified universally accepted system of units, tradition as well as practical convenience have made us heir to a variety of systems of units discussed above, and still others, which, however, do not have universal acceptance. The student, therefore, must acquaint himself with these systems of units, and must learn to express a given physical quantity in various systems of units. We shall not pursue this here. The following examples, however, illustrate the essential points involved, and provide additional insight.

EXAMPLE 3.1

In some engineering practice, a four-unit system is used, in which the primary quantities are mass, force, length, and time. Newton's second law then is no longer used to define, for example, force in terms of mass and acceleration, but rather it is regarded as a physical law. Since force, mass, length, and time each now have (arbitrary) independent standard units, this law must be written as

$$F = kma$$  \hspace{1cm} (3.1)

where $k$ is a physical constant with the physical dimension $[k] = \frac{F T^2}{ML}$

---

1 Rankin, William John MacQuorn (1820-1872), Scottish civil engineer and physicist who wrote the first formal treatise on thermodynamics.
where $F$, $M$, $L$, and $T$ are, respectively, the physical dimensions of primary quantities force, mass, length, and time. For instance, if kgm, kgf, m, and sec, are used as standard units of mass, force, length, and time, $k$ will be

$$
k = 9.80665 \frac{\text{kgf sec}^2}{\text{kgm m}}.
$$

If, on the other hand, lbm, lbf, ft, and sec, are used for the standard units, $k$ becomes

$$
k \approx 32.2 \frac{\text{lbf sec}^2}{\text{lbm ft}}.
$$

**EXAMPLE 3.2**

From the preceding example it is seen that a four-unit system can be reduced to a three-unit system by invoking a relevant physical law. In the above case this is done by setting $k = 1$ in Eq. (3.1) and then using this equation to define a fourth physical quantity in terms of the other three. This observation suggests that a three-unit system can equally well be reduced to a two-unit system by invoking another physical law, for example, Newton's law of gravitation, which states that the gravitational attraction, $F$, of two point-masses with masses $m_1$ and $m_2$, separated from each other by a distance $r$, is given by

$$
F = G \frac{m_1 m_2}{r^2}, \quad (3.2)
$$

where $G$ is a universal constant whose value depends on the units used; in the absolute MKS system, $G = (6.670 \pm 0.015) \times 10^{-11} \frac{\text{N m}^2}{\text{kgm}^2}$. 

To obtain a two-unit system, set $G = 1$ in (3.2) and use the resulting equation as an expression which defines, together with Eq. (3.1) with $k = 1$, mass and force in terms of length and time. In this manner obtain the following physical dimensions for mass and force:

$$[m] = L^3 T^{-2}, \quad [F] = L^4 T^{-4},$$

where $[m]$ and $[F]$ now designate the physical dimensions of mass and force which are no longer primary quantities. For example, in the MS system of units, the unit of mass becomes $m^3/\text{sec}^2$, and the unit of force becomes $m^4/\text{sec}^4$. Other secondary units can be established similarly.

**EXAMPLE 3.3**

The units of energy are: erg (erg) in CGS, Joule (J) in MKS, and foot pound-force ($\text{ft \; lbf}$) in the gravitational engineering system. The relation between these units can be established as follows:

\[
\text{erg} = \text{dyne cm} \\
= (10^{-5} \text{ N})(10^{-2} \text{ m}) = 10^{-7} \text{ N m} = 10^{-7} \text{ J} \\
\approx 10^{-7} \left( \frac{1}{0.4536} \times \frac{1}{0.3048} \times \frac{1}{32.17} \text{ lbf} \right) \left( \frac{1}{0.3048} \text{ ft} \right) \\
\approx 7.38 \times 10^{-8} \text{ ft lbf}.
\]
PROBLEMS FOR CHAPTER 1

1.1 Avogadro's number $N = (6.02650 \pm 0.00016) \times 10^{23} \approx 6.03 \times 10^{23}$ is the number of molecules of a gas per its 1 g mole. The volume of 1 g mole of a gas at $0^\circ$C and 760 mm mercury (1 atm) is 22,400 cm$^3$.

(a) Find Loschmidt's number, which gives the number of molecules per cm$^3$, at $0^\circ$C and 1 atm.

(b) Find the number of molecules at $1000^\circ$C and $10^{-3}$ atm, assuming that the gas follows the perfect gas law.

(c) Find the average distance between the molecules at $0^\circ$C and 1 atm.

(d) Find the average distance between the molecules at $1000^\circ$C and $10^{-3}$ atm.

(e) Discuss whether the basic assumption of the continuum theory can be applied in the above four cases to describe with a reasonable accuracy the gross behavior of the considered gas.

1.2 The average radius of the earth is about $6.37 \times 10^6$ meters. Suppose we have a hollow sphere with the earth's radius. Estimate the average number of molecules of a gas at $0^\circ$C which would be contained per m$^3$ of this sphere if 1 m$^3$ of a gas at $0^\circ$C and 1 atm were released in it. Do you think that the continuum theory may still be applied in this case? Justify your answer.

1.3 How many molecules of water ($H_2O$) are there in a droplet 0.001 mm in diameter? How many molecules are there in the same volume of air at $20^\circ$C and 1 atm?
2.1 Define the mass-density at a particle which is on the boundary of a rain droplet (viewed as a continuum). Use the same procedure as outlined in the text for the interior particles.

2.2 From the kinetic interpretation of temperature mentioned in the text,

\[ \theta = \frac{2}{3} \frac{m_0 v^2}{k} \]

i.e. \( \theta = \frac{2}{3} \frac{m_0 v^2}{k} \), deduce that

(a) \( \bar{v} = \sqrt{\frac{3p}{\rho}} \), where \( \rho \) is the mass-density, and \( p \) is the pressure of the gas.

(b) Show that the average speed \( \bar{v} \) of the random molecular translational motion varies with the gas temperature as

\[ \frac{\bar{v}_1}{\bar{v}_2} = \sqrt{\frac{\theta_1}{\theta_2}} \]

where \( \bar{v}_1 \) is the speed corresponding to temperature \( \theta_1 \), and \( \bar{v}_2 \) corresponds to \( \theta_2 \).

(c) For air at 0°C and 1 atm, calculate \( \bar{v} \) and compare your result with the corresponding speed of sound.

3.1 With mass, \( M \), length, \( L \), and time, \( T \), as primary physical dimensions, establish the physical dimensions of acceleration, force, energy, pressure, and mass-density.

3.2 Define the unit of pressure in various systems of units discussed in the text, and establish the corresponding conversion factors.

3.3 In a system of units, the primary quantities are force, mass, and length. If the corresponding standard units are lbf, lbm, and ft,
respectively, establish the unit of time which is then viewed as a secondary quantity, and which may be called "era". Find the conversion factor between era and sec. What is the corresponding conversion factor if kgf, kgm, and m are the primary standard units of the new system units?
CHAPTER 2

DEFORMATION AND FLOW

2.1 NEWTONIAN POINT-MASS, RECTILINEAR MOTION

Some, hopefully, familiar concepts in Newtonian mechanics are reviewed in this and the following two sections. In the Newtonian particle mechanics a particle is viewed as a geometrical point endowed with finite mass, and is called point-mass. The motion of a point-mass under a force-field\(^1\) constitutes the subject of single-particle mechanics.

**POSITION VECTOR**

Consider a point-mass moving along straight line \(x_1x_1\) which constitutes a one-dimensional coordinate system. Mark a fixed origin \(O\) on \(x_1x_1\), see Fig. 1.1a, and measure the position of point-mass \(M\) with mass \(m\), from \(O\) as a function of time, using an arbitrary but fixed unit of length. If this unit of length is defined by the length of the line segment \(OA\) in Fig. 1.1a, then \(OA\) may be viewed as a directed line segment, a vector, of unit length, pointing from \(O\) to \(A\) and in this manner defining the positive direction along \(x_1x_1\). This vector is the unit base vector. It will be denoted by \(e_\sim_1\).

Suppose now that, at instant \(t\), the point-mass \(M\) is at the distance \(x_1\) from \(O\), where a positive value of \(x_1\) indicated that \(M\) is to the right of \(O\), and negative \(x_1\) indicates that it is to the left. The position of

\(^1\) Consider a region of space at each point of which a force is defined which would act on a point-mass if it were placed there. The collection of these forces constitute a force-field.
Figure 1.1

(a) The $X_1X_1$-axis defines a one-dimensional coordinate system. Point $O$ is the origin, and the directed (arrow) line segment $OA = e_{11}$ is the unit base vector; its length is the unit of length. The position of a point-mass $M$ can be defined by the position vector $x = x_1 e_{11}$ which is an arrow joining $O$ to $M$, and pointing to the latter. For positive $x_1$, $x$ has the same direction as $e_{11}$, and for negative $x_1$, it has an opposite direction. The absolute value of $x_1$ measures the length of $OM$ in terms of the length of $e_{11}$.

(b) The time rate of change of $x$ is the velocity vector $v = V_1 e_{11}$.

(c) If the point-mass moves from A to B in an interval of time $\Delta t$, the work done by the external force $\mathbf{F} = F_1 e_{11}$ is $\mathbf{F} \cdot (e_{11} \Delta x_1) = F_1 \Delta x_1$. The rate of this work then is $\lim_{\Delta t \to 0} \frac{F_1 \Delta x_1}{\Delta t} = F_1 V_1$ which is positive if $\mathbf{F}$ and $\mathbf{v}$ point in the same direction, and negative otherwise.
M may be identified by an arrow which connects 0 along \( x_1 \) to \( M \), and points to the latter; in Fig. 1.1a, this is shown by the arrow \( OM \). This arrow is called the position vector of \( M \). Its length gives the distance of \( M \) from the origin, and its direction indicates whether \( M \) is to the right or left of 0. Since \( e_1 \) has unit length, the position vector \( \mathbf{x} \) of \( M \) can be expressed as

\[
\mathbf{x} = x_1 e_1 .
\]  

(1.1)

Note that, since \( e_1 \) points to the right, (1.1) implies the convention that the product of \( e_1 \) with the real number \( x_1 \) is a vector which points in the same direction as \( e_1 \) if \( x_1 \) is positive, and in the opposite direction if \( x_1 \) is negative, and has a magnitude equal to the absolute value of \( x_1 \). This is an example of the multiplication of vectors by scalars, which will be discussed in more detail later on.\(^1\)

Returning to the point-mass \( M \), observe that since time \( t \) is by definition an increasing parameter, the motion of \( M \) along \( x_1 \) can be defined by giving its position vector, \( \mathbf{x}(t) = e_1 x_1(t) \), as a function of time for all \( t^0 \leq t < \infty \), where at the initial time \( t^0 \) the particle is assumed to be at point \( \mathbf{x}^0 \). Henceforth, the expression "point \( \mathbf{x}^0 \)", for example, is used to indicate a point whose position vector is given by \( \mathbf{x}^0 \).

**VELOCITY**

The velocity of the particle is given by the time rate of change of its position vector. If this velocity is denoted by \( \dot{\mathbf{x}} \), then direct differentiation with respect to time of Eq. (1.1) leads to

\(^1\) A brief account of vectors is given in Appendix A.
\[ \dot{\mathbf{y}} = \frac{d}{dt} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{e}_1 \end{bmatrix} = \mathbf{e}_1 \frac{d\mathbf{x}_1}{dt}, \quad (1.2) \]

where the fact that \( \mathbf{e}_1 \) is constant is used. The absolute value of \( \frac{d\mathbf{x}_1}{dt} \) is the speed of the particle. Its physical dimension is length/time, \( L/T \). From (1.2) observe that velocity can also be represented by a directed line segment whose magnitude is the speed of the particle, see Fig. 1.1b. Since the particle can have different velocities at different times, \( \dot{\mathbf{y}} \) is generally a function of time. This is expressed as \( \dot{\mathbf{y}} = \dot{\mathbf{y}}(t) \).

**ACCELERATION**

The acceleration \( \ddot{\mathbf{y}} \) of a particle is defined by the time rate of change of its velocity. From (1.2) it follows that

\[ \ddot{\mathbf{y}} = \mathbf{e}_1 \frac{d^2\mathbf{x}_1}{dt^2}, \quad (1.3) \]

which is also a vector quantity, generally dependent on time; it is thus written as \( \ddot{\mathbf{y}} = \ddot{\mathbf{y}}(t) \). Note that, if \( \frac{d^2\mathbf{x}_1}{dt^2} \) is positive, \( \ddot{\mathbf{y}} \) then points to the right along the \( \mathbf{x}_1 \mathbf{x}_1 \) axis, and to the left, if this quantity is negative.

**FORCE**

If the force \( \mathbf{F} \) that acts on the point-mass along \( \mathbf{x}_1 \mathbf{x}_1 \) is \( \mathbf{F} \), Newton's second law states that \( \mathbf{F} = m \ddot{\mathbf{x}} \). The entire history of motion is then prescribed if \( \mathbf{F} \) is known, and if the initial position \( \mathbf{y}^0 \) and initial velocity \( \dot{\mathbf{y}}^0 \) of the particle are prescribed at a fixed instant \( t^0 \).

---

\(^1\) See Sec. 1.3 for the operational as well as other definitions of force.
LINEAR MOMENTUM

Newton's law is more appropriately states as $F = \frac{d}{dt} (m \vec{v})$ which reduces to the one previously stated, if mass is constant. The quantity $\vec{F} = m \vec{v}$ is called the linear momentum of the particle. In this latter form, therefore, Newton's law states that the force acting on the particle equals at each instant, the time rate of change of its linear momentum. For problems in which the mass of an object is changing as it moves, for example a rocket during its burning stage, the latter form of Newton's law must be employed.

WORK AND ENERGY

In Fig. 1.1c the point-mass moves from point A to point B, a distance $\Delta x_1$, during an interval of time $\Delta t$. The external force $\vec{F}$ hinders or accelerates this movement, depending on whether it points opposite or in the direction of motion. In the first case the work done by the external force is negative, i.e. it is at the expense of the kinetic energy of the point-mass, whereas in the second case the work is positive, increasing the energy of the point-mass. This work can be expressed as

$$\vec{F} \cdot (\Delta x_1 \vec{e}_1) = F_1 \vec{e}_1 \cdot (\Delta x_1 \vec{e}_1)$$

$$= F_1 \Delta x_1 (\vec{e}_1 \cdot \vec{e}_1)$$

$$= F_1 \Delta x_1, \quad \vec{e}_1 \cdot \vec{e}_1 = 1. \quad (1.4)$$
The rate of the work hence is

$$\lim_{\Delta t \to 0} \frac{F \cdot \Delta x}{\Delta t} = F \cdot v$$

which is positive or negative according to whether \( F \) and \( v \) point in the same direction or not.

For the one-dimensional motion of the point-mass the work of the external force in a given interval of time must equal the corresponding change in the kinetic energy. This is the law of the conservation of energy, which can be obtained from Newton's second law as follows:

$$\int_{t_0}^{t^*} F \cdot v \, dt = \int_{t_0}^{t^*} \frac{d^2 x}{dt^2} \frac{dx}{dt} \, dt$$

$$= \int_{t_0}^{t^*} \frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 \right] \, dt$$

$$= \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 \right]_{t_0}^{t^*}$$

$$= \mathcal{K}(t^*) - \mathcal{K}(t_0).$$

Here \( \mathcal{K} \) is the kinetic energy. The last expression thus is the change in the kinetic energy of \( M \) during the time interval from \( t_0 \) to \( t^* \).

**EXAMPLE 1.1**

Consider a projectile of mass \( m \) shot vertically from the surface of the earth. The minimum velocity required to give the projectile sufficient kinetic energy in order to escape the gravitational pull of the earth is called the escape velocity. Neglect air resistance

\footnote{Note that \( \mathcal{K}(t^*) \) is the value of the kinetic energy, \( \mathcal{K} = \frac{1}{2} m v^2 \), at \( t = t^* \).}
and the gravitational effect of other heavenly bodies, in order to estimate the escape velocity.

To this end assume that the mass of the earth is distributed in a spherically symmetrical manner about its center. Then it can be shown that its gravitational effect on a point-mass outside of it is the same as if its entire mass were given to a point-mass located at the center of this (spherical) earth. Denote by $M_e$ the mass, and by $R_e$ the radius of the earth. Then, if $r$ measures radial distance from the center of the earth, the gravitational force acting on a projectile of mass $m$ is

$$ F = - \frac{G M_e m}{r^2}, $$

where the minus-sign accounts for the fact that this force points toward the center, and where $G$ is a universal constant; see Example 3.2, Ch. 1. The equation of motion then is

$$ - \frac{G M_e m}{r^2} = m \frac{d^2 r}{dt^2}, $$

which can be written as

$$ - G \frac{M_e}{r^2} \frac{d^2 r}{dt^2} = \frac{d^2 \mathbf{r}}{dt^2}, $$

where $\mathbf{r} = \frac{dr}{dt}$, and where the chain rule of differentiation is used. Equation (1.8) can be integrated directly, yielding

$$ \left[ G \frac{M_e}{r} \right]_{R_e}^R = \left[ \frac{1}{2} \frac{\mathbf{r}^2}{r^2} \right]_{V^0}^V, $$

where the point-mass, assumed to be given velocity $V^0$ at the radial distance $R_e$ from the center, attains the velocity $V$ at distance $R$ from the center. Observe that the left-hand side in (1.9) is the work done by the gravitational force per unit mass from $R_e$ to $R$,.
whereas the right-hand side defines the corresponding change in the kinetic energy.

The quantity \( -G \frac{M_e}{r} \) is commonly referred to as the gravitational potential of the earth. If it is denoted by \( \Phi = \Phi(r) \), so that

\[
\Phi = -G \frac{M_e}{r} ,
\]

(1.10)

it then follows that the gravitational force acting on a unit mass is

\[
f = -\frac{d\Phi}{dr} .
\]

(1.11)

Equation (1.9) then can be written as

\[
\Phi(R_e) + \frac{1}{2} V^2 = \Phi(R) + \frac{1}{2} V^2
\]

(1.12)

which states that the sum of the kinetic and potential energies remains constant; \( \Phi(R) = -G \frac{M_e}{R} \) is \( \Phi(r) \) evaluated at \( r = R \).

To obtain the escape velocity, let \( R \) become very large as \( V \) approaches zero. Equation (1.12) then yields

\[
V^2 = \left[ G \frac{M_e}{R_e} \right]^{1/2} .
\]

(1.13)

If the mass of the earth is taken as \( 6.0 \times 10^{24} \) kgm and the radius as \( 6.37 \times 10^6 \) m, then (1.13) yields \( V = 11,200 \) m/sec which is about 25,000 miles/hour.

**EXAMPLE 1.2**

A jet of fluid strikes a stationary flat wall normal to the direction of the stream at a velocity \( V^0 \). Assume that the rebound velocity of the fluid particles in the direction of the impinging
jet, is negligible. Find the force exerted on the wall, if the jet carries fluid at the constant rate of $\dot{Q}^0$ kgm/sec.

During the interval $\Delta t$ the mass flow is $\Delta t \dot{Q}^0$. The corresponding linear momentum is $\Delta t \dot{Q}^0 V^0$. As the fluid impinges on the wall, the entire linear momentum in the direction normal to the wall, is lost. Hence the time rate of change of this linear momentum is $-\frac{\Delta t \dot{Q}^0 V^0}{\Delta t} = -\dot{Q}^0 V^0$ which must equal the force exerted by the wall on the fluid, i.e. $F = -\dot{Q}^0 V^0$. The force on the wall thus is equal to $\dot{Q}^0 V^0$.

Suppose now that the fluid jet is turned back $180^\circ$ by means of a vane as sketched in Fig. 1.2. Then the change in the linear momentum during the interval $\Delta t$ will be $\Delta t \dot{Q}^0 \left[ -V^0 - \dot{V}^0 \right]$, and the corresponding force becomes $-2\dot{Q}^0 V^0$. Hence the jet exerts a force on the wall equal to $2\dot{Q}^0 V^0$.

![Figure 1.2](image)

A fluid jet carrying fluid at a constant rate $\dot{Q}^0$ impinges on a $180^\circ$ vane at velocity $\sim V^0 = e_1 V^0$, and reverses its direction of flow. The time rate of change of linear momentum is $-e_1 2\dot{Q}^0 V^0$ which equals the force exerted by the vane on the jet.
2.2 NEWTONIAN POINT-MASS, PLANAR MOTION

The motion of a point-mass is called planar, if it occurs in a plane. Denote this plane by $\Pi$, and consider the rectangular Cartesian coordinate system consisting of the origin $O$ and coordinate axes $X_1X_1$ and $X_2X_2$; see Fig. 2.1. Select the unit base vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ along the axes $X_1X_1$ and $X_2X_2$, in the manner discussed in the preceding section. The position of point-mass $M$ in plane $\Pi$ may be identified by the position vector $\mathbf{x}$ which connects the origin $O$ to $M$, pointing toward the latter. As $M$ moves in $\Pi$ tracing a curve $C$, its orthogonal projections $M_1$ and $M_2$ on the axes $X_1X_1$ and $X_2X_2$ describe rectilinear motion which can be analyzed in exactly the same manner as discussed in the preceding section. Hence, denote at each instant $t$, the coordinates of $M_1$ and $M_2$ by $x_1(t)$ and $x_2(t)$, and recalling the parallelogram law of vector addition, obtain

$$\mathbf{x}(t) = x_1(t) \mathbf{e_1} + x_2(t) \mathbf{e_2} .$$

From this expression it is clear that the entire motion of the point-mass can be discussed in terms of the rectilinear motion of its projections along the coordinate axes.

1 Descartes, René (1596 - 1650). He is considered the greatest of French philosophers who invented the coordinate systems named after him and sought to discuss physical sciences in terms of geometry. Descartes' most lasting impact was on methodology which led to the practice of "methodical doubt" whose first certitude was "proof" of self-existence: "I think, therefore I am".

2 For a brief discussion, see Appendix A.
Planar Motion: Point-mass $M$ of mass $m$ traces curve $C$ in plane $\Pi$. $X_1X_1$ and $X_2X_2$ are the orthogonal Cartesian coordinates with the respective unit base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. $\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2$ is the position vector of $M$, $\mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2$ its velocity, and $\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2$, its acceleration. The orthogonal projections $M_1$ and $M_2$ describe rectilinear motion so that $F_1 = m \frac{d^2x_1}{dt^2}$ and $F_2 = m \frac{d^2x_2}{dt^2}$, where $F = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2$ is the external force.
VELOCITY

The time rate of change of the position vector $\mathbf{x}(t)$ defines the velocity of the point-mass $M$, which is a vector tangent to the curve $C$ given by

$$\mathbf{v} = \frac{d}{dt} \left[ x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 \right]$$

$$= \mathbf{e}_1 \frac{dx_1}{dt} + \mathbf{e}_2 \frac{dx_2}{dt},$$

where the fact that $\mathbf{e}_1$ and $\mathbf{e}_2$ are constant vectors, is used. The magnitude of the velocity vector is the speed of the particle and, as is evident, is given by

$$v = \left\{ \left( \frac{dx_1}{dt} \right)^2 + \left( \frac{dx_2}{dt} \right)^2 \right\}^{\frac{1}{2}},$$

where $V$ denotes the magnitude of $\mathbf{v}$.

ACCELERATION

In a similar manner the acceleration is defined as the time rate of change of the velocity of the particle, given by

$$\mathbf{a} = \mathbf{e}_1 \frac{d^2x_1}{dt^2} + \mathbf{e}_2 \frac{d^2x_2}{dt^2}.$$  

It should be noted that in all these developments it is assumed that the indicated derivatives exist, i.e. the corresponding functions are at least twice differentiable for $t^0 \leq t < \infty$, where at $t = t^0$ a one-sided derivative is implied.
FORCE

The force $F$ that acts on the point-mass $M$ at each instant $t$ is decomposed to its components $F_1$ and $F_2$ along the respective coordinate axes, so that

$$F = e_1 F_1 + e_2 F_2$$

In this manner Newton's second law can be applied to the imaginary point-masses $M_1$ and $M_2$ which are endowed with the mass $m$ of the point-mass $M$, and which are assumed to be acted upon by forces $e_1 F_1$ and $e_2 F_2$, respectively. This leads to

$$F_1 = m \frac{d^2 x_1}{dt^2}, \quad F_2 = m \frac{d^2 x_2}{dt^2},$$

which, together with the initial particle positions and initial velocities given by

$$\begin{cases}
  x_1 = x_1^o \\
  x_2 = x_2^o
\end{cases} \quad \text{and} \quad \begin{cases}
  \frac{dx_1}{dt} = v_1^o \\
  \frac{dx_2}{dt} = v_2^o
\end{cases} \quad \text{at } t = t^o,$$

define completely the motion of the point-mass $M$.

LINEAR MOMENTUM

For a planar motion the linear momentum $\mathcal{L} = m \mathbf{V}$ becomes

$$\mathcal{L} = e_1 \mathcal{L}_1 + e_2 \mathcal{L}_2 = e_1 m V_1 + e_2 m V_2,$$

so that

$$\mathcal{L}_1 = m V_1, \quad \text{and} \quad \mathcal{L}_2 = m V_2.$$
Newton's second law may therefore be more appropriately stated as
\[ \mathbf{F} = \frac{d \mathbf{\xi}}{dt} \]
which leads to
\[ \mathbf{F}_1 = \frac{d}{dt} (m \mathbf{v}_1), \quad \mathbf{F}_2 = \frac{d}{dt} (m \mathbf{v}_2). \tag{2.10} \]

**WORK AND ENERGY**

The work of the external force \( \mathbf{F} \) acting on the point-mass \( M \) is the sum of the work of its components which may be regarded as acting on the corresponding projections \( M_1 \) and \( M_2 \) of \( M \); i.e., it is \( F_1 v_1 + F_2 v_2 \).

The scalar or inner product of two vectors \( \mathbf{F} \) and \( \mathbf{V} \) which make an angle \( \beta \) as shown in Fig. 2.1, is defined by
\[ \mathbf{F} \cdot \mathbf{V} = F V \cos \beta, \tag{2.11} \]
where \( F \) is the magnitude of \( \mathbf{F} \). With this definition the scalar product of the unit base vectors is
\[ e_1 \cdot e_1 = e_2 \cdot e_2 = 1, \quad e_1 \cdot e_2 = e_2 \cdot e_1 = 0. \tag{2.12} \]
Equation (2.11) hence takes on the form
\[ \mathbf{F} \cdot \mathbf{V} = F_1 v_1 + F_2 v_2. \tag{2.13} \]

The integration of (2.13) from \( t^0 \) to \( t^* \) gives the work that is done by force \( \mathbf{F} \) on the point-mass \( M \) during this interval,
\[ \int_{t^0}^{t^*} \mathbf{F} \cdot \mathbf{V} \, dt = \int_{t^0}^{t^*} m \left[ \frac{d^2 x_1}{dt^2} \frac{dx_1}{dt} + \frac{d^2 x_2}{dt^2} \frac{dx_2}{dt} \right] dt. \]
\[ t^* \int_{t^0}^{t^*} \frac{d}{dt} \left[ \frac{1}{2} m \left( \frac{dx_1}{dt} \right)^2 + \frac{1}{2} m \left( \frac{dx_2}{dt} \right)^2 \right] dt \]

\[ = \left[ \frac{1}{2} m \left( \frac{dx_1}{dt} \right)^2 + \frac{1}{2} m \left( \frac{dx_2}{dt} \right)^2 \right]_{t^0}^{t^*} \]

\[ = \left[ \frac{1}{2} m v_i^2 \right]_{t^0}^{t^*} = K(t^*) - K(t^0), \quad (2.14) \]

where, as before, \( K \) denotes the kinetic energy. Hence, the change in the kinetic energy is equal to the corresponding work of the applied force.

**EXAMPLE 2.1**

Let a fluid jet hit a vane at a velocity \( v_i^0 = e_i v_i^0 \), and be deflected as shown in Fig 2.2. If the jet carries fluid at a constant rate \( q_i^0 \) kgm/sec, find the force exerted on the vane.

The velocity by which fluid is carried after deflection will be denoted by \( v_i^* = e_i v_i^* + e_i v_2^* \). If the angle with the \( x_1 x_1 \)-axis of the jet after deflection is \( \beta \), then \( v_1^* = v_i^0 \cos \beta \), and \( v_2^* = v_i^0 \sin \beta \). In an interval \( \Delta t \) the change in the \( x_1 x_1 \)-component of the linear momentum, therefore is

\[ \Delta J_1 = \Delta t q_i^0 (v_1^* - v_i^0) \]

\[ = \Delta t q_i^0 v_i^0 (\cos \beta - 1). \]

The \( x_1 x_1 \)-component of the vane's reaction hence is

\[ F_1 = \lim_{\Delta t \to 0} \frac{\Delta J_1}{\Delta t} = q_i^0 v_i^0 (\cos \beta - 1). \]
The $X_1X_1$-component of the force exerted by the jet on the vane then is
\[ -F_1 = \varphi^0 v^0 (1 - \cos \beta). \]

In a similar manner it is deduced that the $X_2X_2$-component of the reaction of the vane is
\[ F_2 = \varphi^0 v^0 (\sin \beta - 0) \]
\[ = \varphi^0 v^0 \sin \beta \]
whose negative gives the corresponding component of the force on the vane.

---

**Figure 2.2**

A fluid jet carrying fluid at a constant rate $\varphi^0$ impinges on a vane at velocity $\mathbf{v}^0 = \mathbf{e}_1 v^0$, and is deflected by angle $\beta$. The time rate of change of linear momentum is
\[ \dot{\mathbf{v}} = e_1 [\varphi^0 v^0 \sin \beta] + e_2 [\varphi v^0 (\cos \beta - 1)] \]
which equals the force $\mathbf{F}$ exerted by vane on the jet.
2.3 NEWTONIAN POINT-MASS, MOTION IN SPACE

The extension of the results of the two preceding sections to the more general case of the motion of a point-mass in space, is quite evident. This is particularly facilitated, if a rectangular Cartesian coordinate system is used.

RIGHT-HANDED RECTANGULAR CARTESIAN COORDINATE SYSTEM

To construct a rectangular Cartesian coordinate system, consider plane $\Pi$, and erect perpendicular to it at the origin $O$ a third coordinate axis $X_3$, as shown in Fig. 3.1a. To make this coordinate system right-handed, choose the unit base vector $e_3$ along the $X_3$-axis such that it points in the direction of the advancement of a right-handed screw which lies along $X_3$ and turns $X_1X_1$ toward $X_2X_2$, as shown in Fig. 3.1a.

POSITION VECTOR, VELOCITY, AND ACCELERATION

The position vector of point-mass $M$ which traces curve $C$ in space (Fig. 3.1b), is given by

$$\vec{x} = e_1 x_1 + e_2 x_2 + e_3 x_3 .$$

(3.1)

The corresponding velocity and acceleration vectors then are readily defined as

$$\vec{v} = \frac{dx_1}{dt} e_1 + \frac{dx_2}{dt} e_2 + \frac{dx_3}{dt} e_3$$

(3.2)

and

$$\vec{a} = \frac{d^2x_1}{dt^2} e_1 + \frac{d^2x_2}{dt^2} e_2 + \frac{d^2x_3}{dt^2} e_3 .$$

(3.3)
(a) A Right-Handed Rectangular Cartesian Coordinate System: \( \mathbf{e}_1, \mathbf{e}_2, \) and \( \mathbf{e}_3 \) form an orthogonal unit triad such that \( \mathbf{e}_3 \) points in the direction of the advancement of a right-handed screw which lies along \( X_3X_3 \), and turns \( X_1X_1 \) toward \( X_2X_2 \), as shown.

(b) Motion in Space: Point-mass \( M \) of mass \( m \) traces curve \( C \) in space. \( OX_1, OX_2, \) and \( OX_3 \) are the orthogonal Cartesian coordinates with the respective unit base vectors \( \mathbf{e}_1, \mathbf{e}_2, \) and \( \mathbf{e}_3 \). \( \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \) is the position vector of \( M \). \( \mathbf{V} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 \) is its velocity, and \( \mathbf{A} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \) is its acceleration. The orthogonal projections \( M_1, M_2, \) and \( M_3 \) describe rectilinear motion so that

\[
F_1 = m \frac{d^2x_1}{dt^2}, \quad F_2 = m \frac{d^2x_2}{dt^2}, \quad \text{and} \quad F_3 = m \frac{d^2x_3}{dt^2}, \quad \text{where} \quad \tilde{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3
\]

is the external force.
Note that the velocity vector $\mathbf{\dot{\mathbf{v}}}$ is tangent to the curve $C$, its magnitude $\mathbf{\dot{v}}$ being defined by

$$\mathbf{v} = \left\{ \left( \frac{dx_1}{dt} \right)^2 + \left( \frac{dx_2}{dt} \right)^2 + \left( \frac{dx_3}{dt} \right)^2 \right\}^{1/2}.$$  \hspace{1cm} (3.4)

**LINEAR MOMENTUM, FORCE, AND ENERGY**

The linear momentum of $M$ with mass $m$ and velocity $\mathbf{\dot{v}}$ is defined by

$$\mathbf{\dot{p}} = m \mathbf{\dot{v}} = e_1 \mathbf{\dot{v}}_1 + e_2 \mathbf{\dot{v}}_2 + e_3 \mathbf{\dot{v}}_3,$$

$$= e_1 m \mathbf{\dot{v}}_1 + e_2 m \mathbf{\dot{v}}_2 + e_3 m \mathbf{\dot{v}}_3.$$ \hspace{1cm} (3.5)

If the force that acts on $M$ is $\mathbf{\dot{F}} = e_1 F_1 + e_2 F_2 + e_3 F_3$, Newton's second law can be stated as $\mathbf{\dot{F}} = \frac{d\mathbf{\dot{p}}}{dt}$ which immediately leads to

$$F_1 = \frac{d}{dt} \left( m \frac{dx_1}{dt} \right), \quad F_2 = \frac{d}{dt} \left( m \frac{dx_2}{dt} \right), \quad F_3 = \frac{d}{dt} \left( m \frac{dx_3}{dt} \right).$$ \hspace{1cm} (3.6)

Finally, the work of the external force in the interval from $t^0$ to $t^*$ may be expressed as

$$\int_{t^0}^{t^*} \mathbf{\dot{F}} \cdot \mathbf{\dot{v}} \, dt = \left[ \frac{1}{2} m \mathbf{\dot{v}}^2 \right]_{t^0}^{t^*} = \mathbf{\dot{H}}(t^*) - \mathbf{\dot{H}}(t^0).$$ \hspace{1cm} (3.7)

Note that in Eq. (3.7) the inner product of $\mathbf{\dot{F}} \cdot \mathbf{\dot{v}}$ is defined by

$\mathbf{\dot{F}} \cdot \mathbf{\dot{v}} = \mathbf{F} \cdot \mathbf{v} \cos \beta$, where $\beta$ is the angle formed by the vectors $\mathbf{\dot{F}}$ and $\mathbf{\dot{v}}$.

Observe that the inner product of the base vectors is

$$e_1 \cdot e_1 = e_2 \cdot e_2 = e_3 \cdot e_3 = 1, \quad e_1 \cdot e_2 = e_2 \cdot e_3 = e_3 \cdot e_1 = 0.$$ \hspace{1cm} (3.8)
2.4 INDEX NOTATION AND SUMMATION CONVENTION

A great deal of economy in writing and manipulating various equations is achieved by the adoption of the so-called index notation. According to this notation Eqs. (3.6), for example, can be collectively written as

$$ F_i = \frac{d}{dt} \left( m \frac{dx_i}{dt} \right), \quad i = 1, 2, 3. \quad (4.1) $$

Here, the letter $i$ is the index, and the letter $F$, for example, is often called the kernel. Equation (4.1) stands for three independent equations, each corresponding to one value of the index $i$ which, as stated, takes on values of 1, 2, and 3.

In a similar manner Eqs. (2.9) can be expressed as

$$ S_\alpha = m V_\alpha, \quad \alpha = 1, 2. \quad (4.2) $$

Here the index $\alpha$ has only two values, 1 and 2, and therefore (4.2) stands for two independent equations.

Note that Greek letters such as $\alpha, \beta, \ldots$, or $\Delta, \phi, \ldots$, Roman letters such as $a, b, \ldots$, or $A, B, \ldots$, etc. can equally well be used for indices. The range of these indices may be different and hence must be clearly stated.

SUMMATION CONVENTION

Equation (2.1) can be restated as

$$ x = x_1 e_1 + x_2 e_2 = \sum_{a=1}^{2} x_a e_a. \quad (4.3) $$
If one adopts the convention that a repeated subscript index is to be summed over the entire range of its values, then the summation sign \( \sum \) may be dropped without causing confusion. If this is done, (4.3) becomes

\[
\sum = x_a e_a , \quad a = 1, 2 .
\]  

(4.4)

Note that the range of values of the index \( a \) must be clearly stated, or understood from the context.

Similarly, Eq. (2.13) can be written as

\[
F \cdot V = F_a V_a , \quad a = 1, 2 .
\]  

(4.5)

The summation convention can thus be stated as follows:

**Summation Convention:** A repeated subscript index in a monomial is summed by assigning to it the values 1 and 2, if the index has the range 1 and 2, and 1, 2, and 3, if the index has the range 1, 2, and 3.

A repeated subscript index is called a **dummy** subscript.

A non-repeated subscript index is called a **live** subscript.

An equation consisting of sum of monomials must be such that all monomials possess the same live subscripts.

For example, a system of two linear equations may be written as

\[
x_\alpha A_{\alpha\beta} = y_\beta , \quad \alpha, \beta = 1, 2 ,
\]  

(4.6)

where \( x_1 \) and \( x_2 \), (or, in short, \( x_\alpha \)), are two unknowns, \( A_{11}, A_{21}, A_{12}, \) and \( A_{22} \) (or, in short, \( A_{\alpha\beta} \)), are given constants, and \( y_1 \) and \( y_2 \) are given numbers. The expanded form of (4.6) is

\[
x_1 A_{11} + x_2 A_{21} = y_1 ,
\]

\[
x_1 A_{12} + x_2 A_{22} = y_2 .
\]
In Eqs. (4.6), $\alpha$ is a dummy index, whereas $\beta$ is a live index. In addition to (4.6) consider
\[ x_\gamma B_{\gamma \theta} = z_\theta \quad , \quad \gamma, \theta = 1, 2 \quad , \tag{4.7} \]
and suppose that for some reason the sum of Eqs. (4.6) and (4.7) is needed. Since the live index $\beta$ in (4.6) is not the same as the live index $\theta$ in (4.7), direct addition is inadmissible. Hence, change, say, $\theta$, first to $\beta$, to obtain
\[ x_\gamma B_{\gamma \beta} = z_\beta \quad , \quad \gamma, \beta = 1, 2 \quad , \]
and then add to arrive at
\[ x_\alpha A_{\alpha \beta} + x_\gamma B_{\gamma \beta} = y_\beta + z_\beta \quad , \quad \alpha, \beta, \gamma = 1, 2 \quad . \tag{4.8} \]
This is a perfectly legitimate equation, although the dummy index in the first monomial is different from that in the second monomial. These dummy indices may be equated to any other index letter which is otherwise not used in the same expression. Hence, replace, for example, the dummy index $\gamma$ by $\alpha$, to obtain
\[ x_\alpha A_{\alpha \beta} + x_\alpha B_{\alpha \beta} = x_\alpha \left( A_{\alpha \beta} + B_{\alpha \beta} \right) = y_\beta + z_\beta \quad . \]
Note that the factorization is admissible only after the common dummy indices are identified with the same letter; this letter need not be any one of the dummy indices involved, i.e. the above equation can equally well be written as
\[ x_\theta \left[ A_{\theta \beta} + B_{\theta \beta} \right] = y_\beta + z_\beta \quad , \quad \theta, \beta = 1, 2 \quad , \]
in which $\theta$ is now the dummy index.
KRONECKER DELTA

With the aid of the index notation, the inner product of the unit base vectors can be collectively written as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i,j = 1,2,3 . \quad (4.9)$$

For two dimensions this becomes

$$
\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}, \quad \alpha,\beta = 1,2 . \quad (4.10)
$$

It is often convenient to introduce a new notation, as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad i,j = 1,2,3 , \quad (4.11)$$

for three dimensions, and

$$
\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}, \quad \alpha,\beta = 1,2 . \quad (4.12)
$$

for two dimensions. $\delta_{ij}$ or $\delta_{\alpha\beta}$ is called the Kronecker\footnote{Kronecker, Leopold (1823-1891), German mathematician who made significant contributions to higher algebra, including elliptic functions, the theory of algebraic equations, and the theory of numbers. He sought to construct the entire number system from the properties of the natural numbers.} delta.

EXAMPLE 4.1

Calculate $\delta_{AB}\delta_{AB'}$, $A,B = 1,2$ .

It is observed that $\delta_{AB}$ has the value 1 if the two indices are the same, and 0 otherwise. Hence it follows that
\[ \delta_{AB} \delta_{AB} = \delta_{1B} \delta_{1B} + \delta_{2B} \delta_{2B} \]
\[ = \delta_{11} \delta_{11} + \delta_{12} \delta_{12} + \delta_{21} \delta_{21} + \delta_{22} \delta_{22} \]
\[ = 1 + 0 + 0 + 1 = 2. \]  
\[(4.13)\]

**EXAMPLE 4.2**

Calculate \( \delta_{kl} \delta_{kl} \), \( k, l = 1, 2, 3 \).

As in the previous example, it readily follows from the summation convention and the definition of the Kronecker delta that
\[ \delta_{kl} \delta_{kl} = 3. \]  
\[(4.14)\]

**EXAMPLE 4.3**

With the aid of the Kronecker delta show that \( \sim \cdot \sim = F_{\sim} V_{\sim} \), \( i = 1, 2, 3 \).

In terms of their components, \( \sim \) and \( \sim \) are
\[ \sim = F_k e_k, \quad \sim = V_k e_k, \quad k, l = 1, 2, 3. \]

Hence,
\[ \sim \cdot \sim = F_k e_k \cdot V_k e_k = F_k V_k e_k \cdot e_k. \]

But, from (4.9) and (4.11) it follows that
\[ e_k \cdot e_l = \delta_{kl}, \quad k, l = 1, 2, 3. \]  
\[(4.15)\]

Hence
\[ \sim \cdot \sim = F_k V_k \delta_{kl} \]
\[ = F_k V_k F_{i} V_{i}, \quad k, l, i = 1, 2, 3. \]  
\[(4.16)\]

Observe that \( F_k V_k \) follows from the fact that \( V_k \delta_{kl} = V_{kl} \) since \( \delta_{kl} \) is 0 unless \( k \) is equal to \( l \). Also, since in \( F_k V_k \), \( k \) is a dummy index, it can be replaced by any other letter such as \( i \).
2.5 **RECTILINEAR MOTION OF A CONTINUUM**

Consider a very long cylindrical tube whose central line coincides with the $X_1X_1$-axis. By the *cross-section* of the tube it is meant the portion cut out by the internal surface of the tube from an imaginary plane drawn perpendicular to the $X_1X_1$-axis. Assume that this cross-section is uniform along the $X_1X_1$-axis.

Consider the flow of a continuum through this tube, and assume that this flow is such that the mass-density at all particles instantaneously located on a cross-section, is the same, and that these particles move along the tube on straight lines parallel to the $X_1X_1$-axis in such a manner that they remain on a common cross-section. Thus, particles instantaneously located on a cross-section, have equal velocities and equal accelerations, Fig. 5.1a. To study the motion of this continuum, therefore, one may consider only the motion of particles moving on the $X_1X_1$-axis.

Particles moving on the $X_1X_1$-axis form a one-dimensional continuum.

At the instant $t^0$ which may arbitrarily be chosen as the initial time, consider material particles contained within two cross-sections drawn at points $Q^{(1)}$ and $Q^{(2)}$ of the $X_1X_1$-axis, see Fig. 5.1a. Suppose that one wishes to study the motion and deformation of this particular portion of the continuum for all $t$ greater than $t^0$; assume that the tube is "infinitely" extended.\(^1\)

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\(^1\) This really means that the tube is pretty long.
Figure 5.1

(a) One-Dimensional Flow of a Continuum in a Tube with Uniform Cross-Section: At each instant, particles located on a common cross-section have equal mass-density, velocity, and acceleration, and continue to remain on a common cross-section and have equal kinematical quantities as they move down the tube.

(b) One studies the motion of particles which are located on the line segment Q\(^{(1)}\)Q\(^{(2)}\) at time \(t^0\). The material segment F\(^{(1)}\)F\(^{(2)}\) is a one-dimensional continuum. The particles of this material segment are labeled by their coordinates at \(t^0\), Eq. (5.1).

(c) The material line segment F\(^{(1)}\)F\(^{(2)}\) has moved to a new location at time \(t > t^0\). The particle \(X_1\) has moved to the point \(x_1\) on the \(X_1X_1\)-axis.
Instead of considering all the particles between the above two cross-sections, one studies as representative the motion and deformation of the material line segment \( p^{(1)}p^{(2)} \) which consists of all the particles located on the \( X_1X_1 \)-axis between points \( Q^{(1)} \) and \( Q^{(2)} \) at the initial time \( t^0 \), see Fig. 5.1b.

Since the material line segment \( p^{(1)}p^{(2)} \) is densely and continuously populated by its particles, at the instant \( t^0 \) every point of the axis \( X_1X_1 \) between \( Q^{(1)} \) and \( Q^{(2)} \) is occupied by a material point. If a fixed origin \( O \) is chosen, the points on the \( X_1X_1 \)-axis between \( Q^{(1)} \) and \( Q^{(2)} \) can be identified by their position vector \( X \), and then each material point (particle) of the material line segment \( p^{(1)}p^{(2)} \) can be labeled by its position vector. One then writes

\[
\tilde{X} = X_1 e_1 \quad , \quad X_1^{(1)} \leq X_1 \leq X_1^{(2)} \quad , \quad t = t^0 , \tag{5.1}
\]

where \( e_1 \) is the unit base vector along \( X_1X_1 \), \( X_1 \) is the coordinate of a typical particle at time \( t = t^0 \), and \( X_1^{(1)} \) and \( X_1^{(2)} \) are the coordinates of particles \( p^{(1)} \) and \( p^{(2)} \), respectively.

The material line segment \( p^{(1)}p^{(2)} \) moves along the \( X_1X_1 \)-axis as \( t \) increases, and its particles occupy various positions on this axis at various times. Consider a typical particle \( X_1 \) -- this means the particle whose coordinate at time \( t^0 \) is \( X_1 \) -- and let its coordinate at time \( t \) be denoted by \( x_1 \), Fig. 5.1c. The coordinate \( x_1 \) is, therefore, a function of \( X_1 \), as well as time \( t \), i.e.

\[
x_1 = \Xi_1(X_1, t) \quad , \tag{5.2}
\]

\(^{1}\) Mathematically this means that every interval of the \( X_1X_1 \)-axis, between \( Q^{(1)} \) and \( Q^{(2)} \), no matter how small, contains an unbounded number of particles.
where the symbol $\Xi_1$ is used to denote the function; it should not be confused with $x_1$ which represents the value of this function for specific values of its arguments $X_1$ and $t$. Observe that (5.2) must be such that

$$\Xi_1(X_1, t^0) = X_1 \text{ for all } X_1^{(1)} \leq X_1 \leq X_1^{(2)} ,$$  \hspace{1cm} (5.3)

which indicates that at time $t = t^0$ the considered train of particles occupy the segment $Q^{(1)}Q^{(2)}$ of the $X_1X_1$-axis, as required by (5.1).

Equation (5.2) can be considered as defining a one-parameter family of mappings of points on the $X_1X_1$-axis onto itself. That is, at each fixed $t > t^0$, (5.2) defines a point on the $X_1X_1$-axis corresponding to a given $X_1$ on the same axis.

It will be assumed that this mapping is one-to-one and invertible, so that, to every admissible value of $X_1$, there corresponds one and only one value $x_1$ at a fixed time $t$. This assumption is very fundamental. It assures that two material points (particles) initially occupying two distinct points on the $X_1X_1$-axis, will continue occupying distinct points during the course of their entire motion. Also that a material line segment having nonzero length at a given instant, continues to have nonzero length at all future times, and corresponds to segments which in the past have never had zero length.

The mapping (5.2) will be assumed to be sufficiently smooth so that all the indicated derivatives exist over the domain of the definition of the arguments. The student should keep this in mind, since it will be implied in all subsequent developments.

Since the mapping (5.2) is assumed to be invertible, $X_1$ can be expressed in terms of $x_1$ as
\[ x_1 = \xi_1(x_1, t) \]  

(5.4)

which defines, for a given \( x_1 \), the particle \( X_1 \) that at time \( t \) is located at \( x_1 \). Observe that substitution for \( x_1 \) from (5.2) into (5.4) leads to an identity, i.e.

\[ \xi_1 \left[ \xi_1(X_1, t), t \right] = x_1. \]

**PARTIAL DERIVATIVES**

The partial derivative of a function of several variables will be signified by a round "d", i.e. \( \partial \). For example, when the parameter \( t \) is kept fixed, the derivative of (5.4) with respect to \( x_1 \) will be denoted by \( \frac{\partial \xi_1}{\partial x_1} \), or, equivalently, by \( \frac{\partial X_1}{\partial x_1} \). Also, the partial derivative of (5.4) with respect to \( t \) will be denoted by \( \frac{\partial \xi_1}{\partial t} \), or, equivalently, \( \frac{\partial X_1}{\partial t} \). In this latter case, in the expression \( \xi_1(x_1, t) \), \( x_1 \) is treated as a constant, and then the derivative with respect to \( t \) is taken. As an example, suppose

\[ \xi_1 = \frac{1}{2} \left[ e^{x_1 t/K^0} + 1 \right] x_1. \]

Then

\[ \frac{\partial X_1}{\partial x_1} = \frac{1}{2} \left[ e^{x_1 t/K^0} + 1 \right] - \frac{1}{2} \frac{x_1 t}{K^0} e^{x_1 t/K^0}, \]

and

\[ \frac{\partial X_1}{\partial t} = -\frac{x_1^2}{2 K^0} e^{-x_1 t/K^0}. \]

**STRETCH**

From Eq. (5.4) a small material line element \( dX_1 \) is given by

\[ dX_1 = \frac{\partial X_1}{\partial x_1} dx_1 = \frac{\partial \xi_1(x_1, t)}{\partial x_1} dx_1. \]  

(5.5)
As a nonzero $dX_1$ corresponds to a nonzero $dx_1$, $\frac{\partial x_1}{\partial x_1}$ is finite for all admissible values of $x_1$ and $t$. Moreover, this quantity is positive, since the sign of $dx_1$ must correspond to that of $dX_1$. Hence it follows that

$$0 < \frac{\partial x_1}{\partial x_1} < \infty \quad (5.6)$$

which is both a necessary and sufficient condition that the mapping (5.2) and its inverse (5.4) be one-to-one.

In a similar manner, $dx_1$ can be expressed as

$$dx_1 = \frac{\partial x_1}{\partial X_1} dX_1 = \frac{\partial \xi_1(X_1,t)}{\partial X_1} dX_1 \quad (5.7)$$

It is customary to set

$$\frac{\partial x_1}{\partial X_1} = \Lambda_1(X_1,t) \quad (5.8)$$

$\Lambda_1$ is called the stretch. From (5.6) it follows that

$$0 < \Lambda_1 < \infty \quad (5.9)$$

By the chain rule of differentiation one observes that

$$\frac{\partial x_1}{\partial X_1} \frac{\partial X_1}{\partial x_1} = \frac{\partial \xi_1(X_1,t)}{\partial X_1} \frac{\partial \xi_1(X_1,t)}{\partial x_1} = 1 \quad (5.10)$$

If the length of an element does not change in the course of its motion, the corresponding stretch will have the value 1.

**MATERIAL DESCRIPTION**

When the initial particle position $X_1$ is used as the independent variable (in addition to time $t$), and all other quantities which describe the motion and deformation of the continuum (for example, the stretch)
are expressed in terms of $X_1$ and $t$, it is said that the material description of motion is given. It is common to refer to such a description as Lagrangian.\footnote{Lagrange, Joseph Louis (1736-1813), French-Italian mathematician who made great contributions to mechanics and various areas in mathematics. His genius was even recognized by political leaders of his time, such as Frederick the Great who invited Lagrange to Berlin where he lived for 20 years; after that, Louis XVI invited him to Paris where he was honored throughout the French Revolution; then by Napoleon who made him a count, senator, and a high official of the Legion of Honor.} Here, however, the term "material description" will be used.

**SPATIAL DESCRIPTION**

When the current particle position $x_1$ at current time $t$, and $t$ are used to describe all other quantities of motion, it is said that the motion is given by its spatial description. The term Eulerian\footnote{Euler, Leonhard (1707-1783), Swiss mathematician who made great contributions to analysis, and who perhaps was not surpassed by any other 18th Century mathematician. He made numerous contributions which bear his name, such as Euler number, Euler equations in the calculus of variations, integrals defining the gamma and beta functions, and numerous other quantities. Although he lost the sight in one eye at the age of 28 and became totally blind at 59, his productivity was not diminished by these and other tragedies.} is commonly used for this purpose in the literature, but here the term "spatial description" will be employed.

Observe that when spatial description is employed, one fixes the spatial point $x_1$ and then watches material points which are labeled by their initial coordinate, $X_1$, pass through that point as time $t$ increases. On the other hand, when material description is used, one fixes a material point and follows its motion by moving with it.
NOTATION

As a general practice, lower case letters will be used to denote quantities which are expressed in terms of the spatial variables \( x_1 \) and \( t \), and upper case letters when the material description is used; there will occur a few exceptions however, which will be stated explicitly. For example, the stretch \( \lambda_1(x_1,t) \) will be written as

\[
\lambda_1(x_1,t) = \Lambda_1[X_1(x_1,t),t]
\]

when spatial description is employed. Observe that while this expression is an identity, \( \lambda_1(x_1,t) \) is neither equal to nor identical with \( \Lambda_1(X_1,t) \), but that these quantities are equivalent. In the sequel this equivalence will be denoted by \( \Rightarrow \), which indicates that upon the appropriate change of the variables, the left-hand side of this sign reduces to the right-hand side. For example,

\[
\lambda_1(x_1,t) \Rightarrow \Lambda_1(X_1,t).
\]

Note that the inverse of this equivalence is also valid, so that

\[
\Lambda_1(X_1,t) \Rightarrow \lambda_1(x_1,t).
\]

In the first case the spatial variables are changed to the material ones, whereas in the second case the material variables are changed to the spatial ones.

VELOCITY

Consider a typical material point, say \( X^0_1 \). Similarly to the case of the point-mass in Newtonian mechanics, the velocity of \( X^0_1 \) is given
by the time rate of change of its position vector. Since at time \( t \) the position vector of \( X_1^o \) is defined by \( \Xi_1(X_1^o,t) \) -- keep in mind that \( X_1^o \) is fixed -- the velocity of this particle is defined by \( \frac{\partial \Xi_1(X_1^o,t)}{\partial t} \).

In general, therefore, the velocity of a typical particle \( X_1 \) is given by

\[
V_1 = \frac{\partial \Xi_1(X_1,t)}{\partial t}.
\] (5.11)

The right-hand side of this equation is a function of \( X_1 \) and \( t \). This function will be denoted by \( V_1(X_1,t) \). This is actually not very good notation, since it confuses the function, i.e. \( V_1(X_1,t) \), with its value \( V_1 \) corresponding to specific values of \( X_1 \) and \( t \). But, since in continuum mechanics one deals with a large number of functions, it is more confusing and impractical to employ different symbols for functions and their values.

Equation (5.11) will be written as

\[
V_1 = \frac{\partial \Xi_1(X_1,t)}{\partial t} \equiv V_1(X_1,t) ,
\] (5.12)

where \( V_1 = V_1 \xi_1 \) is the velocity vector. This equation gives the material description of particle velocities. For a given particle \( X_1 \), it gives the velocity as a function of time, in exactly the same manner as that for a point-mass.

To obtain the spatial description of the particle velocities, one substitutes for \( X_1 \) from (5.4) into (5.12) to arrive at

\[
V_1 \left[ X_1(x_1,t),t \right] = v_1(x_1,t)
\]
or

\[ v_1 = v_1(x_1, t) . \]  \hspace{1cm} (5.13)

For a fixed spatial point \( x_1 \), (5.13) defines the velocity of that particle which at time \( t \) is located at this spatial point. Since different particles pass through \( x_1 \) at different times, \( v_1 \) as a function of time specifies their velocity which they attain as they pass through \( x_1 \). Observe again that in (5.13) the same letter, namely \( v_1 \), is used to denote the \textit{function} as well as its \textit{value}.

For a fixed time \( t \), \( v = v_1 \mathbf{e}_1 \) defines at each admissible point along the \( X_1 X_1 \)-axis, a vector which represents the velocity of the corresponding particle located there at this instant. If one draws these vectors at their corresponding spatial points, one obtains the \textit{velocity field} of the one-dimensional continuum at instant \( t \). In general this velocity field changes with time. When the velocity field does not depend on time, it is called \textit{stationary}. In this case, the velocity attached to each spatial point remains the same as time goes on; i.e. all particles attain a common velocity as they pass through a fixed point. Different velocities may however, correspond to different spatial points.

In the case of the one-dimensional flow in the considered tube, the velocity field defines a field of \textit{parallel} vectors.

\textbf{ACCELERATION}

Consider again the material point \( X_1^0 \), and denote its velocity as a function of time by \( V_1(x_1^0, t) \). The time rate of change of this velocity
gives the acceleration of the material point \( X_1^O \). Since \( X_1 \) is fixed, its acceleration is defined by the following partial time derivative:

\[
\frac{\partial V_1(X_1^O,t)}{\partial t}.
\]

In general, for a typical particle \( X_1 \), the acceleration is given by

\[
A_1 = \frac{\partial V_1(X_1,t)}{\partial t} \equiv A_1(X_1,t).
\]  

(5.14)

In order to emphasize that the particle is fixed while taking partial differentiation with respect to time in (5.14) some authors write

\[
\frac{\partial V_1(X_1,t)}{\partial t} \bigg|_{X_1 \text{ fixed}}.
\]

This notation will not be used here, since the fact that \( X_1 \) is kept fixed in (5.14) is implied by the very definition of the partial time derivative.

For a fixed \( X_1 \) characterizing a given particle, (5.14) gives the acceleration \( A = A_1 \varepsilon_1 \) as a function of time in exactly the same manner as in the case of a point-mass in Newtonian particle mechanics.

**Spatial Description of Acceleration**

Direct substitution for \( X_1 \) from (5.4) into (5.14) results in

\[
a_1 = a_1(x_1,t) = A_1 \left[ \xi_1(x_1,t), t \right]
\]  

(5.15)

which gives the spatial description of the acceleration.

The same result is obtained if one takes the material time derivative of the spatial form of the velocity given by Eq. (5.13). Before taking the time rate of change of the velocity as described by Eq. (5.13), one must keep in mind that, since the acceleration is defined for a fixed
particle, the corresponding spatial position \( x_1 \) changes with time, i.e. it is a function of time through Eq. (5.2). Hence, using the chain rule of differentiation, one writes

\[
\frac{Dv_1}{Dt} = \frac{\partial v_1(x_1, t)}{\partial t} + \frac{\partial v_1(x_1, t)}{\partial x_1} \frac{Dx_1}{Dt} = \frac{\partial v_1(x_1, t)}{\partial t} + v_1(x_1, t) \frac{\partial v_1}{\partial x_1} v_1(x_1, t),
\]

(5.16)

where \( \frac{D}{Dt} \) denotes the material time derivative, as contrasted with the partial time derivative which, as was agreed, is designated by \( \frac{\partial}{\partial t} \). The material time derivative is therefore the time rate of change as observed by the moving particle. For this material time derivative, some authors use \( \frac{d}{dt} \); in this book, however, notation (5.16) will be used. The acceleration, therefore, is

\[
a_1 = a_1(x_1, t) = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1}.
\]

(5.17)

The second term in the right-hand side of (5.17) is called the convected part of acceleration. For a fixed \( x_1 \), Eq. (5.17) gives the acceleration of that particle which at time \( t \) is located at point \( x_1 \). Observe the difference between this equation and Eq. (5.14). In (5.14), if the particle \( x_1 \) is fixed, then its acceleration is defined as a function of time throughout its entire motion. This particle moves to different spatial positions at different times, these spatial positions being
defined by (5.2). In (5.17), on the other hand, it is the spatial position which is fixed, and hence, at different times, different particles are located there, each having the acceleration given by Eq. (5.17). Since this distinction is very important in the development of the continuum theory, the student should make sure that he understands it, before proceeding further.

ON MATERIAL TIME DERIVATIVE

A kinematical or physical quantity of a continuum can be described either in terms of the spatial coordinates $x_1$ and $t$, or in terms of the material coordinates $X_1$ and $t$, or sometimes in terms of all three variables $x_1$, $X_1$, and $t$. The material time derivative of such a quantity defines its time rate of change as is experienced by the material points. Suppose that a kinematical quantity is described in such a manner that it is an explicit function of $X_1$, $x_1$, and $t$, for example,

$$ q = q(X_1, x_1, t) $$  \hspace{1cm} (5.18)

It is possible to substitute for, say, $x_1$ in terms of $X_1$ and $t$ in (5.18), and express it in material coordinates. In a similar manner it is possible to express (5.18) in terms of the spatial variables. Suppose, however, one wishes to retain the form (5.18), and wants to calculate its material time derivative. In doing so, one then keeps in mind the fact that the particle $X_1$ is kept fixed in the process of taking the material time derivative, while the particle position $x_1$ at the current time $t$ is not fixed. In this manner one arrives at
\[ \frac{Dg}{Dt} = \frac{\partial g}{\partial t} + v_1 \frac{\partial g}{\partial x_1} \]

\[ = f(X_1, x_1, t) . \]

In summary, the operator
\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} \] (5.19)

characterizes the time rate of change as viewed by the material particles, and for this reason is called the material time derivative operator.

With respect to this operator, the variable \( X_1 \) is always treated as constant, i.e. \( \frac{DX_1}{Dt} = 0 \), but that \( \frac{Dx_1}{Dt} = v_1 \).

**STRAIN**

In the one-dimensional motion considered so far, material line elements change their length as they move down the tube. Consider a material line element \( \Delta X_1 \) which emanates from point \( X_1 \) along the \( X_1 \)-axis at time \( t^0 \), and let its length be \( \Delta L \). At a later time \( t \), particle \( X_1 \) moves to \( x_1 \), and the material line element \( \Delta X_1 \) occupies the position \( \Delta x_1 \), along the \( X_1 \)-axis, having length \( \Delta L \). The ratio \( \frac{\Delta L}{\Delta L} \) is the average stretch for the considered line element. The stretch \( \Lambda_1(X_1, t) \) at particle \( X_1 \) at time \( t \) is given by the following limit \(^1\)

\[ \Lambda_1(X_1, t) = \lim_{\Delta L \to 0} \frac{\Delta L}{\Delta L} = \frac{\frac{\partial X_1}{\partial X_1}}{\frac{\partial x_1}{\partial X_1}} = \frac{\partial x_1}{\partial X_1} . \] (5.20)

\(^1\) Note that this limiting process is to be interpreted in the same sense as that of a nested sequence discussed in Sec. 1.2; this will be amplified later on.
As is clear from its definition, stretch is a measure of elongation or contraction of material line elements. At the initial time \( t^0 \), the value of stretch is equal to 1. Moreover, it remains equal to 1 if the continuum moves down the tube as a rigid-body.

As is shown later on in connection with two- and three-dimensional motion of continua, the stretch squared is actually a more suitable measure of deformation.

There is another measure of deformation called the \textit{Lagrangian strain}. It is commonly denoted by \( E_1(X_1,t) \).

The Lagrangian strain is defined in terms of the stretch squared by

\[ E_1 = \frac{1}{2} \left[ \Lambda_1^2 - 1 \right] . \quad (5.21) \]

Observe that for rigid motion or at the initial time \( t^0 \), this strain is zero. Moreover, in contrast to \( \Lambda_1 \), which is always positive, the strain can have \textit{positive} or \textit{negative} values, depending on whether the considered line element \textit{elongates} or \textit{contracts}. To show the significance of the Lagrangian strain, consider the following limit

\[ \lim_{\Delta L \to 0} \frac{(\Delta L)^2 - (\Delta L')^2}{(\Delta L)^2} = \Lambda_1^2 - 1 \]

which is twice the Lagrangian strain in (5.21). Thus, if the current length \( \Delta L \) is larger than the initial length \( \Delta L' \) of a material line element, the strain takes on a positive value; this indicates that the line element is elongated.
DISPLACEMENT AND DISPLACEMENT-GRADIENT

It is often convenient to consider the displacement of a particle from its initial position $X_1$ to the current position $x_1$. Denote this by $U_1(X_1,t)$, and write

$$U_1(X_1,t) = [x_1(t) - X_1] = x_1 - X_1.$$ \hspace{1cm} (5.22)

In terms of $U_1$, the stretch becomes

$$\Lambda \equiv \frac{\partial U_1}{\partial x_1} + 1,$$ \hspace{1cm} (5.23)

and the Lagrangian strain reduces to

$$E = \frac{\partial U_1}{\partial x_1} + \frac{1}{2} \left( \frac{\partial U_1}{\partial x_1} \right)^2.$$ \hspace{1cm} (5.24)

The quantity $\frac{\partial U_1}{\partial x_1}$ is called the displacement-gradient.

The displacement may equally well be expressed in terms of the spatial variables $x_1$ and $t$. Denote it then by $u_1 = u_1(x_1,t)$ and write

$$u_1 = u_1(x_1,t) \equiv U_1 \left[ x_1(t), t \right].$$

Then

$$u_1(x_1,t) = x_1 - \xi(x_1,t) = x_1 - X_1.$$ \hspace{1cm} (5.25)

The quantity $\frac{\partial u_1}{\partial x_1}$ is also called the displacement-gradient. Note that $\frac{\partial u_1}{\partial x_1}$ is not equivalent to $\frac{\partial U_1}{\partial x_1}$, but by the chain rule of differentiation one obtains
\[
\frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} \frac{\partial x}{\partial x} = \frac{\partial u}{\partial x} \left[ 1 + \frac{\partial u}{\partial x} \right]^{-1},
\]

\[
\frac{\partial u_1}{\partial x_1} \Rightarrow \frac{\partial u_1}{\partial x_1} \frac{\partial x_1}{\partial x_1} = \frac{\partial u_1}{\partial x_1} \left[ 1 - \frac{\partial u_1}{\partial x_1} \right]^{-1}. \tag{5.26}
\]

**VELOCITY-GRADIENT AND RATE OF STRETCH**

Similarly to the displacement-gradient one can define a velocity-gradient. While the displacement-gradient relates to the total deformation of a material element from the initial to the current state, the velocity-gradient relates to the time rate of such deformation.

In spatial description, the velocity-gradient is obtained by taking the **partial** derivative of the velocity-component \( v_1 = v_1(x_1, t) \) with respect to \( x_1 \), obtaining

\[
d_1(x_1, t) = \frac{\partial v_1(x_1, t)}{\partial x_1}. \tag{5.27}
\]

At time \( t \) consider now an element \( dx_1 \), and calculate the time rate of change of its square as

\[
\frac{D}{Dt} (dx_1)^2 = 2 \ dx_1 \ \frac{D}{Dt} (dx_1)
\]

\[
= 2 \ dx_1 \ d \left( \frac{Dx_1}{Dt} \right)
\]

\[
= 2 \ dx_1 \ \frac{\partial v_1}{\partial x_1} \ dx_1 = 2 \ d_1 \ dx_1 \ dx_1.
\]
Thus it follows that
\[ \frac{\partial}{\partial t} \left( \frac{dx_1}{dx_1} \right) = \dot{d}_1, \tag{5.28} \]
from which one concludes that \( d_1 \) is the time rate of change of the current length of an element per its current unit of length. It is called the rate of stretch, or stretching.

To give additional physical interpretation to the rate of stretch, consider at time \( t \) two adjacent particles, one located at \( x_1 \), the other at \( x_1 + dx_1 \). Denote the velocity of the second particle by \( v_1(x_1 + dx_1, t) \), and write
\[ v_1(x_1 + dx_1, t) = v_1(x_1, t) + dx_1 \frac{\partial v_1(x_1, t)}{\partial x_1} + \ldots \]
\[ = v_1(x_1, t) + dx_1 \dot{d}_1 + \ldots, \tag{5.29} \]
where \( v_1(x_1, t) \) is the velocity of the particle at \( x_1 \), and where a Taylor series expansion is used. To the first order of approximation in \( dx_1 \), Eq. (5.29) indicates that the instantaneous motion of a small material neighborhood about a particle instantaneously situated at \( x_1 \), consists of a rigid-body translation with the velocity of the particle at \( x_1 \), and a pure deformation defined by the rate of stretch \( \dot{d}_1 \).

**LINEARIZATION**

When the displacement-gradient \( \frac{\partial u_1}{\partial x_1} \) is small, so that its square can be neglected, the Lagrangian strain becomes
\[ E_{11} \approx \frac{\partial u_1}{\partial x_1}, \quad (5.30) \]

where

\[ \left| \frac{\partial u_1}{\partial x_1} \right| \ll 1. \quad (5.31) \]

In this case it can be shown that \( \left| \frac{\partial u_1}{\partial x_1} \right| \ll 1 \), and that \( v_1 = \frac{\partial u_1}{\partial t} \).

To this end consider (5.26) and obtain \( \frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial x_1} \). Hence if (5.31) holds, so does

\[ \left| \frac{\partial u_1}{\partial x_1} \right| \ll 1. \quad (5.32) \]

Now take the material time derivative of (5.4) and, noting that \( \frac{Dx_1}{Dt} = 0 \), arrive at

\[ 0 = \frac{\partial s_1}{\partial t} + v_1 \frac{\partial s_1}{\partial x_1} \]

which leads to

\[ v_1 = - \left( \frac{\partial s_1}{\partial t} \right) / \left( \frac{\partial s_1}{\partial x_1} \right). \quad (5.33) \]

But from (5.25) it follows that

\[ \frac{\partial s_1}{\partial x_1} = 1 - \frac{\partial u_1}{\partial x_1}, \quad \text{and} \quad \frac{\partial s_1}{\partial t} = - \frac{\partial u_1}{\partial t}, \]

which lead to

\[ v_1 = \frac{\partial u_1}{\partial t} \left[ 1 - \frac{\partial u_1}{\partial x_1} \right]^{-1} = \frac{\partial u_1}{\partial t}, \quad (5.34) \]

as asserted.
Next consider the linearization of the acceleration \( a_1 = a_1(x_1,t) \), and by direct calculation obtain
\[
\begin{align*}
a_1(x_1,t) &= \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} \\
&= \left[ 1 - \frac{\partial u_1}{\partial x_1} \right]^{-3} \left\{ \left( \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial u_1}{\partial x_1} \right)^2 \\
&+ 2 \frac{\partial u_1}{\partial t} \frac{\partial^2 u_1}{\partial t \partial x_1} \left[ 1 - \frac{\partial u_1}{\partial x_1} \right] \\
&+ \left( \frac{\partial u_1}{\partial t} \right)^2 \frac{\partial^2 u_1}{\partial x_1^2} \right\} \quad (5.35)
\end{align*}
\]

which, because of (5.32), leads to
\[
\begin{align*}
a_1(x_1,t) &\approx \frac{\partial^2 u_1}{\partial t^2} + 2 \frac{\partial u_1}{\partial t} \frac{\partial^2 u_1}{\partial t \partial x_1} + \left( \frac{\partial u_1}{\partial t} \right)^2 \frac{\partial^2 u_1}{\partial x_1^2} . \quad (5.36)
\end{align*}
\]

Now, if it is also assumed that
\[
\left| \frac{\partial u_1}{\partial t} \right| \ll 1 , \quad (5.37)
\]

and that the second partial derivatives of \( u_1 \) are of the same order of magnitude, it then follows that
\[
\begin{align*}
a_1(x_1,t) &= \frac{\partial^2 u_1}{\partial t^2} \quad (5.38)
\end{align*}
\]

Observe that without the assumption (5.37), (5.38) does not follow from (5.32) or (5.31), except for a very special class of problems which will be discussed later on; see Sec.
EXAMPLE 5.1

Consider the rectilinear motion of a continuum described by

\[ x_1 = \left( \frac{t^2}{X_1^2} + 1 \right) X_1, \quad 1 < X_1 < 2, \quad 0 \leq t < 1, \]

where all the quantities are assumed to have been rendered physically dimensionless, so that \( x_1, X_1, \) and \( t \) represent real numbers.\(^1\) To see whether the mapping is one-to-one and admissible, calculate the stretch

\[ \Lambda_1 = \frac{\delta x_1}{\delta X_1} = \frac{-t^2}{X_1^2} + 1 \]

which ceases to be positive for \( X_1 = 1 \) at \( t = 1 \). Since \( t < 1 \), the mapping is admissible.

The inverse transformation is obtained by solving the equation

\[ x_1^2 - x_1 X_1 + t^2 = 0 \]

which has two solutions, \( X_1 = \frac{1}{2} \left[ x_1 \pm \left( x_1^2 - 4t^2 \right)^{\frac{1}{2}} \right] \).

Since at \( t = 0 \), \( x_1 = X_1 \), it follows that

\[ X_1 = \frac{1}{2} \left[ x_1 + \left( x_1^2 - 4t^2 \right)^{\frac{1}{2}} \right], \quad 1 < X_1 < 2\frac{1}{2}, \quad 0 \leq t < 1 \quad (5.39) \]

The velocity field is

\[ V_1 = \frac{Dx_1}{Dt} = \frac{2t}{X_1} \quad (5.40) \]

The acceleration field is

\[ A_1 = \frac{D^2 x_1}{Dt^2} = \frac{2}{X_1} \quad (5.41) \]

---

\(^1\) Since physically meaningful equations consisting of the sum of monomials should be such that these monomials possess the same physical dimension, see Sec. 1.3, the terms entering these equations can be rendered dimensionless by means of a suitable choice of units of length, time, etc.
The velocity and the acceleration fields may be expressed in terms of the spatial variables by direct substitution for $X_1$ from Eq. (5.39). This gives

$$v_1 = 4t \left[ x_1 + \left( x_1^2 - 4t^2 \right)^{\frac{1}{2}} \right]^{-1}, \quad a_1 = 4 \left[ x_1 + \left( x_1^2 - 4t^2 \right)^{\frac{1}{2}} \right]^{-1}.$$

It is instructive to note that the material time derivative of both sides of (5.39) gives directly the velocity field in terms of the spatial variables (check by direct calculation; see Eq. (5.33)).

The displacement field is

$$U_1 = \frac{t^2}{x_1}, \quad u_1 = \frac{1}{2} \left[ x_1 - \left( x_1^2 - 4t^2 \right)^{\frac{1}{2}} \right],$$

from which the displacement-gradients are calculated to be

$$\frac{\partial U_1}{\partial x_1} = -\frac{2t^2}{x_1^2}, \quad \frac{\partial u_1}{\partial x_1} = \frac{1}{2} \left[ \frac{(x_1^2 - 4t^2)^{\frac{1}{2}} - x_1}{(x_1^2 - 4t^2)^{\frac{1}{2}}} \right].$$

The Lagrangian strain therefore becomes

$$E_1 = - \frac{t^2}{x_1^2} + \frac{1}{2} \frac{t}{x_1^4}.$$

The velocity-gradient is

$$\frac{\partial v_1}{\partial x_1} = -4t \left( x_1^2 - 4t^2 \right)^{-\frac{1}{2}} \left[ x_1 + \left( x_1^2 - 4t^2 \right)^{\frac{1}{2}} \right]^{-1},$$

which is also the stretch-rate $d_1$.

**EXAMPLE 5.2**

Calculate the material time derivative of the stretch.

To this end observe that $\Lambda_1 = \frac{\partial x_1}{\partial x_1} = \frac{1}{\frac{\partial x_1}{\partial X_1}}$. Since $\frac{DX_1}{dt} = 0$, it follows that
\[
\frac{DA_I}{Dt} = \frac{\partial \Lambda_I}{\partial t} = \frac{\partial^2 x_1}{\partial t \partial x_1}.
\]

On the other hand,

\[
\frac{DA_I}{Dt} \Rightarrow \frac{D}{Dt} \left[ \frac{\partial x_1}{\partial X_1} \right] = \frac{\partial^2 x_1}{\partial x_1 \partial x_1} \Rightarrow d_I \lambda_I.
\]

**EXAMPLE 5.3**

Calculate the material time derivative of \(\frac{\partial^2 x_1}{\partial x_1^2}\).

To this end first observe that \(\frac{\partial x_1}{\partial x_1} = \frac{1}{\lambda_I}\). Hence,

\[
\frac{\partial^2 x_1}{\partial x_1^2} = \frac{\partial}{\partial x_1} \left[ \frac{1}{\lambda_I} \right] \Rightarrow \frac{\partial}{\partial x_1} \left[ \frac{1}{\lambda_I} \right] \frac{1}{\partial x_1} = - \lambda_I^{-3} \frac{\partial^2 x_1}{\partial x_1^2}.
\]

Since the last quantity is expressed in terms of material coordinates, its material time derivative reduces to the partial time derivative. 1 Thus

\[
\frac{D}{Dt} \left[ \frac{\partial^2 x_1}{\partial x_1^2} \right] = \frac{\partial}{\partial t} \left[ - \lambda_I^{-3} \frac{\partial^2 x_1}{\partial x_1^2} \right]
\]

\[
= 3 \lambda_I^{-4} \frac{\partial^2 x_1}{\partial \tau \partial x_1} \frac{\partial^2 x_1}{\partial x_1^2} - \lambda_I^{-3} \frac{\partial^3 x_1}{\partial t \partial x_1^2}.
\]

---

1 Note that while the partial time derivative with respect to \(X_1\) and the material time derivative are commutative for sufficiently smooth functions of \(X_1\) and \(t\), the partial derivative with respect to \(x_1\) and the material time derivative are *not* commutative, i.e., \(\frac{D}{Dt} \left( \frac{\partial}{\partial x_1} \right) \neq \frac{\partial}{\partial x_1} \left( \frac{D}{Dt} \right)\); see Prob. 5.7.
2.6 **Planar Motion of a Continuum**

Suppose a continuum moves and deforms in such a manner that particles situated on planes parallel to a given plane \( \Pi \) remain in the same planes, and that particles instantaneously located on an axis perpendicular to \( \Pi \) possess identical kinematical quantities, and perform identical motion for all considered time. The motion is then called **planar**.

To study a planar motion it is sufficient to consider the motion of particles situated in a typical plane, say, \( \Pi \). Define on this plane the rectangular Cartesian coordinate system \( OX_1X_2 \), and consider at time \( t^0 \) material points in a region \( R \) with boundary \( \partial R \); see Fig. 6.1. At a later time \( t \) these material points move to new positions in plane \( \Pi \), occupying region \( r \) with boundary \( \partial r \).

As in the case of rectilinear motion it is convenient to label the particles by their rectangular Cartesian coordinates at an arbitrarily chosen initial time, say, \( t^0 \). Consider a typical particle \( P^{(1)} \), and denote its coordinates at the initial time \( t^0 \) by \( \{X_1^{(1)}, X_2^{(1)}\} \), where the superscript letter in parentheses identifies the particle. At a later time \( t \) this particle has moved to its new position defined by the coordinates \( \{x_1^{(1)}, x_2^{(1)}\} \); where, as in the preceding section, lower case letters will be used to denote quantities at the current time \( t \). The motion of this particle as a function of time can be defined by a set of two functions, \( x_1^{(1)}(t) \) and \( x_2^{(1)}(t) \), in exactly the same manner as for a point-mass. These two functions then give the position of the considered particle at a given instant. Different functions will be involved when
Figure 6.1

Planar Motion: The particles are labeled by their coordinates at initial time $t^0$. At this time, a portion of the continuum occupies the region $R$ with the boundary $\partial R$. At a later time, the particles initially in $R$, are moved into new positions forming the region $r$ with boundary $\partial r$. 
different particles are considered, i.e. these functions depend on the particles which are identified by their initial coordinates. To state this mathematically, one may write

\[ x_1^{(1)} = \Xi_1(x_1^{(1)}, x_2^{(1)}, t), \quad x_2^{(1)} = \Xi_2(x_1^{(1)}, x_2^{(1)}, t). \]

In general, therefore, it can be written that

\[ x_1 = \Xi_1(X_1, X_2, t) \quad x_2 = \Xi_2(X_1, X_2, t) \]

\[ \equiv \Xi_1(\sim X, t) \quad \equiv \Xi_2(\sim X, t), \quad (6.1) \]

which define at time \( t \) the position of a particle whose coordinates at the initial time \( t^0 \) are \( \{X_1, X_2\} \). If these coordinates are the same as that of point \( P^{(1)} \), then (6.1) defines as a function of time the path of this particle; this is shown by curve \( C \) in Fig. 6.1.

In order to simplify the notation, in the second line of (6.1) the dependency of \( \Xi_1 \) and \( \Xi_2 \) on \( X_1 \) and \( X_2 \), is denoted collectively by the argument \( \sim X \). This is a notational convenience, and the student should keep in mind that \( x_1 \) and \( x_2 \) are explicit functions of \( X_1, X_2 \), and \( t \).

Equation (6.1) can be considered as defining a one-parameter family of mappings of points in \( \mathbb{R} \) into \( r \).

In continuum mechanics it is accepted on physical grounds that two distinct particles will remain distinct throughout the motion of the continuum, and therefore the mapping (6.1) is one-to-one and invertible. Hence, similarly to the one-dimensional case, it will be assumed that (6.1) admits the following inverse:
\[ x_1 = \xi_1(x_1, x_2, t) \quad \quad x_2 = \xi_2(x_1, x_2, t) \]

\[ \equiv \xi_1(x, t) \quad , \quad \quad \equiv \xi_2(x, t) \quad , \quad (6.2) \]

which identify the particle that at the current time \( t \) is located at \( \{ x_1, x_2 \} \). Note again that in the second line of (6.2) the explicit dependency of \( \xi_1 \) and \( \xi_2 \) on \( x_1 \) and \( x_2 \) is collectively denoted by the argument \( x \).

The mapping (6.1) and its inverse (6.2) will be assumed to be sufficiently smooth so that all the indicated derivatives exist over the domain, of the definition of the arguments. The student should keep this in mind, since it will be implied in all subsequent developments.

Observe that substitution for \( x_1 \) and \( x_2 \) from (6.1) into (6.2) leads to identities, i.e.

\[ \xi_1 \left[ \Xi_1(x_1, x_2, t), \Xi_2(x_1, x_2, t), t \right] = x_1 \quad , \]

\[ \xi_2 \left[ \Xi_1(x_1, x_2, t), \Xi_2(x_1, x_2, t), t \right] = x_2 \quad . \]

CONDITION FOR ONE-TO-ONE MAPPING

The set of material points that occupy at the initial time \( t^0 \) the line element \( dL \) form a material line element; Fig. 6.1. Let the components of this line element be denoted by \( dX_1 \) and \( dX_2 \). This material element is mapped by deformation (6.1) into a new line element \( d\tilde{L} \) with components \( dx_1 \) and \( dx_2 \). From (6.1) it follows that
\[ dx_1 = \left( \frac{\partial x_1}{\partial x_1} \right) dx_1 + \left( \frac{\partial x_1}{\partial x_2} \right) dx_2 , \]
\[ dx_2 = \left( \frac{\partial x_2}{\partial x_1} \right) dx_1 + \left( \frac{\partial x_2}{\partial x_2} \right) dx_2 . \]

For the mapping to be one-to-one, one must be able to solve Eqs. (6.3) for \( dx_1 \) and \( dx_2 \) in terms of \( dx_1 \) and \( dx_2 \), obtaining a unique solution. The solution of (6.3) is
\[ dx_1 = \frac{1}{J} \left[ \left( \frac{\partial x_2}{\partial x_1} \right) dx_1 - \left( \frac{\partial x_1}{\partial x_2} \right) dx_2 \right] , \]
\[ dx_2 = \frac{1}{J} \left[ -\left( \frac{\partial x_2}{\partial x_1} \right) dx_1 + \left( \frac{\partial x_1}{\partial x_2} \right) dx_2 \right] , \quad (6.4) \]

where
\[ J = \begin{vmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} \end{vmatrix} = \left( \frac{\partial x_1}{\partial x_1} \right) \left( \frac{\partial x_2}{\partial x_2} \right) - \left( \frac{\partial x_1}{\partial x_2} \right) \left( \frac{\partial x_2}{\partial x_1} \right) \quad (6.5) \]

is called the Jacobian determinant of the transformation. From (6.4) it follows that if the Jacobian \( J \) is nonzero having finite values, the system (6.3) admits a unique solution for given \( dx_1 \) and \( dx_2 \). Moreover that if and only if \( dx_1 = 0 \) and \( dx_2 = 0 \), then the solution is \( dx_1 = dx_2 = 0 \).

The condition for a one-to-one mapping, therefore, is that the Jacobian \( J \) remain nonzero and finite everywhere in \( \mathbb{R} \). Since a single set of coordinates is used, it should be required that \( J \) remain positive, i.e.
\[ 0 < J < \infty . \quad (6.6) \]
A similar condition is obtained if one considers the inverse mapping (6.2). This leads to

\[ dX_1 = \left( \frac{\partial X_1}{\partial x_1} \right) dx_1 + \left( \frac{\partial X_1}{\partial x_2} \right) dx_2 , \]

\[ dX_2 = \left( \frac{\partial X_2}{\partial x_1} \right) dx_1 + \left( \frac{\partial X_2}{\partial x_2} \right) dx_2 . \quad (6.7) \]

For given \( dX_1 \) and \( dX_2 \) which are not both zero, these equations must yield a unique solution for \( dx_1 \) and \( dx_2 \) which also are not both zero.\(^1\) This requires that the Jacobian of the inverse transformation given by

\[ j^{-1} = \left| \begin{array}{cc} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} \end{array} \right| = \left( \frac{\partial X_1}{\partial x_1} \right) \left( \frac{\partial X_2}{\partial x_2} \right) - \left( \frac{\partial X_1}{\partial x_2} \right) \left( \frac{\partial X_2}{\partial x_1} \right) \quad (6.8) \]

be neither zero nor infinity; it must be

\[ 0 < j^{-1} < \infty . \quad (6.9) \]

Observe that \( J = j^{-1} \), and conversely.

**INDEX NOTATION**

With the aid of index notation and the summation convention a great deal of economy is achieved in the description of the planar, as well as

\[ 1 \text{ As stated before, this then guarantees that two distinct material points will remain distinct in the course of deformation and motion of the continuum.} \]
the three-dimensional motion of a continuum. For a beginner, however, it is more understandable if relevant expressions are first written out in expanded form.

Consider a typical particle which at initial time \( t^0 \) has the coordinates \( \{X_1, X_2, \ldots \} \), and which moves to the point \( \{x_1, x_2, \ldots \} \) at a later time \( t \). In index notation the initial and current particle positions can be expressed as \( X_A \), and \( x_a \), \( A, a = 1, 2, \ldots \), so that \( X_A \) corresponds to \( \{X_1, X_2\} \) and \( x_a \) to \( \{x_1, x_2\} \).

The lower case letters will be used for indices to lower case kernels, and the upper case letters for upper case kernels. There will be some exceptions which will not cause any confusion, since the index letter, whether upper or lower case, always takes on the values 1, 2, or 1, 2, 3, which will be clearly stated.

Equations (6.3) now become

\[
dx_a = \frac{\partial x_a}{\partial X_A} dX_A, \quad a, A = 1, 2 \quad (6.10)
\]

Similarly, (6.7) become

\[
dX_A = \frac{\partial X_A}{\partial x_a} dx_a \quad (6.11)
\]

In (6.10) the index \( A \) is a dummy index and therefore the right-hand side should be summed on \( A \), leading to two terms. The index \( a \) in (6.10), however, is a live index and therefore this equation stands for two equations given by (6.3). In (6.11), on the other hand, the \( a \) is a dummy index and \( A \) is the live one.
From the chain rule of differentiation the following identities are obtained:

\[
\frac{\partial X_A}{\partial x_a} \frac{\partial x_a}{\partial X_B} = \delta_{AB}, \quad \frac{\partial x_a}{\partial x_b} \frac{\partial X_b}{\partial X_A} = \delta_{ab}, \quad A, B, a, b = 1, 2, \quad (6.12)
\]

where \(\delta_{AB}\), or \(\delta_{ab}\) is the Kronecker delta defined before; see Eq. (4.12).

**PERMUTATION SYMBOL**

To express the Jacobian \(J\) defined in (6.5), or its inverse \(J^{-1}\) given by (6.8) in index notation, one needs the so-called two-dimensional permutation symbol \(e_{AB}\), or \(e_{ab}\) defined as follows:

\[
e_{AB} = \begin{cases} 
1 & \text{if } A = 1, B = 2 \\
-1 & \text{if } A = 2, B = 1 \\
0 & \text{if } A = B = 1, 2 
\end{cases} \quad (6.13)
\]

with exactly the same definition applying to \(e_{ab}\).

The Jacobian \(J\) now becomes

\[
J = \frac{1}{2} e_{ab} e_{AB} \frac{\partial x_a}{\partial X_A} \frac{\partial x_b}{\partial X_B} \quad (6.14)
\]

Similarly, (6.8) becomes

\[
J^{-1} = \frac{1}{2} e_{AB} e_{ab} \frac{\partial X_A}{\partial x_a} \frac{\partial X_B}{\partial x_b}, \quad A, B, a, b = 1, 2 \quad (6.15)
\]

---

1 Note, for example, that \(\frac{\partial X_1}{\partial X_2} = 0 = \frac{\partial X_1}{\partial x_1} \frac{\partial x_1}{\partial X_2} + \frac{\partial X_1}{\partial x_2} \frac{\partial x_2}{\partial X_2}\), where the last expression follows from the fact that \(X_1 = \xi_1(x_1, x_2, t)\). Similarly, \(\frac{\partial X_4}{\partial X_1} = 1 = \frac{\partial X_4}{\partial x_4} \frac{\partial x_4}{\partial X_1} + \frac{\partial X_4}{\partial x_1} \frac{\partial x_1}{\partial X_1}\).
Equations (6.14) and (6.15) can be verified by direct expansion. Note that all indices in these equations, are dummy.

**VELOCITY**

As for a point-mass in Newtonian mechanics, the velocity of a particle \( X_A, A = 1,2 \), is represented by a vector \( \vec{v} = e_1 v_1 + e_2 v_2 \). Different particles at a given instant possess, in general, different velocities. Therefore, in addition to time \( t \), the velocity vector \( \vec{v} \) depends on the particle considered, and therefore is a function of the initial coordinates of the particles. To express this mathematically one writes

\[
V_1 = V_1(x_1, x_2, t) \quad V_2 = V_2(x_1, x_2, t) \\
\equiv V_1(x, t) , \quad \equiv V_2(x, t) . \quad (6.16)
\]

Since the velocity of a particle is given by the time rate of change of its position, the material description of particle velocities is obtained if one forms the time rate of change of particle positions. From (6.1) this is given by

\[
V_1 = \frac{Dx_1}{Dt} = \frac{Dx_1}{Dt} = \frac{\partial x_1}{\partial t} , \\
V_2 = \frac{Dx_2}{Dt} = \frac{Dx_2}{Dt} = \frac{\partial x_2}{\partial t} . \quad (6.17)
\]

To obtain the spatial description of the particle velocity, one needs to substitute for \( x_1 \) and \( x_2 \) from (6.2) into (6.16), arriving at
\[ v_1 = v_1(x_1, x_2, t) \]

\[ \equiv v_1(x, t) = V_1(g_1(x_1, x_2, t), g_2(x_1, x_2, t), t) \]

\[ v_2 = v_2(x_1, x_2, t) \]

\[ \equiv v_2(x, t) = V_2(g_1(x_1, x_2, t), g_2(x_1, x_2, t), t) \]  \hspace{1cm} (6.18)

Equations (6.18) define the components of the velocity of the particle which at time \( t \) is situated at point \( x_a, a = 1,2 \).

**STREAMLINES**

At a given instant \( t, v = e_1 v_1(x, t) + e_2 v_2(x, t) \), defines at each spatial point \( x_a, a = 1,2 \), a vector whose components are \( v_a(x, t) \), \( a = 1,2 \). The totality of all these vectors at this instant comprises the so-called velocity field of the considered planar motion. Consider the collection of planar curves which are tangent to the velocity vector associated with each point. These curves are called the *streamlines*.

In general the streamlines change in time, since the velocity vector associated with a spatial point changes in time as different material points pass through this point. For a particular case in which the velocity vectors associated with spatial points do not change with time, the streamlines then remain stationary. The corresponding velocity field is called *stationary* (or *steady*).

Since at each instant one and only one particle can occupy a given spatial point, and since on physical grounds the velocity of this particle
at this instant is a unique vector which must be tangent to the path of
this particle as well as to the streamline passing through the con-
sidered spatial point, it follows that a streamline can never cross
itself, and that at a given instant the streamline of a given spatial
point is tangent to the path of the particle which at that instant is
at that point.

For a fixed particle Eqs. (6.1) with \( t \) as a parameter define the
particle path. To obtain the equations characterizing the streamlines
one observes that at each point the streamline is tangent to the veloc-
ity vector. Hence, with \( t \) fixed, the integral curves of the system

\[
\frac{dx_1}{v_1} = \frac{dx_2}{v_2} \quad (6.19)
\]

are the streamlines at this instant. Note that (6.19) can be written as

\[
\frac{dx_2}{dx_1} = \frac{v_2(x_1, x_2, t)}{v_1(x_1, x_2, t)} \quad (6.20)
\]

which in the \( x_1, x_2 \)-plane gives the slope of the tangents of the stream-
lines.

**ACCELERATION**

The time rate of change of the particle velocity is its accelera-
tion. Fix the particle, and from (6.16) obtain

\[
A_1 = A_1(x_1, x_2, t) \quad A_2 = A_2(x_1, x_2, t)
\]

\[
A_1 = \frac{\partial V_1(x, t)}{\partial t}, \quad A_2 = \frac{\partial V_2(x, t)}{\partial t} \quad (6.21)
\]
which is the material description of the particle acceleration.

The corresponding spatial description is obtained if one substitutes for \( x_1 \) and \( x_2 \) from (6.2) into (6.21). This gives

\[
a_1 = a_1(x_1, x_2, t) = a_1(x, t) = A_1[\xi_1(x_1, x_2, t), \xi_2(x_1, x_2, t), t],
\]

\[
a_2 = a_2(x_1, x_2, t) = a_2(x, t) = A_2[\xi_1(x_1, x_2, t), \xi_2(x_1, x_2, t), t]. \tag{6.22}
\]

The spatial description of acceleration may equally well be obtained if one forms the material time derivative of the velocity field in spatial description given by (6.18). To do this one should keep in mind that \( x_1 \) and \( x_2 \) are functions of time through Eqs. (6.1), since it is the particle which is kept fixed not its spatial position. Thus by the chain rule of differentiation one obtains

\[
a_1 = \frac{Dv_1}{Dt} = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_1} \frac{Dx_1}{Dt} + \frac{\partial v_1}{\partial x_2} \frac{Dx_2}{Dt}
\]

\[
= \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2},
\]

\[
a_2 = \frac{Dv_2}{Dt} = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2}. \tag{6.23}
\]

The index notation and the summation convention can be used to reduce the length of equations. With this in mind the acceleration can be written as

\[
a_1 = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}, \quad i, j = 1, 2. \tag{6.24}
\]
The first term in the right-hand side is called the *local* acceleration, and the second term, the *convected* acceleration.

Note that the material time derivative operator takes on the form

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + v_j \frac{\partial}{\partial x_j} \]

\[ = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \quad . \]  

(6.25)

**DEL OPERATOR**

It is convenient to introduce the following differential operator called the "del" operator:

\[ \nabla_x = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} = e_a \frac{\partial}{\partial x_a} \quad , \quad a = 1,2 \quad , \]  

(6.26)

in which the subscript \( x \) is used to denote that differentiation with respect to spatial coordinates \( x_a, a = 1,2 \), is involved.

Let \( \varphi = \varphi(x_1, x_2) \) be a function defined in a region \( r \) in the \( x_1, x_2 \)-plane, and differentiable there. The vector-valued quantity

\[ \nabla_x \varphi = e_a \frac{\partial \varphi}{\partial x_a} \]

\[ = e_1 \frac{\partial \varphi}{\partial x_1} + e_2 \frac{\partial \varphi}{\partial x_2} \quad . \]  

(6.27)

is called the gradient of \( \varphi \), and is often denoted by \( \text{grad} \varphi \).

At a typical point in \( r \) consider a direction defined by the unit vector \( \mathbf{n} = e_1 n_1 + e_2 n_2 \), and let \( n \) measure length in this direction. If \( \alpha \) is the angle formed by \( \mathbf{n} \) and the \( X_1 X_1 \)-direction, then \( n_1 = \frac{dx_1}{dn} = \cos \alpha \), and
\[ n_2 = \frac{dx_2}{dn} = \sin \alpha . \] The directional derivative of \( \varphi \) along \( n \) is denoted by \( \frac{d\varphi}{dn} \), and is given by

\[
\frac{d\varphi}{dn} = \frac{\partial \varphi}{\partial x_1} \frac{dx_1}{dn} + \frac{\partial \varphi}{\partial x_2} \frac{dx_2}{dn}
\]

\[ = n_1 \frac{\partial \varphi}{\partial x_1} + n_2 \frac{\partial \varphi}{\partial x_2} = \mathbf{n} \cdot \nabla_x \varphi . \quad (6.28) \]

Hence, the directional derivative operator is defined by

\[ \mathbf{n} \cdot \nabla_x \equiv n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} \equiv n_a \frac{\partial}{\partial x_a} . \quad (6.29) \]

Since \( \nabla_x \) is a vector operator, one may \textbf{formally} perform vector operations on it. Suppose, for example, that \( \mathbf{F} = \mathbf{F}(x_1, x_2) \) is a differentiable vector field defined on \( \mathbf{r} \). Then its divergence is defined by

\[
\nabla_x \cdot \mathbf{F} = e_a \frac{\partial}{\partial x_a} \cdot (e_b \mathbf{F}_b)
\]

\[ = (e_a \cdot e_b) \frac{\partial \mathbf{F}_b}{\partial x_a} = \delta_{ab} \frac{\partial \mathbf{F}_b}{\partial x_a} \]

\[ = \frac{\partial F_a}{\partial x_a} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} . \quad (6.30) \]

The divergence of \( \mathbf{F} \) is often denoted by \( \text{div} \mathbf{F} \). Note that the divergence of a differentiable vector field is a scalar field.

Return now to the planar motion of a continuum and observe that, with the aid of the del operator, the material time derivative operator (6.25) becomes

\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{n} \cdot \nabla_x . \quad (6.31) \]
EXAMPLE 6.1

As an exercise consider the hypothetical planar motion characterized by the following velocity field:

\[ v_1 = t x_1 x_2, \quad v_2 = -x_2^2, \quad (6.32) \]

where dimensionless quantities are used. Find the corresponding mapping, the acceleration field, and the expression for the streamlines.

To obtain the mapping which corresponds to the above velocity field, integrate the following equations:

\[ v_1 \, dt = dx_1, \quad v_2 \, dt = dx_2. \quad (6.33) \]

The second equation yields \( t = \frac{1}{x_2} + K_0 \), where \( K_0 \) is the constant of integration. Since at \( t = 0 \), \( x_2 = X_2 \), it follows that

\[ x_2 = X_2 \left(1 + t x_2\right). \quad (6.34) \]

Now substitute this expression and \((6.32)_1\) into \((6.33)_1\), to arrive at \(dx_1/x_1 = x_2 \, t \, dt / \left(1 + t \, x_2\right)\) which upon integration yields

\[ x_1 = X_1 \left(1 + t \, x_2\right)^{-\frac{1}{x_2}} e^t, \quad (6.35) \]

where the fact that, at \( t = 0 \), \( x_1 = X_1 \), is used. Equations (6.34) and (6.35) define the corresponding mapping.

To obtain the acceleration field, take the material time derivative of (6.32), which gives
\[ a_1 = x_1 x_2 + (t x_1 x_2)(t x_2) + (-x_2^2)(t x_1) \]
\[ = x_1 x_2 \left[ 1 + (t^2 - t) x_2 \right] , \]
\[ a_2 = x_1 x_2 (0) + (-x_2^2)(2x_2) \]
\[ = 2x_2^3 . \]  
\[(6.36)\]

Equations (6.32) and (6.36) may, of course, be expressed in terms of the material coordinates by direct substitution from (6.34) and (6.35). On the other hand, partial differentiation of (6.34) and (6.35) with respect to time directly yields the material description of the velocity field.

To obtain the streamlines at a given instant \( t \), fix \( t \) and consider the differential equations (6.19) which now become \( \frac{dx_1}{t x_1 x_2} = \frac{dx_2}{-x_2^2} \). This equation can be integrated directly. The result is

\[ K_1 = x_1 x_2^t , \quad K_1 = \text{constant} . \]  
\[(6.37)\]

At a fixed time \( t \), Eq. (6.37) defines, for each value of \( K_1 \), a curve in the \( X_1, X_2 \)-plane, i.e. a streamline. Suppose the streamline which, at \( t = 1 \), passes through the point \( x_1 = 1, x_2 = 2 \), is required. From (6.37), \( K_1 = 2 \), and hence the corresponding equation is \( x_2 = 2/x_1 \) which is a hyperbola.

**EXAMPLE 6.2**

Show that \( \frac{\partial \mathbf{u}}{\partial x} \) is \( A \), where \( J \) is the Jacobian, Eq. (6.5), and

\[ x_{AA} = \frac{\partial x_a}{\partial x_A} \quad \text{and} \quad x_{Aa} = \frac{\partial x_A}{\partial x_a} . \]
Direct differentiation of $J$ yields

\[
\frac{\partial J}{\partial x_{11}} = \frac{\partial x_2}{\partial x_2}, \quad \frac{\partial J}{\partial x_{12}} = -\frac{\partial x_2}{\partial x_1}, \quad \frac{\partial J}{\partial x_{22}} = \frac{\partial x_1}{\partial x_1}, \quad \frac{\partial J}{\partial x_{21}} = -\frac{\partial x_1}{\partial x_2}.
\]

(6.38)

On the other hand, from Eqs. (6.12) it follows that

\[
\frac{\partial x_1}{\partial x_1} \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_1} \frac{\partial x_2}{\partial x_1} = 1, \quad \frac{\partial x_2}{\partial x_1} \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2} \frac{\partial x_2}{\partial x_1} = 0,
\]

which can be solved for $\frac{\partial x_1}{\partial x_1}$ and $\frac{\partial x_2}{\partial x_1}$ yielding

\[
\frac{\partial x_1}{\partial x_1} = \frac{1}{J} \frac{\partial x_2}{\partial x_2}, \quad \frac{\partial x_2}{\partial x_1} = -\frac{1}{J} \frac{\partial x_2}{\partial x_1}.
\]

(6.39)

In a similar manner, from (6.12) one can show that

\[
\frac{\partial x_2}{\partial x_2} = \frac{1}{J} \frac{\partial x_1}{\partial x_1}, \quad \frac{\partial x_1}{\partial x_2} = -\frac{1}{J} \frac{\partial x_1}{\partial x_2}.
\]

(6.40)

Now, if it is observed that

\[
\frac{\partial \ln J}{\partial x_{aA}} = \frac{1}{J} \frac{\partial J}{\partial x_{aA}},
\]

(6.41)

from (6.38), (6.39), and (6.40) then follow the sought results.
2.7 \textbf{PLANAR DEFORMATION OF A CONTINUUM}

As the continuum occupying region $R$ moves into its current configuration and occupies region $r$, its material elements deform. Material line elements elongate or shorten and the angle between two such elements increases or decreases. Although there are various ways that the deformation of a material neighborhood can be defined quantitatively, the change in length of an element relative to its initial length, and the change of angle of initially orthogonal elements, are common measures of deformation.

\textbf{STRETCH}

A material line element emanating from a particle $P$ can be identified by giving its length and orientation at each instant. Its orientation is defined by its tangent unit vector.

Consider the line element $\Delta L$ whose unit tangent at initial time $t^0$ is

\[ \mathbf{M} = \mathbf{e}_1 M_1 + \mathbf{e}_2 M_2, \] \[ \text{where } M_1 = \cos \phi, \ M_2 = \sin \phi; \text{ see Fig. 7.1.} \]

At the current time $t$ the line element has changed to $\Delta l$ with the unit tangent vector $\mathbf{m} = \mathbf{e}_1 m_1 + \mathbf{e}_2 m_2$, where $m_1 = \cos \varphi$, $m_2 = \sin \varphi$. The stretch of the material line element initially tangent to $\mathbf{M}$ is then defined by the following limit:

\[ \Lambda (\mathbf{M}) = \lim_{\Delta L \to 0} \frac{\Delta l}{\Delta L} = \frac{dl}{dL}, \quad (7.1) \]

where the limiting process should be interpreted in the sense of nested intervals mentioned before. The subscript (M) in (7.1) signifies that the stretch depends not only on the considered particle but also on the orientation of the material line element.
A material line element $\Delta L$ with unit tangent vector $\tilde{M} = \tilde{e}_1 \cos \tilde{\phi} + \tilde{e}_2 \sin \tilde{\phi}$ is mapped into $\Delta \tilde{L}$ with unit tangent vector $\tilde{m} = \tilde{e}_1 \cos \varphi + \tilde{e}_2 \sin \varphi$. Similarly, the element $\Delta L^* \tilde{M}$ which makes a right angle with $\Delta L$ at $t = t^0$, is mapped into $\Delta l$ which makes an angle $\frac{\pi}{2} - \Gamma(M^* \tilde{M})$ with $\Delta L$, at a later time $t$. 

Figure 7.1
Let the initial and current components of a line element be denoted by \( dX_A \), and \( dx_a \), \( A,a = 1,2 \), respectively. It then follows that

\[
(dL)^2 = (dx_1)^2 + (dx_2)^2 = dx_a dx_a , \quad A,a = 1,2 ,
\]

where in the last line the summation convention is used. Substitution from (6.10) into (7.2) yields

\[
(dL)^2 = \frac{\partial x_a}{\partial X_A} \frac{\partial x_a}{\partial X_B} dX_A dX_B , \quad A,B,a = 1,2 .
\]

Hence, the stretch squared becomes

\[
\left( \frac{dL}{dL} \right)^2 = \frac{\partial x_a}{\partial X_A} \frac{\partial x_a}{\partial X_B} \frac{dX_A}{dL} \frac{dX_B}{dL} = \frac{\partial x_a}{\partial X_A} \frac{\partial x_a}{\partial X_B} M_A M_B , \quad A,B,a = 1,2 .
\]

Set

\[
C_{AB} = \frac{\partial x_a}{\partial X_A} \frac{\partial x_a}{\partial X_B} , \quad A,B,a = 1,2 ,
\]

and from (7.1) and (7.3) obtain

\[
\lambda(M)^2 = C_{AB} M_A M_B \\
= C_{11} \cos^2 \phi + C_{22} \sin^2 \phi + 2C_{12} \cos \phi \sin \phi \\
= \frac{1}{2} (C_{11} + C_{22}) + \frac{1}{2} (C_{11} - C_{22}) \cos 2\phi + C_{12} \sin 2\phi ,
\]
where in the last two lines the fact that \( M_1 = \cos \phi \) and \( M_2 = \sin \phi \),
is used.

The functions \( C_{AB} = C_{AB}(X,t) \), \( A,B = 1,2 \), constitute what is
commonly called Green's \(^1\) deformation tensor.

Consider a material line element initially parallel to the \( X_1X_1 \)-
axis. For this element \( M_1 = 1 \), and \( M_2 = 0 \). Hence, from (7.5) it fol-
lows that \( C_{11} = \left( \frac{\partial x_1}{\partial x_1} \right)^2 + \left( \frac{\partial x_2}{\partial x_1} \right)^2 \) is the stretch squared, \( \Lambda^{(1)} \), of
such element. Similarly, if a line element is initially parallel to the
\( X_2X_2 \)-axis, its stretch squared, \( \Lambda^{(2)} \), becomes \( C_{22} = \left( \frac{\partial x_1}{\partial x_2} \right)^2 + \left( \frac{\partial x_2}{\partial x_2} \right)^2 \).

When, on the other hand, a line element makes an angle \( \phi \) with the \( X_1X_1 \)-
axis, its stretch squared is given by Eq. (7.5). Note that the quantities
\( C_{AB}, A,B = 1,2 \), are functions of \( X_1, X_2 \), and \( t \), so that the stretch of
elements having a common initial direction, changes from point to point
as well as in time. Note also that Green's deformation tensor can equally
well be expressed in terms of the spatial coordinates \( x_1, x_2 \), and \( t \), by
direct substitution for \( X_1 \) and \( X_2 \).

Green's deformation tensor, characterized by \( C_{AB} \), is symmetric,
i.e. \( C_{12} = C_{21} = \left( \frac{\partial x_1}{\partial x_1} \right) \left( \frac{\partial x_1}{\partial x_1} \right) + \left( \frac{\partial x_2}{\partial x_1} \right) \left( \frac{\partial x_2}{\partial x_2} \right) \). This last quantity
relates to the change of the angle of two material line elements emanating
from the considered point.

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\(^1\) Green, George (1793-1841), English mathematician who was essentially
self-taught, and made substantial contributions to the potential theory in
which he produced theorems and relations which bear his name.
At the particle $P$ consider another material line element $dL^*$ with unit tangent vector $\mathbf{M}^*$, which is orthogonal to the element $dL$ with unit tangent vector $\mathbf{M}$, i.e. $\mathbf{M} \cdot \mathbf{M}^* = 0$. Let the component of the starred material line element in the initial and the current states be denoted by $d\mathbf{X}_a^*$ and $d\mathbf{x}_a^*$, $A, a = 1, 2$, respectively, and designate the corresponding unit tangent in the current state by $\mathbf{m}^* = e_1^* \mathbf{m}_1^* + e_2^* \mathbf{m}_2^*$; see Fig. 7.1.

Now if the current angle between these elements is $\frac{\pi}{2} - \Gamma_{(MM^*)}$, so that $\Gamma_{(MM^*)}$ is the decrease in the initially right angle, then one has

$$m \cdot m^* = \cos \left( \frac{\pi}{2} - \Gamma_{(MM^*)} \right) = \sin \Gamma_{(MM^*)} = \frac{dx_a}{dl} \frac{dx_a^*}{dl^*}, \quad a = 1, 2, \quad (7.6)$$

where $dl^*$ is the current length of material line element $dL^*$. The last expression in Eq. (7.6) can be rewritten as follows:

$$\frac{dx_a}{dl} \frac{dx_a^*}{dl^*} = \left\{ \frac{\partial x_a}{\partial \mathbf{X}_a} \frac{d\mathbf{X}_A}{dl} \frac{dL}{dl} \right\} \left\{ \frac{\partial x_a^*}{\partial \mathbf{X}_B} \frac{d\mathbf{X}_B}{dl^*} \frac{dL^*}{dl^*} \right\}$$

$$= \frac{\partial x_a}{\partial \mathbf{X}_A} \frac{\partial x_a^*}{\partial \mathbf{X}_B} \frac{M_A^*}{\Lambda_{(M)}} \frac{M_B^*}{\Lambda_{(M^*)}}.$$

Hence, (7.6) becomes

$$\sin \Gamma_{(MM^*)} = \frac{C_{AB} M_A^* M_B^*}{\Lambda_{(M)} \Lambda_{(M^*)}}, \quad A, B = 1, 2. \quad (7.7)$$

Suppose now that $\mathbf{M}$ is parallel to the $X_1X_1^*$-axis, and $\mathbf{M}^*$ is parallel to the $X_2X_2^*$-axis, respectively. Equation (7.7) then yields $\sin \Gamma_{(12)} = \frac{C_{12}}{\sqrt{C_{11} C_{22}}}$.
which defines the decrease in the initially right angle of two material line elements initially parallel to the corresponding coordinate axes.

**DISPLACEMENT AND DISPLACEMENT-GRADIENT**

The displacement of a particle which has moved from initial position \( X_A \) to a current position \( x_a \), \( A,a = 1,2 \), is defined by

\[
U_1(X,t) = \Xi_1(X,t) - X_1 = x_1 - X_1, \quad U_2(X,t) = \Xi_2(X,t) - X_2 = x_2 - X_2, \tag{7.8}
\]

when the material description is used. In the spatial description this becomes

\[
u_1(x,t) = x_1 - \xi_1(x,t), \quad u_2(x,t) = x_2 - \xi_2(x,t). \tag{7.9}
\]

Observe that \( u_a(x,t) \) corresponds to \( U_A(X,t) \) in the sense discussed in Sec. 2.5; see p. 2-32.

The partial derivative of \( U_A \) with respect to the initial particle position, \( \frac{\partial U_A}{\partial X_B} \), \( A,B = 1,2 \), is called the **displacement-gradient**. The quantity \( \frac{\partial u_a}{\partial X_b} \), \( a,b = 1,2 \), is also called the displacement-gradient.

**STRAIN**

Consider a material line element initially parallel to the \( X_1 \) axis.

The stretch squared of this element according to (7.5) is \( \lambda^2_{(1)} = C_{11} \).

Similarly to the one-dimensional case (see Eq. (5.21)), the Lagrangian strain for this material line element is given by

\[
E_{11} = \frac{1}{2} \left[ \lambda_{(1)}^2 - 1 \right] = \frac{1}{2} \left( C_{11} - 1 \right). \tag{7.10}
\]

In the same manner, if an element is initially parallel to the \( X_2 \) axis, its strain becomes \( E_{22} = \frac{1}{2} \left( C_{22} - 1 \right) \).

---

1 The double indices are used so that the Lagrangian strain corresponds to the Green's deformation tensor; see Eq. (7.10).
Consider now an element initially tangent to \( \mathcal{M} \). Its strain then is

\[
\frac{1}{2} \lim_{\Delta L \to 0} \frac{(\Delta l)^2 - (\Delta L)^2}{(\Delta L)^2} = \frac{1}{2} \left[ \Lambda_\mathcal{M}^2 - 1 \right]
\]

\[
= \frac{1}{2} \left[ C_{AB}^\mathcal{M} M_A M_B - 1 \right]
\]

\[
= \frac{1}{2} \left[ C_{AB} - \delta_{AB} \right] M_A M_B, \quad A, B = 1, 2.
\]

This suggests that one introduce the quantity

\[
E_{AB} = \frac{1}{2} \left( C_{AB} - \delta_{AB} \right) \tag{7.10}
\]

which is called the Lagrangian strain. In this manner the strain of an element initially tangent to \( \mathcal{M} \) is given by

\[
E_{\mathcal{M}} = E_{AB} M_A M_B, \quad A, B = 1, 2. \tag{7.11}
\]

The Lagrangian strain can be expressed in terms of the displacement-gradient by direct substitution, similarly to the one-dimensional case. In this way one obtains

\[
E_{AB} = \frac{1}{2} \left[ \frac{\partial U_A}{\partial X_B} + \frac{\partial U_B}{\partial X_A} \right] + \frac{1}{2} \left( \frac{\partial U_D}{\partial X_A} \right) \left( \frac{\partial U_D}{\partial X_B} \right), \quad A, B, D = 1, 2. \tag{7.12}
\]

Observe that \( E_{AB} \) is symmetric with respect to the exchange of \( A \) and \( B \), and therefore (7.12) represents three equations which are

\[
E_{11} = \frac{\partial U_1}{\partial X_1} + \frac{1}{2} \left[ \left( \frac{\partial U_1}{\partial X_1} \right)^2 + \left( \frac{\partial U_2}{\partial X_1} \right)^2 \right],
\]

\[
E_{22} = \frac{\partial U_2}{\partial X_2} + \frac{1}{2} \left[ \left( \frac{\partial U_1}{\partial X_2} \right)^2 + \left( \frac{\partial U_2}{\partial X_2} \right)^2 \right],
\]
\[ E_{12} = E_{21} = \frac{1}{2} \left[ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right] + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right) \left( \frac{\partial u_1}{\partial x_2} \right) + \left( \frac{\partial u_2}{\partial x_1} \right) \left( \frac{\partial u_2}{\partial x_2} \right) \right] . \] (7.13)

The quantities \( E_{11} \) and \( E_{22} \) are called the normal components, and \( E_{12} = E_{21} \) is called the shear component of Lagrangian strain.

**VELOCITY-GRADIENT AND RATE OF STRETCH**

As in the case of one-dimensional motion, the velocity-gradient is defined by

\[ \frac{\partial v_a(x,t)}{\partial x_b} , \quad a,b = 1,2 . \] (7.14)

Since in planar motion there are two independent velocity components, each depending on two coordinate variables \( x_1 \) and \( x_2 \), as well as on \( t \), there are, in general, four independent functions which characterize the velocity-gradient (7.14). These quantities relate to the time rate of deformation as well as rotation of material neighborhoods.

Consider in the current state the material line element \( dl \) with components \( dx_a \), \( a = 1,2 \). The time rate of change of its squared length is

\[
\frac{D}{Dt} (dl)^2 = 2 \frac{d}{dt} \frac{D(dl)}{dt} \\
= \frac{D}{Dt} (dx_a dx_a) = 2 \frac{D(dx_a)}{dt} dx_a \\
= 2 d \left( \frac{Dx_a}{dt} \right) dx_a \\
= 2 \frac{\partial v_a}{\partial x_b} dx_a dx_a , \quad a,b = 1,2 .
\]
It hence follows that
\[
\frac{D}{Dt} \frac{d}{dl} = \frac{\partial v_a}{\partial x_b} m_a m_b - \frac{1}{2} \frac{\partial v_a}{\partial x_b} m_a m_b + \frac{1}{2} \frac{\partial v_a}{\partial x_b} m_b m_a,
\]
where \( m_a = \frac{dx_a}{dl} \).

The left-hand side of (7.15) is the time rate of change of the current length of a material line element per its unit current lengths, i.e. its rate of stretch or stretching. The right-hand side of this equation can equally be written as
\[
\frac{\partial v_a}{\partial x_b} m_a m_b = \frac{1}{2} \frac{\partial v_a}{\partial x_b} m_a m_b + \frac{1}{2} \frac{\partial v_a}{\partial x_b} m_b m_a
\]
\[
= \frac{1}{2} \left[ \frac{\partial v_a}{\partial x_b} + \frac{\partial v_b}{\partial x_a} \right] m_a m_b , \quad a, b = 1, 2 ,
\]
so that the rate of stretch depends only on the symmetric part of the velocity-gradient given by
\[
d_{ab} = \frac{1}{2} \left[ \frac{\partial v_a}{\partial x_b} + \frac{\partial v_b}{\partial x_a} \right] = d_{ba} , \quad a, b = 1, 2 . \quad (7.16)
\]

The three functions \( d_{ab} = d_{ab}(x,t) \) are the components of the so-called deformation rate tensor.

Consider a material line element instantaneously parallel to the \( X_1X_1 \)-axis. The unit tangent to this element has components \( m_1 = 1, m_2 = 0 \). For this element the rate of change of length per its unit current length is given by \( d_{11} \). Similarly, the stretch rate of an element instantaneously parallel to the \( X_2X_2 \)-axis is given by \( d_{22} \). For an element
instantaneously tangent to the unit vector \( \bar{m} \) the stretch rate, \( d_{(m)} \), is then given by

\[
d_{(m)} = d_{ab} \bar{m}_a \bar{m}_b = d_{11} m_1^2 + d_{22} m_2^2 + 2d_{12} m_1 m_2 ,
\]

so that, if this element makes an angle \( \varphi \) with the \( X_1X_1' \)-axis, one has \( m_1 = \cos \varphi \) and \( m_2 = \sin \varphi \), from which it follows that

\[
d_{(m)} = d_{11} \cos^2 \varphi + d_{22} \sin^2 \varphi + 2d_{12} \cos \varphi \sin \varphi
\]

\[
= \frac{1}{2} (d_{11} + d_{22}) + \frac{1}{2} (d_{11} - d_{22}) \cos 2\varphi + d_{12} \sin 2\varphi .
\]

The component \( d_{12} = d_{21} \) of the deformation rate tensor gives one half of the decrease in the instantaneously orthogonal angle of two material line elements emanating from a point in the directions of the coordinate axes.

To see this, consider material line elements \( dl \) and \( dl^* \) with respective components \( dx_a \) and \( dx_a^* \), and unit tangent vectors \( \bar{m} \) and \( \bar{m}^* \), which emanate from the same point and form an angle \( \Theta \) with each other.

Now calculate as follows:

\[
\frac{1}{dl} \frac{D}{Dt} (dl \ dl^* \cos \Theta) = \left[ d_{(m)} + d_{(m)^*} \right] \cos \Theta - \frac{D\Theta}{Dt} \sin \Theta
\]

\[
= \frac{1}{dl} \frac{D}{Dt} (dx_a \ dx_a^*) = 2 \ d_{ab} \bar{m}_a \bar{m}_b^* , \quad a,b = 1,2 .
\]

Hence,

\[
\left[ d_{(m)} + d_{(m)^*} \right] \cos \Theta - \frac{D\Theta}{Dt} \sin \Theta = 2 \ d_{ab} \bar{m}_a \bar{m}_b^* .
\]

(7.19)
For instantaneously orthogonal material line elements, $\theta = \frac{\pi}{2}$ and (7.19) yields

\[ -\frac{1}{2} \frac{D\theta}{Dt} = d_{ab} \mathbf{m}_a \mathbf{m}_b^* \]  

(7.20)

which is called the shear-rate or the rate of shearing of these orthogonal directions.\(^1\) If $\mathbf{m}$ makes an angle $\varphi$ with the $X_1X_1$-axis, then $\mathbf{m}^*$ makes an angle $\varphi^* = \varphi + \frac{\pi}{2}$ with the same axis, and the corresponding shear-rate becomes

\[ -\frac{1}{2} \frac{D\theta}{Dt} = d_{11} \cos \varphi \cos \varphi^* + d_{22} \sin \varphi \sin \varphi^* 
+ d_{12} \left[ \cos \varphi \sin \varphi^* + \sin \varphi \cos \varphi^* \right]. \]

But, since $\cos \varphi^* = \cos \left( \varphi + \frac{\pi}{2} \right) = -\sin \varphi$ and similarly, $\sin \varphi^* = \cos \varphi$, the above equation reduces to

\[ \frac{D\theta}{Dt} = (d_{11} - d_{22}) \sin 2\varphi - 2d_{12} \cos 2\varphi. \]

(7.21)

When $\mathbf{m}$ is in the $X_1X_1$, and $\mathbf{m}^*$ in the $X_2X_2$-direction, respectively, $\varphi = 0$, and it follows from (7.21) that the corresponding shear-rate is $-\frac{1}{2} \frac{D\theta}{Dt} = d_{12}$, as asserted before.

**INSTANTANEOUS ROTATION**

In addition to pure deformation which is characterized by the symmetric part of the velocity-gradient, Eq. (7.16), a material neighborhood has an instantaneous rotation as a rigid-body, that is, a neighborhood of a given particle rotates with an angular velocity like a rigid-body.

---

\(^1\) Some authors refer to $-\frac{D\theta}{Dt}$ as the shear-rate.
To characterize this rotation consider a particle instantaneously at a point with coordinates \( x_1 \) and \( x_2 \), and let \( d_1^* \) be a line element with unit tangent vector \( m^* \), emanating from this point and making an angle \( \varphi^* \) with the \( X_1X_1 \)-axis. With \( dx_1^* \) and \( dx_2^* \) denoting the components of \( d_1^* \), one has
\[ \frac{dx_1^*}{d_1^*} = \cos \varphi^* \quad \text{and} \quad \frac{dx_2^*}{d_1^*} = \sin \varphi^*. \]
It follows then that
\[
\frac{D \cos \varphi^*}{Dt} = \frac{D \varphi^*}{Dt} \sin \varphi^*
\]
\[
= \frac{D}{Dt} \left( \frac{dx_1^*}{d_1^*} \right) = \frac{1}{d_1^*} \frac{d}{dt} \left( \frac{Dx_1^*}{dt} \right) - \frac{D(dl^*)}{dl^*} \frac{dx_1^*}{dl^*}
\]
\[
= \frac{1}{d_1^*} \left[ \frac{\partial v_1}{\partial x_1} dx_1^* + \frac{\partial v_1}{\partial x_2} dx_2^* \right] - \frac{d}{dt} \left( m^* \right) \frac{dx_1^*}{dl^*}
\]
\[
= \frac{\partial v_1}{\partial x_1} \cos \varphi^* + \frac{\partial v_1}{\partial x_2} \sin \varphi^* - \frac{d}{dt} \left( m^* \right) \cos \varphi^*. \quad (7.22)
\]
Set \( \varphi^* = \frac{\pi}{2} \) and from (7.22) obtain
\[ \frac{D\varphi^*}{Dt} = \frac{\partial v_1}{\partial x_2}. \]

In a similar manner, for an element which makes an angle \( \varphi \) with the \( X_1X_1 \)-axis, one shows that
\[
\frac{D \sin \varphi}{Dt} = \frac{D\varphi}{Dt} \cos \varphi
\]
\[
= \frac{\partial v_2}{\partial x_1} \cos \varphi + \frac{\partial v_2}{\partial x_2} \sin \varphi - \frac{d}{dt} \left( m \right) \sin \varphi. \quad (7.23)
\]
Set \( \varphi = 0 \) and deduce that 
\[ \frac{D\varphi}{Dt} = \frac{\partial v_2}{\partial x_1}. \]
Now, for an instantaneous rigid rotation, the material line element parallel to the $X_1X_1$-axis, for which $\varphi = 0$, and the material element parallel to the $X_2X_2$-axis, for which $\varphi^* = \frac{\pi}{2}$, or, for that matter, any other element emanating from the same point, will have a common angular velocity, say $\omega$.

Since the line element instantaneously in the $X_1X_1$-direction rotates toward the $X_2X_2$-direction (counterclockwise) with an angular velocity
\[
\frac{D\varphi}{Dt} = \frac{\partial v_2}{\partial x_1},
\]
and the element instantaneously in the $X_2X_2$-direction rotates in the same direction with an angular velocity
\[
\frac{D\varphi^*}{Dt} = -\frac{\partial v_1}{\partial x_2},
\]
(note the negative sign), it follows that, one half of the rate of decrease in the instantaneously right angle of these elements, that is the corresponding shear-rate, is given by
\[
\frac{1}{2} \left( \frac{D\varphi}{Dt} - \frac{D\varphi^*}{Dt} \right) = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right),
\]
and that the angular velocity of the instantaneously rigid rotation of this material neighborhood is
\[
\omega = \frac{1}{2} \left( \frac{D\varphi}{Dt} + \frac{D\varphi^*}{Dt} \right) = \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right).
\]

Twice of this angular velocity is called the vorticity.

To obtain further insight into the nature of the instantaneous motion and deformation of a material neighborhood, consider at a fixed instant $t$ two neighboring particles, one at a point with coordinates $x_a$ and the other at a point with coordinates $x_a + dx_a$, $a = 1, 2$. If the velocity of the first particle is $v_a(x, t)$, and that of the second is $v_a(x + dx, t)$, and if the velocity field is sufficiently smooth (as it has consistently been
assumed to be the case), it then follows by a Taylor series expansion that
\[ v_a(\mathcal{X} + \mathcal{X}, t) = v_a(x, t) + \frac{\partial v_a(x, t)}{\partial x_b} dx_b + \ldots \]
\[ = v_a(x, t) + \frac{1}{2} \left[ \frac{\partial v_a(x, t)}{\partial v_b} + \frac{\partial v_b(x, t)}{\partial v_a} \right] dx_b \]
\[ + \frac{1}{2} \left[ \frac{\partial v_a(x, t)}{\partial v_b} - \frac{\partial v_b(x, t)}{\partial v_a} \right] dx_b + \ldots \]
\[ = v_a(x, t) + d_{ab}(x, t) dx_b + \frac{1}{2} \left[ \frac{\partial v_a(x, t)}{\partial v_b} - \frac{\partial v_b(x, t)}{\partial v_a} \right] dx_b + \ldots \]
(7.25)

The first term in the right-hand side is the velocity of the instantaneous rigid translation of the material neighborhood.

The second term in the last line of (7.25) corresponds to the instantaneous deformation of this neighborhood, as discussed before.

The last term in the last line of (7.25) represents the instantaneous rigid-body rotation of this material neighborhood. To see this more clearly consider the components
\[ v_1(x + dx_1, t) = v_1(x, t) + d_{11}(x, t) dx_1 + d_{12}(x, t) dx_2 - \omega(x, t) dx_2 + \ldots \]
\[ v_2(x + dx_2, t) = v_2(x, t) + d_{21}(x, t) dx_1 + d_{22}(x, t) dx_2 + \omega(x, t) dx_1 + \ldots \]
(7.26)

where \(\omega\) is the angular velocity of the rigid rotation given by (7.24).

Consider now a local coordinate system at point \(x_a\), \(a = 1, 2\), and assume that \(d_{ab}\), \(a, b = 1, 2\), is identically zero. With respect to this local
coordinate system, the velocity of the particle at $\tilde{x} + dx$ is

$$\begin{align*}
    v_1 &= -\omega dx_2 = -\omega \frac{dx}{dl} \sin \varphi , \\
    v_2 &= \omega dx_1 = \omega \frac{dx}{dl} \cos \varphi ,
\end{align*}$$

(7.27)

where $\sin \varphi = \frac{dx_2}{dl}$ and $\cos \varphi = \frac{dx_1}{dl}$, $dl$ being the distance between $\tilde{x}$ and $\tilde{x} + dx$. Hence the particle at $\tilde{x} + dx$ rotates with an angular velocity $\omega$ about the particle at $\tilde{x}$. The corresponding relative velocity makes an angle $\varphi + \pi/2$ with the $x_1,x_1$-direction, and has the magnitude $\omega \frac{dx}{dl}$.

From the above discussion it follows that the instantaneous motion of a typical material neighborhood in general consists of a rigid-body translation, a pure deformation, and a rigid-body rotation.

When the components of the deformation rate tensor, $d_{ab}$, vanish at a material point, the neighborhood of this point undergoes only an instantaneous rigid-body motion, i.e. a rigid-body translation and rotation. The instantaneous rotation is completely defined by the angular velocity $\omega$ given in (7.24). If, on the other hand, this angular velocity vanishes at a material point, the neighborhood of this point performs an instantaneous rigid-body translation and a pure deformation. The instantaneous pure deformation, in general, involves instantaneous change in the direction of the material line elements; there are however, at least two instantaneously orthogonal directions which are left unaltered by an instantaneous pure deformation, as will be discussed in the sequel.
PRINCIPAL VALUES OF STRETCH-RATE

At a given instant \( t \) and at a given point \( x \), the rate of stretch of a material line element which instantaneously makes an angle \( \varphi \) with the \( X_1 X_2 \)-axis, is given by Eq. (7.18). Since this stretch depends on the orientation of the considered line element at a fixed point and fixed time, one may seek to establish the orientation of line elements whose rates of extension have extreme values, i.e. either a maximum, or a minimum. To do this one takes the derivative of the quantity in the right-hand side of (7.18) with respect to \( \varphi \), and sets the resulting equation equal to zero, arriving at

\[
\tan 2\varphi = \frac{2d_{12}}{d_{11} - d_{22}}
\]  

(7.28)

which gives the sought direction.

Since \( \tan 2\varphi = \tan 2(\varphi + \pi/2) \), it follows that there are two orthogonal directions for which the rate of extension is the extremum. Substitution from (7.28) into (7.21) now reveals that \( \frac{D}{Dt} \) which is the rate of change of the instantaneously orthogonal directions of the considered material element, is zero.

The directions defined by (7.28) are called the principal directions of the stretch-rate. The corresponding stretch-rates are called the principal stretch-rates. Substitution from (7.28) into (7.18) yields the following values for the principal stretch-rates:

\[
d_{I, II} = \frac{1}{2} \left( d_{11} + d_{22} \right) \pm \left[ d_{12}^2 + \frac{1}{4} \left( d_{11} - d_{22} \right)^2 \right]^{1/2},
\]

(7.29)

where the principal stretch-rates are denoted by \( d_I \) and \( d_{II} \).
GRAPHICAL REPRESENTATION OF PLANAR MOTION

At a given instant $t$ the motion of the material points relative to a point in a given material neighborhood can be given a graphical interpretation. Consider at an instant a typical material neighborhood, choose an origin in that neighborhood, and erect there a new system of rectangular Cartesian coordinates $X'_1X'_1$ and $X'_2X'_2$ in the direction of the principal stretch-rates in such a manner that the principal stretch-rate $d_1$ of the $X'_1X'_1$-direction, and $d_{II}$ of the $X'_2X'_2$-direction satisfy the condition $d_1 \geq d_{II}$. Then, for a line element with the unit tangent vector $\vec{m}$ which makes an instantaneous angle $\varphi'$ with the $X'_1X'_1$-axis, the stretch-rate becomes

$$d_{(m)} = \frac{1}{2} (d_1 + d_{II}) + \frac{1}{2} (d_1 - d_{II}) \cos 2\varphi' ;$$

(7.30)

note that, since the coordinate axes are along the principal directions, the shear-rate for these directions is zero.

Consider now another element with the unit tangent vector $\vec{m}^*$ which makes the angle $\varphi' + \frac{\pi}{2}$ with the $X'_1X'_1$-axis. The shear-rate then is

$$d_{(mm^*)} = -\frac{1}{2} \frac{D \theta}{Dt} = -\frac{1}{2} (d_1 - d_{II}) \sin 2\varphi' .$$

(7.31)

Since the angular velocity of the rigid-body rotation of the neighborhood is $\omega$, Eq. (7.24), the line element with unit vector $\vec{m}$ has the instantaneous angular velocity $\Omega$ given by

$$\Omega = \omega - \frac{1}{2} (d_1 - d_{II}) \sin 2\varphi' ;$$

(7.32)

note that this equation can be directly obtained from (7.25).
Equations (7.30) and (7.32) completely define the instantaneous motion of the material points in a neighborhood of a point relative to this point. They have the following graphical interpretation.

In a plane, called the plane of relative velocities, represent angular velocity and stretch-rate of an element instantaneously along the unit vector \( \mathbf{m} \) by the abscissa \( \Omega \) and the ordinate \( d \) of a point \( M \), respectively. As the angle \( \varphi' \) of this direction varies, the representative point \( M \) describes a circle, called the circle of relative velocities, whose center \( O \) has the coordinates \( \Omega = w \) and \( d = \frac{1}{2} (d_1 + d_{II}) \), and whose radius is \( \frac{1}{2} (d_1 - d_{II}) \), see Fig. 7.2. On this circle the highest point \( A_1 \) and the lowest point \( A_{II} \) have the maximum \( d_1 \) and the minimum \( d_{II} \) stretch-rates, and they correspond, respectively, to the principal \( X'_1 \) - and \( X'_2 \) - directions. The abscissa of the point \( O \) represents the angular velocity \( w \) of the rigid-body rotation. If the \( d \)-axis is translated parallel to itself so that it passes through the point \( O \), a construction known as Mohr's circle is obtained. This gives only the stretch-rates of line elements and shear-rates for pairs of orthogonal line elements, but no angular velocities.

To locate on the circle the point \( M \) that corresponds to the direction \( \mathbf{m} \), measure in the counterclockwise direction the angle \( \angle A_1OM \) equal to \( 2\varphi' \). A line drawn through \( M \) in the direction of \( \mathbf{m} \) intersects the circle at a point \( P^* \) which is the pole of this circle. Since the angle \( \angle A_1P^*M \) is equal to \( \varphi' \), the line \( A_1P^* \) is in the principal \( X'_1 \) - direction. The stretch-rate and the angular velocity of a given line element is now defined, respectively, by the ordinate and the abscissa of the second

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1 Mohr, Otto (1835-1918), German structural engineer and educator who designed some of the first steel trusses in Germany.
intersection of the circle with a line through \( P^* \) that has the direction of the considered line element. To obtain the angular distortion-rate of the two line elements emanating from the considered particle, first draw lines through \( P^* \) in the respective directions of these elements to intersect the circle. The difference between the abscissae of these intersection points now is the desired value of the angular distortion-rate. Since this quantity is a maximum for points \( B_1 \) and \( B_2 \) in Fig. 7.2 one concludes that the orthogonal directions represented by these points correspond to the directions of the maximum shear-rate.

\[
\frac{1}{2}(d_1 + d_{II})
\]

\( \mu \) \( \theta' \)

\( \phi' \)

\( \Omega \)

\( d \)

\( d_1 \)

\( d_{II} \)

\( A_I \)

\( M \)

\( B_I \)

\( B_{II} \)

\( \omega \)

\( X_1' \)

\( X_2' \)

Figure 7.2
BASIC INVARIANTS

From Eqs. (7.29) it follows that

\[ d_I + d_{II} = d_{11} + d_{22}. \]

In fact, if a line element with unit tangent vector \( \tilde{m}^* \) which makes an angle \( \varphi^* = \frac{\pi}{2} + \varphi' \) with the \( X_1'X_1' \)-axis, (i.e. \( \tilde{m}^* \) is normal to \( \tilde{m} \)), is considered, from Eq. (7.30) one obtains

\[
d_{(m^*)} = \frac{1}{2} (d_I + d_{II}) + \frac{1}{2} (d_I - d_{II}) \cos 2\left( \frac{\pi}{2} + \varphi' \right)
\]

\[ = \frac{1}{2} (d_I + d_{II}) - \frac{1}{2} (d_I - d_{II}) \cos 2\varphi', \]

so that, independently of the value of \( \varphi' \) the stretch-rates \( d_{(m)} \) and \( d_{(m^*)} \) of the orthogonal directions \( \tilde{m} \) and \( \tilde{m}^* \) always satisfy the following relation:

\[ I_d = d_{(m)} + d_{(m^*)} = d_{11} + d_{22} = d_I + d_{II}. \quad (7.33) \]

The quantity \( I_d \) is therefore independent of the particular instantaneously orthogonal material line elements chosen at a point, and hence represents a basic measure of the instantaneous deformation of the material neighborhood of the considered point. It is a basic invariant of the deformation-rate tensor characterized by \( d_{ab}, a,b = 1,2 \).

Consider now the product of the principal stretch-rates, Eqs. (7.29), and, in a similar way as above, conclude that

\[ II_d = d_I d_{II} = d_{11} d_{22} - d_{12}^2 = d_{(m)} d_{(m^*)} - d_{(mm^*)}^2 \quad (7.34) \]

is a basic invariant of the deformation-rate tensor.

For planar motion there are, in general, two basic invariants defined by Eqs. (7.33) and (7.34). These invariants can be given the following interpretation.
Since the orthogonal principal directions of stretch-rate at each point have no angular distortion (shear-rate is zero), it follows that the pure deformation-rate characterized by $d_{ab}$ leaves these directions unaltered. Hence, if $\sim m$ is a principal direction at a point, it follows from the second term in the right-hand side of (7.25) that

$$d_{ab} m_b = d m_a , \quad a,b = 1,2 ,$$  \hspace{1cm} (7.35)

where $d$ is the proportionality factor. This equation states that, if $m_b$ is the component of a unit vector defining a principal direction at a point, the instantaneous pure deformation does not rotate this element, but simply changes its length. Equations (7.35) can be written as

$$\begin{bmatrix} d_{ab} - \delta_{ab} d \end{bmatrix} m_b = 0 , \quad a,b = 1,2 ,$$  \hspace{1cm} (7.36)

which are two homogeneous equations for two unknowns $m_1$ and $m_2$ which define the corresponding principal direction. These equations admit nontrivial solutions if and only if the determinant of the coefficients of $m_1$ and $m_2$ vanish, i.e. if and only if

$$\det \begin{bmatrix} d_{ab} - \delta_{ab} d \end{bmatrix} = \begin{vmatrix} d_{11} - d & d_{12} \\ d_{21} & d_{22} - d \end{vmatrix} = (d_{11} - d)(d_{22} - d) - d_{12}^2 = 0$$

which can be rewritten as

$$d^2 - I_d d + II_d = 0 ,$$  \hspace{1cm} (7.37)

where $I_d$ and $II_d$ are the basic invariants. Equation (7.37), in general, has two real roots which are the corresponding principal stretch-rates given by Eq. (7.29).
NOTE ON LAGRANGIAN STRAIN

Referring to Eqs. (7.10) and (7.11), observe that all the results presented above for the deformation-rate tensor $d_{ab}$ can be extended to the Lagrangian strain tensor defined by $E_{AB}$, $A,B = 1,2$. Hence, at each material point in the initial state there are two orthogonal material directions which have extreme values of strain, $E_I$ and $E_{II}$, which are the principal values of strain, the corresponding directions being the principal directions of strain. The shear strain of the orthogonal principal directions is zero. The principal strains are given by the roots of the equation

$$E^2 - I_E E + II_E = 0,$$  
(7.38)

where

$$I_E = E_I + E_{II} = E_{11} + E_{22},$$

$$II_E = E_I E_{II} = E_{11} E_{22} - E_{12}^2,$$  
(7.39)

are the corresponding principal values of the Lagrangian strain in planar motion. Moreover, in terms of the principal values of strain, the strain of any direction is

$$E(M) = \frac{1}{2} \left( E_I + E_{II} \right) + \frac{1}{2} \left( E_I - E_{II} \right) \cos 2\gamma'$$  
(7.40)

where $\gamma'$ is the angle between $M$ and the principal direction $X'_1 X'_1$, which corresponds to the principal value $E_I$. Moreover, the shear strain of the initially orthogonal directions $M$ and $M^*$ becomes

$$E(MM^*) = -\frac{1}{2} \left( E_I - E_{II} \right) \sin 2\gamma'$$  
(7.41)
From (7.40) and (7.41) it now follows that the strain-state at a point can be given graphical representation in the same manner as discussed for the strain-rate, except that the center of the Mohr circle for Lagrangian strain lies on the normal strain axis; see Prob. 7.1.

\textbf{EXAMPLE 7.1}

At a given particle find the initial directions of material line elements which have extreme stretches.

To this end consider Eq. (7.5), take the derivative of the right-hand side with respect to $\psi$ (maximize it), and setting the result equal to zero, obtain

$$\tan 2\psi = \frac{2C_{12}}{C_{11} - C_{22}}$$

(7.42)

which defines two orthogonal directions. Substitution from (7.42) into (7.5) gives the corresponding extreme squared stretches which are

$$C_{I,II} = \frac{1}{2} \left( C_{11} + C_{22} \right) \pm \left[ C_{12}^2 + \frac{1}{4} \left( C_{11} - C_{22} \right)^2 \right]^\frac{1}{2}.$$  

(7.43)

These are called the principal stretches (squared). The corresponding directions are called the principal directions of stretch. Substitution from (7.42) into (7.7) immediately reveals that the initially orthogonal angle of the principal directions of stretch remains orthogonal after deformation.

The principal squared stretches $C_I$ and $C_{II}$ are the roots of the following equation:

$$C^2 - I_C C + II_C = 0 ,$$

(7.44)

where

$$I_C = C_I + C_{II} = C_{11} + C_{22}, \quad II_C = C_I C_{II} = C_{11} C_{22} - C_{12}^2,$$

(7.45)
are the basic invariants of Green's deformation tensor $C_{AB}$, $A, B = 1, 2$.

**EXAMPLE 7.2**

Calculate the material time derivative of the Jacobian $J$.

To this end proceed as follows:

$$\frac{D J}{D t} = \frac{\partial J}{\partial x_{a A}} \frac{D}{D t} \left( \frac{\partial x_a}{\partial x_A} \right) = \frac{\partial J}{\partial x_{a A}} \frac{\partial}{\partial x_b} \left( \frac{D x_a}{D t} \right) \frac{\partial x_b}{\partial x_A}$$

$$= J \frac{\partial x_a}{\partial x_b} \frac{\partial x_b}{\partial x_A} = J \frac{\partial x_a}{\partial x_a} = (d_{11} + d_{22}) J = I \frac{d}{d t} J, \quad (7.46)$$

where, in addition to the chain rule of differentiation and Eqs. (6.12) and (7.33), the result of Example 6.2 is used.

**EXAMPLE 7.3**

Consider the velocity field in Example 6.1, Eq. (6.32). Calculate the components of the deformation-rate tensor, and construct the circle of relative velocities at the particle which, at $t = 1$, is at point $x_1 = x_2 = 1$.

The components of the velocity-gradient are

$$\frac{\partial v_1}{\partial x_1} = t x_2, \quad \frac{\partial v_1}{\partial x_2} = t x_1, \quad \frac{\partial v_2}{\partial x_1} = 0, \quad \frac{\partial v_2}{\partial x_2} = -2 x_2,$$

from which one obtains, Eq. (6.16),

$$d_{11} = t x_2, \quad d_{12} = d_{21} = \frac{1}{2} t x_1, \quad d_{22} = -2 x_2.$$

The rigid rotation, moreover, becomes, Eq. (7.24), $\omega = -\frac{1}{2} t x_1$.

For $t = 1$ and $x_1 = x_2 = 1$, one obtains $d_{11} = 1, d_{12} = d_{21} = \frac{1}{2}, d_{22} = -2$, and $\omega = -\frac{1}{2}$.
To construct the circle of relative velocities, observe that in the
$\Omega, d$-plane the points $M$ with coordinates $\Omega = \omega - d_{12} = -\frac{1}{2} - \frac{1}{2} = -1$ and
d = $d_{11} = 1$, and $M^*$ with $\Omega = \omega + d_{12} = -\frac{1}{2} + \frac{1}{2} = 0$ and d = $d_{22} = -2$,
are on the circle, $MM^*$ being a diameter. Since the center of the circle
is on the line $\Omega = \omega = -\frac{1}{2}$ parallel to the d-axis, the intersection of
this line and $MM^*$ defines the center of the circle, see Fig. 7.3.

Figure 7.2
PROBLEMS FOR CHAPTER 2

1.1 A projectile is shot vertically upward from ground level with an initial velocity of 120 ft/sec.

(a) Find the time it takes for the projectile to return if the air resistance is neglected, and \( g = 32.2 \text{ ft/sec.}^2 \)

(b) Find the return time if the air resistance is proportional to the velocity, with the proportionality factor given by \( k = 0.001 \text{ m} \), where \( m \) is the mass of the projectile.

1.2 A point-mass of mass \( m \) is attached to a linear spring with a spring coefficient \( k \), and is suspended under gravity. Suppose it is given an initial vertical velocity of \( V^0 \).

(a) Write the equation of motion when \( k = \) constant, and find the period of vibration.

(b) Write the equation of motion when \( k = k_0 + k_1 x \), where \( x \) measures in the vertical direction, the displacement of the point-mass from its equilibrium position. Find an expression which gives the total energy as a function of \( x \) and \( \frac{dx}{dt} \).

1.3 A steel ball hits a heavy wooden wall at the speed of 60 miles/hr, and is captured by the wall. The ball is \( \frac{1}{4} \) in in diameter, and has a mass density of 7.8 g/cm³.

(a) How much energy is transferred to the wall? Express your answer in cal, BTU, and J.

(b) What is the momentum loss?
1.4 A jet of water moving at 30 ft/sec strikes a stationary flat wall normal to the direction of the stream. Assume that the jet is very fine, and carries water at the rate of 0.1 ft\(^3\)/sec. What is the force on the wall?

1.5 A rocket with its fuel has a total mass of \(m\) at the instant \(t\) during its motion. The mass of the rocket without its fuel is \(m_r\) and it total fuel at the initial time \(t = 0\) has a mass of \(m_f\). The rocket is fired at \(t = 0\) vertically with an initial velocity \(v^0\), and consumes its fuel during its burning period at a constant rate \(c\). Let \(v\) be the absolute value of the velocity of the exhaust gas relative to the rocket.

(a) Show that the motion of the rocket is governed by

\[
m \frac{dV}{dt} + v \frac{dm}{dt} + mg = 0 ,
\]

(*)

where \(V\) is the velocity of the rocket at time \(t\). Assume that \(v\) is constant, and solve (*) to obtain

\[
V = v^0 + v \ln \left\{ \frac{m + m_f}{m} \right\} + \frac{g}{c} (m - m_r - m_f) .
\]

(b) Let \(v = 7000\) ft/sec and assume that the loss of mass is about 1/60th of the initial mass. Find the ratio of the fuel mass to the mass of the empty rocket so that starting from zero initial velocity, the rocket could achieve a velocity of 7 mile/sec.

2.1 A projectile is shot from the ground level with the initial velocity \(v^0\) which makes the angle \(\alpha\) with the \(X_1X_1\)-axis.
(a) Write down the equations of motion.

(b) Find the maximum elevation that can be attained by the projectile for an optimum value of the angle $\alpha$.

(c) Find the maximum distance along the $X_1X_1$-axis, which can be reached for an optimum value of $\alpha$.

(d) Discuss (a), (b), and (c) in terms of the motion of the projections $M_1$ and $M_2$ of the projectile on the $X_1X_1$- and $X_2X_2$-axis, using the corresponding energy conservation equations.

2.2 A small block of mass $m$ is attached to the springs of negligible masses, as shown in Fig. P-2.2. Regard the block as a point-mass.

(a) If the block is given an initial velocity $\mathbf{v}^0$ which makes an angle $\alpha$ with the $X_1X_1$-axis, write down the equations of motion.

(b) In the absence of air resistance, discuss the motion using the energy conservation law.

(c) If the air resistance is proportional to the velocity, find the time which takes until the block attains one-half of its initial speed.

2.3 Consider the projectile in Prob. 2.1 above. Assume that the air resistance is proportional to the velocity with the proportionality factor $mk$, where $m$ is the projectile's mass.

(a) Write down the equations of motion.

(b) Let the angle that the velocity vector $\mathbf{v}$ makes with the $X_1X_1$-axis be $\varphi$. Show that the acceleration is given by

$$\mathbf{a} = \frac{dv}{dt} \mathbf{v} + \mathbf{v} \frac{d\varphi}{dt} \mathbf{v},$$
where \( \mathbf{T} \) is the unit tangent, \( \mathbf{N} \) the unit normal to the path, and \( V \) is the speed.

(c) Show that the equations of motion in terms of \( V \) and \( \varphi \), are

\[
\frac{dV}{dt} = -kV - g \sin \varphi, \\
\frac{d\varphi}{dt} = -\frac{1}{V} g \cos \varphi, \\
(*)
\]

where \( g \) is the gravitational acceleration.

(d) Integrate Eqs. (*) to obtain

\[
V = \cos \alpha \left[ \frac{\cos \varphi}{V^0} - \frac{k}{g} \sin (\varphi - \alpha) \right]^{-1},
\]

\[
\tan \varphi = \tan \alpha - \frac{g}{k} \left( e^{kt} - 1 \right) / \left( V^0 \cos \alpha \right).
\]

(e) Show that the time when the projectile attains its maximum height is

\[ t_{\text{max}} = \frac{1}{k} \ln \left( \frac{kV^0}{g} \sin \alpha + 1 \right). \]

2.4 A jet of water moves at 30 ft/sec in the \( X_1X_1 \)-direction, carrying water at the rate of 0.1 ft\(^3\)/sec.

(a) Find the force exerted on a stationary vane which deflects the jet in a direction which makes a 45\(^o\) angle with the \( X_1X_1 \)-axis.

(b) Find the force exerted on a vane which moves with the speed of 10 ft/sec in the \( X_1X_1 \)-direction, and deflects the jet in a direction which makes a 45\(^o\) angle with the \( X_1X_1 \)-axis.

4.1 Let \( x_a, a = 1,2,3, \) stand for the independent coordinate variables \( x_1, x_2, x_3. \)

(a) Show that \( \frac{\partial x_a}{\partial x_b} = \delta_{ab} \).
(b) Show that the square of the distance from the origin to a point with coordinates \( x_a \) is \( x \cdot x = x_a x_b \delta_{ab} = x_c x_c \), \( a,b,c = 1,2,3 \).

4.2 In accordance with the index notation and the summation convention, discuss the validity of each of the following expressions:

(a) \( K_{ij} y_b \delta_{ik} \Delta_k = \delta_{ij} \), \( i,j,k = 1,2,3 \), \( \delta = 1,2 \).

(b) The same expression as in (a), but with \( \delta = 1,2,3 \), \( i,j,k = 1,2 \).

(c) \( K_{ij} y_b \delta_{ijk} \Delta_k = \delta_{AB} \), \( i,j,k = 1,2,3 \), \( A,B,\delta = 1,2 \).

(d) \( H_{ABC} \Delta_{BC} = \delta_A \), \( A,B,C = 1,2,3 \).

4.3 Solve Eqs. (4.6) for \( x_\alpha \), \( \alpha = 1,2 \), and show that a unique solution exists if and only if \( A_{11} A_{22} - A_{12} A_{21} \neq 0 \).

5.1 The rectilinear motion of a portion of a continuum confined at \( t = 0 \) in the interval \( 1 < X_1 < 2 \), is described by\(^1\)

\[
x_1 = \frac{2t^2 X_1 + 1}{t^2 X_1 + 1} X_1.
\]

(a) Describe this motion in terms of the spatial variables \( x_1 \) and \( t \).

(b) Calculate the stretch, and discuss whether the mapping is one-to-one.

(c) Find the limiting particle positions as \( t \) becomes very large.

(d) Calculate the Lagrangian strain, and find its limit as \( t \) becomes very large.

5.2 Consider the motion defined in Prob. 5.1.

(a) Find the velocity, \( V_1(X_1,t) \), of the particles.

---

\(^1\) Assume that by means of suitable choice of units of length and time, the physical quantities \( x_1, X_1, t \) are made dimensionless; i.e. they are real numbers.
(b) Express \( V_1 \) in terms of the spatial variables.

(c) Consider the spatial description of this motion, i.e. \( X_1 = \xi_1(x_1, t) \), and by direct material time differentiation, namely from

\[
0 = \frac{\partial \xi_1}{\partial t} + \frac{\partial \xi_1}{\partial x_1} \frac{Dx_1}{Dt},
\]

obtain \( v_1 = \frac{Dx_1}{Dt} \). Compare this result with that obtained in (b) above.

(d) Find the acceleration \( A_1 = A_1(X_1, t) \).

5.3 Let a quantity of motion in Prob. 5.1, say, the particle temperature, be given by

\[
\theta = x_1^2 t + 2.
\]

Find its material time rate of change, i.e. its variation in time as is experienced by the material particles.

5.4 Suppose that the rectilinear motion of a continuum is defined in terms of the spatial variables by

\[
X_1 = \frac{1}{2} \left( e^{-\frac{x_1 t}{k^0}} + 1 \right) x_1, \quad t \geq 0.
\]

(a) Find the particle which at \( t = 2 \) sec is at point \( x_1 = 10 \) ft, where \( k^0 = 10 \) ft sec.

(b) Find the velocity field.

(c) Calculate the velocity-gradient, and obtain the rate of stretch at \( x_1 = 10 \) ft, \( t = 2 \) sec.

5.5 Let \( G(X_1, t) \) be a sufficiently smooth quantity of motion, which corresponds to \( g(x_1, t) \) in spatial description, i.e. \( G \Rightarrow g \) and \( g \Rightarrow G \).

(a) Show that

\[
\frac{\partial}{\partial x_1} \left[ \frac{DG}{Dt} \right] = \frac{D}{Dt} \left[ \frac{\partial G}{\partial x_1} \right], \quad \frac{\partial}{\partial t} \left[ \frac{DG}{Dt} \right] = \frac{D}{Dt} \left[ \frac{\partial G}{\partial t} \right].
\]
(b) Show that

\[ \frac{\partial}{\partial t} \left[ \frac{Dg}{Dt} \right] - \frac{D}{Dt} \left[ \frac{\partial g}{\partial t} \right] = \frac{\partial v_1}{\partial t} \frac{\partial g}{\partial x_1}. \]

5.6 Show that

\[ \frac{\partial v_1}{\partial x_1} = \lambda I(x_1, t) d_t(x_1, t). \]

5.7 The temperature-gradient in a rectilinear motion of a continuum is given by

\[ \frac{\partial \theta}{\partial x_1} = g(x_1, t), \]

where \( g \) is a given function of \( x_1 \) and \( t \). Find its material time rate of change, and show that

\[ \frac{\partial}{\partial x_1} \left( \frac{\partial \theta}{\partial t} \right) - \frac{D}{Dt} \left( \frac{\partial \theta}{\partial x_1} \right) = g(x_1, t) d_t(x_1, t). \]

Verify this result with the aid of the example in which \( \theta = x_1^2 t^2 + 2 \).

5.8 Find the material time rate of change of the Lagrangian strain \( \varepsilon_{11} \), and show that in terms of the spatial variables one has

\[ \frac{D \varepsilon_{11}}{Dt} = \lambda I d_t. \]

[Hint: Take the material time derivative of both sides of Eq. (5.21) and use the chain rule of differentiation.]

6.1 Consider a planar motion with the velocity field

\[ v_1 = x_1 (1 + t), \quad v_2 = \frac{1}{2} \frac{x_1}{x_2} e^{-\frac{x_1}{2} t^2}. \]
(a) Find the corresponding mapping.

(b) Calculate the acceleration field.

(c) Find the streamline which at $t = 1$ passes through point $x_1 = 1$, $x_2 = 1$.

6.2 Consider a planar motion defined by

$$v_1 = x_1 / (1 + t), \quad v_2 = -Kx_2^2.$$  

(a) Find the corresponding mapping.

(b) Calculate the acceleration field.

(c) Find the expression for the streamlines.

6.3 Show that $e_{AB}^e e_{BC}^e = \delta_{AC}^e$, $A, B, C = 1, 2$.

6.4 Set $x_{aA} = \frac{\partial x_a}{\partial X_A}$ and $X_{AA} = \frac{\partial X_A}{\partial x_a}$, $a, A = 1, 2$. Show that

(a) $\frac{\partial}{\partial x_b} \left( \frac{x_{Aa}}{J} \right) x_{bA} = 0$. [Hint: Use the result in Example 6.2.]

(b) $\frac{\partial}{\partial X_B} \left( \frac{x_{Aa}}{J} \right) x_{Ba} = 0$. [Hint: First show that $x_{aA} = -\frac{\partial \ln J}{\partial X_{Ac}}$.

6.5 With the same notation as in Prob. 6.4 above, show that

(a) $\frac{\partial (x_{AB})}{\partial (X_{Ab})} = -x_{aA} x_{bB}$. [Hint: Differentiate both sides of (6.12) with respect to $X_{d}$]

(b) $x_{aA} = -x_{bB} \frac{\partial (x_{BB})}{\partial (X_{Aa})}$.
6.6 Verify the validity of the following expression for acceleration:

\[
a_i = \frac{\partial v_i}{\partial t} - v_i \frac{\partial v_i}{\partial x_j} (v_i v_j), \quad i, j = 1, 2.
\]

6.7 Consider a motion whose velocity field is expressed by

\[
v = \nabla H + F \nabla G,
\]

where \( H, F, \) and \( G \) are scalar-valued functions of \( x_1, x_2, \) and \( t. \)

Show that

\[
a_i = \frac{\partial}{\partial x_i} \left( \frac{1}{2} v \cdot v + \frac{\partial H}{\partial t} + F \frac{\partial G}{\partial t} \right) + \frac{\partial G}{\partial x_i} \frac{DF}{Dt} - \frac{\partial F}{\partial x_i} \frac{DG}{Dt}.
\]

7.1 A planar deformation is called **homogeneous** if every initially straight material line element is mapped into another straight line. Consider the following homogeneous deformation:

\[
x_1 = 3X_1 + 2X_2 \quad x_2 = X_1 + 2X_2, \quad 0 \leq X_1 \leq 1, \quad 0 \leq X_2 \leq 1.
\]

(a) Calculate the components of the Lagrangian strain.

(b) Find the initial and final directions which correspond to the extreme values of strain.

(c) Verify that the initially orthogonal directions of extreme values of strain, undergo zero shear strain.

(d) Construct the corresponding Mohr's circle.
7.2 If \( I_C \) and \( II_C \) are the basic invariants of \( C_{AB} \), and \( I_E \) and \( II_E \) are those of \( E_{AB} \), show that

\[
I_C = 2 I_E + 2 , \quad II_C = 4 II_E + 2 I_E + 1 .
\]

7.3 Show that \( \frac{\partial x}{\partial x_a} C_{AB} \), where \( x_a \equiv \frac{\partial x}{\partial X_a} \) and \( x_Ba \equiv \frac{\partial X_B}{\partial x_a} \).

7.4 Consider a planar deformation defined by

\[
x_1 = x_1 + \frac{a}{2} x_2^2 ,
\]

\[
x_2 = x_2 - \frac{a}{2} x_1^2 , \quad 0 \leq x_1 \leq 2 , \quad 0 \leq x_2 \leq 2 .
\]

(a) Show that the mapping is one-to-one.

(b) Sketch the deformed configuration.

(c) Calculate the components of the Lagrangian strain tensor,

\( E_{AB}, A,B = 1,2 \).

(d) Find the principal strains and their directions, as functions of \( X_1 \) and \( X_2 \).

(e) At \( X_1 = X_2 = 1 \), construct the Mohr circle.

7.5 Consider the velocity field in Prob. 6.1.

(a) Calculate the components of the stretch-rate tensor, i.e.

\( d_{ab}, a,b = 1,2 \).

(b) Calculate the angular velocity \( \omega \).

(c) Construct the circle of relative velocity at the material point which at \( t = 1 \) is located at \( x_1 = 1, x_2 = 2 \).

(d) Find the principal stretch-rates.

(e) Calculate the basic invariants of the stretch-rate tensor \( d_{ab} \).
CHAPTER 3

FORCE AND STRESS

3.1 INTRODUCTION

The kinematical ingredient of the continuum theory was discussed in the preceding chapter. In the present chapter the dynamical ingredient of the theory will be discussed. This concerns the relation between forces that represent the action of other continua upon the considered continuum, and the forces of reaction that are induced within the continuum.

There are essentially two ways that a body is affected mechanically by other bodies. One is by means of forces that act at a distance, such as gravitational forces, and the other by means of contact with other bodies, which induces contact forces over the contact area. The first group of forces constitutes what is commonly called body forces, and the second group, the surface tractions.

3.2 BODY FORCES

In a gravitational field such as that of earth, particles of a body are subjected to forces which are proportional to their mass. Since the mass of a continuum is assumed to be continuously distributed throughout the space occupied by the body, the corresponding gravitational forces are likewise continuously distributed. This type of force is defined per unit mass of the body, and is called body force. The symbol \( f \) will be used to denote body forces.
If the body forces acting on a continuum are constant, they are called dead body forces. The gravitational forces acting on a body close to the surface of the earth constitute dead body forces. For example, if the $X_1X_1$-axis is directed toward the center of the earth, the body forces acting on a continuum will be directed in the $X_1X_1$-direction and will have the constant magnitude $g$ which is approximately equal to 980 dyne per gram. At high altitudes, however, this gravitational force changes proportionately to the inverse of the squared distance from the center of the earth\(^1\). In this case the body force depends on the position of the particle, i.e. $\vec{f} = f_1(x_1) \vec{e}_1$.

One may conceive of situations in which body forces not only change with particle position, but also depend on time at a given position. In such more general cases, the body force will be represented by $\vec{f} = f(x, t)$.

In a rectilinear motion of a continuum, the corresponding body forces which affect this motion must by necessity be directed along the direction of motion. In this case they have only one component, say in the $X_1X_1$-direction, and they are denoted by $\vec{f} = f_1(x_1, t) \vec{e}_1$.

For the planar motion of a continuum the body forces which affect this motion must also be planar, i.e. being parallel to the plane of motion, say the plane $\Pi$. In this case the body force at each point has two components, say one in the $X_1X_1$- and the other in the $X_2X_2$-direction. Hence one writes $\vec{f} = f_1(x, t) \vec{e}_1 + f_2(x, t) \vec{e}_2$.

---
\(^1\) See Example 1.1, Ch. 2, p. 2-6.
In the general case the body force at each point may have three independent components, and one writes $\mathbf{f} = f_a(x, t) \mathbf{e}_a$, $a = 1, 2, 3$, where the summation convention is implied.

**POTENTIAL**

In the case of the Newtonian point-mass, one usually defines a **force-field** by assigning to every point in a region in space a force which acts on a point-mass of unit mass, if it is placed there. For example, if the point-mass moves along the $X_1X_1$-axis, one defines at every point on this axis the force that would be acting on the point-mass as soon as it is located there. Suppose this force is denoted by $\mathbf{F} = F_1 \mathbf{e}_1$. If there exists a function $\varphi = \varphi(x_1)$ such that

$$F_1 = -\frac{d\varphi}{dx_1}, \quad (2.1)$$

the force-field is then said to be **derivable from the potential** $\varphi$. The work done by these forces on the particle as it moves from point $x_1^{(1)}$ to point $x_1^{(2)}$ along the $X_1X_1$-axis, is given by

$$\int_{x_1^{(1)}}^{x_1^{(2)}} F_1 \, dx_1 = \int_{x_1^{(1)}}^{x_1^{(2)}} \frac{d\varphi}{dx_1} \, dx_1 \quad = \varphi(x_1^{(1)}) - \varphi(x_1^{(2)}), \quad (2.2)$$

so that the difference between the value of the potential at the initial and final points is equal to the corresponding work.
A similar discussion applies to body forces acting on continua. In the rectilinear motion, for example, the body force $f_1$ measured per unit mass of the continuum, admits a potential when it is defined in accordance with

$$f_1 = -\frac{\partial \varphi}{\partial x_1}.$$  \hspace{1cm} (2.3)

For example, a constant body force is derivable from a potential which is a linear function of position, i.e.

$$\varphi = A_0 x_1 + B_0,$$

where $A_0$ and $B_0$ are constants.

Consider now planar body forces. If there exists a function $\varphi = \varphi(x_1, x_2, t)$ which is sufficiently smooth and is such that

$$\underline{f} = -\nabla_\mathbf{x} \varphi,$$  \hspace{1cm} (2.4)

where the del operator $\nabla_\mathbf{x}$ is defined in Sec. 2.6, the body forces $\underline{f}$ are said to be derivable from the potential $\varphi$, and one has

$$f_1 = -\frac{\partial \varphi}{\partial x_1}, \quad f_2 = \frac{\partial \varphi}{\partial x_2}.$$  \hspace{1cm} (2.5)

A similar definition applies to the general three-dimensional case.
3.3 **INTERNAL FORCES AND CAUCHY'S STRESS HYPOTHESIS**

A solid body offers resistance when it is stretched. For example, one needs to exert some force, and hence do some work in stretching a rubber band. Similarly, in order to stretch a steel rod and elongate it by a small fraction of its original length, heavy equipment (testing machine) is needed, since steel rather strongly resists changes in length. Solids consist of atoms which vibrate about their equilibrium states when viewed on a microscopic scale. Two adjacent atoms attract each other when the average distance, $d$, between them is increased, and repel each other, when this distance is decreased. Figure 3.1 shows schematically these interatomic forces $F$; the distance $OA = d_0$ is of the order of the atomic size. When no attempt is made to deform a solid, the average distance between two atoms corresponds to the abscissa of point A. At the vicinity of point A the curve is almost a straight line, and therefore it is natural to expect that, for sufficiently small deformations, the force of resistance in most solids would be linearly related to the amount of deformation, since a macroscopic deformation is the accumulation of numerous microscopic (interatomic) changes of length.

From the above remarks it is clear that internal forces are usually developed in bodies when they are subjected to external forces (loads). Although the microscopic interpretation of such internal forces is not the same for gases, for example, as that explained above for solids (more will be said about this later on), the macroscopic or phenomenological effects are quite similar. Therefore, a continuum theory that ignores the molecular and atomic structure of materials, must account for internal forces through
Figure 3.1

Force Between two Atoms as a Function of Their Separation Distance: OA is of the order of the atomic size. At the vicinity of the equilibrium state A, the curve is almost a straight line. Hence, for sufficiently small "displacements" the force of resistance in most solids is linearly related to the displacement. This is Hooke's Law.
appropriate assumptions about the behaviour of continua. This is accomplished by the introduction of the stress hypothesis which is commonly named after Cauchy\(^1\), and which is discussed below.

The continuum theory regards matter as continuously distributed throughout the space that a considered body occupies. Therefore, if a rod (a cylindrical body with, for example, circular cross-section) is pulled, the force exerted on one end must be transmitted through the material to the other end, establishing an equilibrium state. Imagine now a plane, see Fig. 3.2a, cutting (this is an imaginary plane, so that one does not actually cut the body) through the rod normal to its axis which is also the direction of the applied force. The material on one face of this plane exerts force (in this case, pulls) on the material located immediately on the other face of the plane. This effect is represented by considering forces (internal forces) distributed over the cut surface of the body. They are called surface tractions, and are measured as force per unit area of the considered cut surface. Depending on the nature of the externally applied loads (forces), the tractions on a given internal surface may or may not be uniformly distributed. For example, in the case of the rod in Fig. 3.2a, it may be assumed that these tractions are uniformly distributed over the cross-section II, as shown in Fig. 3.2b. In this figure the forces applied to the ends (externally applied loads) of the rod are shown by arrows which pull the ends away from each other. Denote this force by \( F \), the area of the cross-section taken normal to the axis of the rod in the

\(^1\) Cauchy, Augustin Louis (1789-1857), great French mathematician who made significant contributions to rigorous mathematical analysis, laid the foundations for complex analysis, and developed numerous results which bear his name. He made significant contributions to astronomy, optics, and laid the foundation for the mechanics of deformable bodies. He was a self-righteous religious bigot.
(a) A rod of circular cross-section is pulled by forces $F$ that are uniformly distributed over the end sections. The material on the left side of the imaginary plane I I pulls on the material on the right side of this plane.

(b) The effect (pull) of part R on part L can be represented by tractions $t_I = \frac{F}{\alpha}$.

(c) On a cross-section which makes an angle $\alpha$ with the $X_1X_1$-axis, the traction is $\tau = \frac{F}{\alpha} \sin \alpha$, and has a normal component (normal stress) $n = t_I \sin^2 \alpha$ and a tangential component (shear stress) $s = \frac{1}{2} t_I \sin 2\alpha$. 
deformed state, by \( a \), and observe that the traction \( t_I \) exerted over cross-section I I I, is

\[
t_I = \frac{F}{a}
\]

(3.1)

where \( F \) is the magnitude of the force \( F \).

By convention, the traction \( t_I \) is regarded positive when tensile, and negative when compressive. In the first case, the material on one side of the cross-section pulls on the material on the other side, whereas in the second case it pushes.

Instead of cross-section I I I, consider a plane which cuts the rod in Fig. 3.2a at an angle \( \alpha \) with the rod's axis\(^1\). The area of this slanted cross-section II II is equal to \( \frac{a}{\sin \alpha} \), as is evident from its geometry. Since the total force (the resultant force) on section II II must equal \( F \) in magnitude so as to maintain the equilibrium, the traction \( \tau \) is not equal to \( t_I \), but one has

\[
at_I = \frac{a}{\sin \alpha} \tau .
\]

(3.2)

The traction \( \tau \) on section II II may be decomposed according to the parallelogram law into a normal, \( n \), and a tangential, \( s \), component. In this way one obtains, see Fig. 3.2a,

\[
n = \tau \sin \alpha , \quad s = \tau \cos \alpha ,
\]

(3.3)

which, together with (3.2) yield

\[
n = t_I \sin^2 \alpha ,
\]

\[
s = \frac{1}{\tau} t_I \sin 2\alpha .
\]

(3.4)

\(^1\) This means that the direction of the normal to the plane II II, makes an angle \( \frac{\pi}{2} - \alpha \) with the \( x_1 x_1 \)-axis.
The tangential component $s$ is called the **shear stress**, while the normal component $n$ is called the **normal stress**.

It is important to observe that the surface tractions acting on the interior surfaces within a continuum at a point, depend on the **orientation** of these surfaces. In the above example one may imagine sections I I and II II passing through the same material point, while the tractions acting on the imaginary section I I are normal to this section, the tractions acting on section II II are not, and have both normal and shear components. In this example the shear stress attains its maximum value $\frac{1}{2}t$ for $\alpha = 45^0$, as is evident from Eq. (3.4).

It is also important to note that the material on one side of, say, cross-section I I in the above example, can actually be removed, and instead, the corresponding surface tractions applied on the cut cross-section, without affecting the equilibrium and deformation of the remaining part. This observation is fundamental in continuum mechanics. Note that, if, in addition to contact forces which are manifested through the surface traction, the two parts of the body affect each other through distant forces, for example, gravitational forces, one cannot just simply remove one part, and replace its effect by surface tractions (contact forces) alone. One must, in addition, adjust the body forces on the remaining part of the body, so as to account for the distant forces as well. Let us now consider another example.

From his elementary physics course, or by personal experience, the student knows that submerged bodies in liquids (for example, when one dives deep into the swimming pool) are subjected to hydrostatic pressure. Hydrostatic pressure is a very special kind of surface tractions. It represents
the effect of the material points outside the person's body, upon the body, over the surface boundary of the body. The direction of this traction happens to be perpendicular to the surface of the body, and its magnitude depends on the distance from the free surface of the fluid, see Fig. 3.3.

As an additional example, let us discuss the internal forces in the one-dimensional motion of the continuum that was considered before.

As the continuum moves in the tube along the \( X_1X_1 \)-axis, the particles located on one side of a cross-section exert forces upon those on the other side of the cross-section. Since every particle moves in the direction of the \( X_1X_1 \)-axis, these forces according to Newton's second law must also act in that direction. This is shown in Fig. 3.4. In Fig. 3.4, the material to the left of cross-section \( II \) which is taken at the point \( x_1 \) on the \( X_1X_1 \)-axis, is removed, and its effect upon the material to the right of this section is represented by the surface tractions \( t_1 \). In this special one-dimensional flow, the tractions \( t_1 \) are uniformly distributed over the cross-section. These tractions are shown as tension, i.e. as if the material on the left of the section is pulling the material on the right. If the converse is in fact the case, the tractions \( t_1 \) will then be negative. As was mentioned before, this is a usual convention. For example, if the tube contains an ideal gas with pressure \( p \) at cross-section \( II \), then

\[
    t_1 = -p
\]

(3.5)
Figure 3.3
Hydrostatic pressure is perpendicular to the surface of the body upon which it acts, and its magnitude depends on the distance from the free surface. If the mass-density of the fluid is equal to \( \rho = \text{constant} \), the pressure at A on surface \( S \) is
\[ p = \rho gh + p_0, \]
where \( p_0 \) is the atmospheric pressure at the free surface.

Figure 3.4
In the one-dimensional motion of a continuum in a tube, the material to the left of cross-section I I is removed and instead uniformly distributed tractions \( t_1 \) are applied.
EXAMPLE 3.1

Structural steel (about 0.3% carbon) begins to yield plastically\(^1\) when the shear stress reaches the value of 21,000 lbf/in\(^2\), or when the tensile stress reaches the value of 35,000 lbf/in\(^2\). If a rod with a 2 \times 2 in square cross-section is subjected to a force \(F\), at what value of \(F\) does yielding occur.

The normal stress is given by (3.1), whereas the maximum shear occurs on a plane that makes a 45° angle with the direction of pull, the corresponding shearing stress being one half the normal stress. Since the yield stress in shear is more than half of that in tension, yielding occurs when \(\sigma = t_1 = 35,000\). This yields \(F = 2 \times 2 \times 35,000 = 140,000\) lbf.

EXAMPLE 3.2

A rod of uniform cross-section and length \(L_o\) hangs under its own weight. Its mass-density varies linearly from \(\rho_0\) at the top to 10 \(\rho_0\) at the free end. Find the maximum shear stress, assuming that the stress is uniformly distributed over each cross-section.

Let \(x\) measure distance along the rod from the free end; see Fig. 3.5. The mass-density at each section then is

\[
\rho = 10 \rho_0 - 9 \rho_0 \frac{x}{L_o}.
\]

If the tensile stress at the section \(x\) from the free end is \(t_1\), at the section \(x + dx\) it will be \(t_1 + dt_1\). The additional stress \(dt_1\) is

\(^1\) For a discussion, see Sec. 4.3, p. 4-18.
caused by the weight of the element of rod of length \( dx \). This additional weight is \( g \rho A \ dx \), where \( A \) is the cross-sectional area. Hence,

\[
dt_I = g \rho \ dx
\]

\[
= g \rho_o \left[ 10 - 9 \frac{x}{L_o} \right] \ dx.
\]

Upon integration from \( x = 0 \) to an arbitrary section \( x \), one obtains

\[
t_I = g \rho_o \left[ 10x - \frac{9x^2}{2L_o} \right] .
\]

The maximum shear occurs at \( x = L_o \) and is one half of the maximum tensile stress, i.e.

\[
s_{\text{max}} = \frac{13}{4} L_o \ g \rho_o .
\]

\[\text{Figure 3.5}\]
3.4 **FINITE DEFORMATION AND RELATIVE STRESS**

Consider a long prismatic solid body of length $L$, with a constant rectangular cross-section of width $W$, and height $H$, having its longitudinal centroidal axis along the $X_1X_1$-axis, as shown in Fig. 4.1a. Assume that the material is uniformly distributed within the body, and consider its finite elongation to length $l$ by means of forces $F$ uniformly distributed over the end cross-sections, as shown in the figure. For the convenience of referencing, let the left end of the body remain at the origin $O$. Assume that each cross-section simply translates to the right upon the application of $F$, without any rotation. Experience shows that, for most materials, the elongation of the prism is usually accompanied by a lateral contraction, and its shortening is accompanied by a lateral extension.

The stretch in the $X_1X_1$-direction is given by $\Lambda_1 = \frac{1}{L}$. It is convenient to consider the directions $X_2X_2$ and $X_3X_3$ in a typical cross-section along the width and height as shown in Fig. 4.1b. If $w$ and $h$ are, respectively, the width and height of the cross-section after deformation, then the stretches in the $X_2X_2$- and $X_3X_3$-directions are $\Lambda_{II} = \frac{w}{W}$, and $\Lambda_{III} = \frac{h}{H}$, respectively. The initial cross-sectional area $A = H W$ changes to $a = h w$ upon deformation, and one has

$$\frac{a}{A} = \frac{h w}{H W} = \Lambda_{II} \Lambda_{III} .$$

(4.1)

The initial volume $V = L A$ changes to $v = l a$, so that

$$\frac{v}{V} = \Lambda_1 \Lambda_{II} \Lambda_{III} .$$

(4.2)
FIGURE 4.1

A prismatic body of length $L$, height $H$, and width $W$ is pulled by uniformly distributed end-loads to a length $l$, height $h$, and width $w$. 
Let the mass of this prism be uniformly distributed throughout its volume, and denote the initial and the current (after deformation) mass-densities by \( \rho_0 \) and \( \rho \), respectively. Now set

\[
J = \Lambda_1 \Lambda_{II} \Lambda_{III},
\]

(4.3)

and since the total mass of the body is \( \rho_0 V = \rho V \), obtain

\[
\rho_0 = J \rho.
\]

(4.4)

The traction transmitted across a cross-section is given by

\[
t_I = \frac{F}{A}
\]

(4.5)

which is the true stress (Cauchy's stress), since it represents force per unit area, transmitted across a cross-section in the deformed state. One could, however, measure the traction transmitted across a cross-section, per unit of its undeformed area, obtaining the so-called nominal or relative stress \( T^R_I \) which is

\[
T^R_I = \frac{F}{A}
= \frac{F}{\alpha A}
= t_I \Lambda_{II} \Lambda_{III}.
\]

(4.6)

Multiplying both sides by \( \Lambda_I \), arrive at

\[
T^R_I \Lambda_I = J t_I.
\]

(4.7)

The results presented above are exact, but they apply to a very simple problem. For very long strings with very small cross-sectional dimensions, one may develop approximate theories based on the above results. This is
discussed in Sec. 4.7. In the sequel, however, some linearized results are presented, which will be useful in connection with the linear constitutive relations that are developed in the next chapter.

**LINEAR STRAIN- AND STRESS-MEASURES**

Consider the deformation of the prismatic body in Fig. 4.1, under the following assumption:

$$
\varepsilon_1 \equiv \frac{L - L'}{L} \ll 1,
$$

(4.8)

where $\varepsilon_1$ is called the extension or the engineering strain. With this assumption, the Lagrangian strain $E_1$ becomes

$$
E_1 = \frac{1}{2} \left( \frac{L - L'}{L} \right) \left( \frac{L}{L} + 1 \right)
= \varepsilon_1 + \frac{1}{2} \varepsilon_1^2 = \varepsilon_1.
$$

(4.9)

Experience shows that the lateral extensions defined by

$$
\varepsilon_2 = \frac{w - W}{W}, \quad \varepsilon_3 = \frac{h - H}{H},
$$

are usually fractions of $\varepsilon_1$. Hence, they are also small, if $\varepsilon_1$ is small. Since $\Lambda_1 = 1 + \varepsilon_1$, $\Lambda_{II} = 1 + \varepsilon_2$, and $\Lambda_{III} = 1 + \varepsilon_3$, one obtains

$$
J = (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) \approx 1 + \varepsilon,
$$

(4.10)

where

$$
\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3
$$

(4.11)

is called the dilatation. To the first order of approximation in $\varepsilon_1$, $\varepsilon_2$, and $\varepsilon_3$, the dilatation represents the change in volume per unit initial volume, since from (4.2) and (4.3) one obtains
\[ \varepsilon = \frac{\nu - V}{V} \quad \text{.} \quad (4.12) \]

From (4.10) and (4.4) it follows that
\[ \rho_0 = (1 + \varepsilon) \rho \quad \text{.} \quad (4.13) \]
which shows that, for the linearized theory, the conservation of mass states that the mass-density should remain unchanged.

From Eq. (4.6) and the fact that the extensions \( \varepsilon_1, \varepsilon_2, \) and \( \varepsilon_3, \) are assumed to be very small, one readily concludes that
\[ T^R_1 = t_1 (1 + \varepsilon_2 + \varepsilon_3) \approx t_1 \quad , \quad (4.14) \]

since \( \varepsilon_2 + \varepsilon_3 \) can be neglected as compared to 1, in the same manner that \( \varepsilon_1^2 \) can be neglected in comparison to \( \varepsilon_1. \) In this connection the student should note that, for most structural materials such as steel, the design (engineering) strains do not exceed 0.1 to 0.2 percent, while the accuracy of most measured quantities is not better than several percent. In this linearized approach the common value of the relative (nominal) stress \( T^R_1 \) and the true (Cauchy) stress \( t_1 \) will be denoted by \( \sigma_1; \) see Ch. 4.
3.5 **Biaxial Stress-State**

Consider a rectangular block in a biaxial state of stress, as shown in Fig. 5.1. The uniformly distributed stresses $t_I$ and $t_{II}$ are applied on the faces of the block which are, respectively, perpendicular to the $X'_1X'_1$- and $X'_2X'_2$-directions; see Fig. 5.1a. Consider an imaginary plane which cuts plane $OX'_1X'_2$ along the axis $OX'_1$. For the sake of simplicity assume that the rectangular block has unit thickness in the direction perpendicular to the plane of Fig. 5.1a. The material on one face of plane $OC'$ (i.e. the plane whose trace on the $OX'_1X'_2$-plane is $OC'$) interacts with the material on the other face through interatomic forces.

If the prism which has triangle $OEC'$ as its base is isolated, shear stresses $t_{21}$ and normal stresses $t_{22}$ must be applied over the face $OC'$. These stresses represent the action of the materials outside this prism upon those inside it. At the right in Fig. 5.1b the resultant forces acting on three faces of the prism are shown; the length of $OC'$ is denoted by $l$, and the angle between $OX'_1$ and $OX'_1$ is designated by $\varphi'$. Sum the forces acting on this prism, and taking their projection along the $OX'_1$-direction, obtain

$$t_{21} = -\frac{1}{2} \left( t_I - t_{II} \right) \sin 2\varphi'. \quad (5.1)$$

Similarly, projecting these forces along the $OX'_2$-direction, one concludes that

$$t_{22} = t_I \sin^2 \alpha + t_{II} \cos^2 \varphi'$$

$$= \frac{1}{2} \left( t_I + t_{II} \right) - \frac{1}{2} \left( t_I - t_{II} \right) \cos 2\varphi'. \quad (5.2)$$

---

1 This and other planes which are normal to the plane $OX'_1X'_2$, will be designated by their "trace" on the $OX'_1X'_2$-plane.
(a) Biaxial State of Stress: A block of unit thick is under uniformly distributed stresses $t_1$ and $t_{II}$ applied on its faces which are respectively orthogonal to the $X_1X_1'$- and $X_2X_2'$-directions.

(b) On imaginary plane OC', the shear stress $t_{21}$ and normal stress $t_{22}$ act.

(c) The stress-state for OABC (left). The state of pure shear (right), where $\varphi' = 45^\circ$ and $t_1 = -t_{II}$.
At the left of Fig. 5.1b, the resultant forces acting on the faces of the prism whose base is the triangle OA'D, are shown. The axis OX₂ makes an angle $\varphi' + \frac{\pi}{2}$ with OX₁. By direct calculation, similarly to above, one obtains

$$t_{12} = -\frac{1}{2} \left( t_I - t_{\text{II}} \right) \sin 2\varphi'$$

(5.3)

and

$$t_{11} = t_I \cos^2 \varphi' + t_{\text{II}} \sin^2 \varphi' = \frac{1}{2} \left( t_I + t_{\text{II}} \right) + \frac{1}{2} \left( t_I - t_{\text{II}} \right) \cos 2\varphi' .$$

(5.4)

In a similar manner the normal and shear stresses transmitted over faces AB and BC of the square with base OABC in Fig. 5.1a, can be calculated. The state of stress is shown in the left-hand side of Fig. 5.1c. As is evident from Eqs. (5.1) and (5.3), the shear stresses $t_{12}$ and $t_{21}$ are equal.

Consider a special state of stress in which

$$t_I = -t_{\text{II}} ,$$

(5.5)

and then conclude that

$$t_{12} = t_{21} = t_{\text{II}} \sin 2\varphi' ,$$

$$t_{11} = t_{\text{II}} (\sin^2 \varphi' - \cos^2 \varphi') = -t_{22} .$$

(5.6)

When $\varphi' = 45^\circ$, Eq. (5.6) reveals that $t_{11} = t_{22} = 0$, and $t_{21} = t_{12} = t_{\text{II}} .$

The state of stress is that of a simple shear, as shown on the right of Fig. 5.1c.

Note here that $t_{11} + t_{22} = t_I + t_{\text{II}}$, and $t_{11}t_{22} - t_{12}^2 = t_I t_{\text{II}} .$

These relations which give the basic invariants of the stress in planar motion, will be examined again later on.
GENERAL BIAXIAL STRESS-STATE

From the preceding discussions it follows that a portion of a continuum may be isolated from the remainder, and the effect of the material points outside of this isolated portion can be represented by surface tractions applied on its boundary, and (if necessary) by suitably adjusting the body forces. Figure 5.2a shows for a planar deformation an isolated portion of the continuum, region \( r \), with boundary \( \partial r \). The surface tractions, \( t \), on \( \partial r \) are shown by arrows which point outward, making an acute angle with the exterior unit normal \( \mathbf{n} \) at each point where the corresponding traction has a normal component, \( t(n) \), and a shear component, \( t(s) \). The normal component is represented as tension in accordance with the previously stated sign convention.

Around a typical point \( P \) in \( r \) consider a rectangular block with unit thickness (slice with unit thickness) having sides parallel to the coordinate axes, as shown in Fig. 5.2a. In Fig. 5.2b this block is isolated, and the corresponding tractions which represent the effect of the material outside upon that within are shown by normal and shear surface tractions applied on its boundary. If the dimensions \( dx_1 \) and \( dx_2 \) of this block are sufficiently small, it can be assumed that the normal and shear tractions on each side are uniformly distributed. The following sign convention is commonly used:

The normal stress is positive when it represents tension (material immediately outside is pulling on the material immediately inside of the considered boundary). The shear stress on the surface whose exterior unit normal points in the positive (negative) direction of the coordinate axis
Figure 5.2

(a) At an instant t, the material in region r is isolated and the corresponding tractions are applied on its surface boundary \( \partial r \). At each point on \( \partial r \), the traction \( \mathbf{t} \) can be decomposed into a normal \( \mathbf{t}_{(n)} \) and a tangential \( \mathbf{t}_{(s)} \) component.

(b) Tensions acting on a block ABCD are shown when they are all positive.

(c) Tensions acting on opposite faces of the same plane are equal in magnitude but opposite in direction.

(d) From the equilibrium of the prism ABP, as \( l \to 0 \), the normal and shear stresses on the plane AB are expressed in terms of \( t_{11} \), \( t_{22} \), and \( t_{12} \).
normal to this surface, is positive if it points in the positive (negative) direction of the other coordinate axis.

In Fig. 5.2b it is assumed that all the components of the stress are positive. On the face AB whose exterior unit normal is \( e_1 \), \( t_{11} \) represents the component in the \( X_1 X_1 \)-direction of the corresponding traction, while \( t_{12} \) is the component in the \( X_2 X_2 \)-direction of the traction. On this face, \( t_{11} \) pointing to the right, and \( t_{12} \) pointing upward, are positive. On the other hand, on face DC whose exterior unit normal is \( -e_1 \), \( t_{11} \) pointing to the left, and \( t_{12} \) pointing downward, are positive. Note that the first index represents the unit normal of the surface on which the traction is transmitted, whereas the second index represents the direction along which the corresponding traction is resolved.

Let the face AB be taken very close to the face CD, as sketched in Fig. 5.2c. At the limit, AB and CD represent the right- and left-hand faces of the same plane passing through point P. The above sign-convention then complies with the requirement of Newton's law of the equality of action and reaction, so that the tractions acting over the opposite faces of the same plane at a given material point, are equal in magnitude and opposite in direction.

The quantities \( t_{11}, t_{12}, t_{21}, \) and \( t_{22}, \) can be represented collectively by \( t_{ab}, a, b = 1,2 \). These are called the components of the stress tensor, or, for short, the stress tensor. In general, they are a function of particle position as well as time, so that \( t_{ab} = t_{ab}(x,t) \). When the stresses do not depend on \( x \), the state of stress is called homogeneous. The state of stress corresponding to Fig. 5.1 is an example of a homogeneous stress-state. Moreover, in a sufficiently small neighborhood of a particle, the state of stress can be regarded locally homogeneous.
It is observed that the stress tensor is symmetric, as was shown in connection with Fig. 5.1, so that \( t_{ab} = t_{ba} \) and there are only three independent stress-components in a planar stress-state.

**Cauchy's Formula**

At a given time the quantities \( t_{ab} \) define the tractions transmitted at a given point across planes through this point that are parallel to the coordinate planes; these planes are drawn normal to the plane of motion. As was discussed above, tractions transmitted across different planes at a given point are different, and therefore depend on the orientation of the considered plane. These tractions are, however, related to the stress-components \( t_{ab} \) by the so-called Cauchy formula which will be developed below.

At a typical point \( P \) consider tractions transmitted across planes I and II which are, respectively, parallel to the coordinate planes whose traces on the plane of motion are \( X_1X_1 \) and \( X_2X_2 \), respectively. Consider another plane whose unit normal is \( n \), at a distance \( l \) from point \( P \). The tractions transmitted across this latter plane will be developed for the limiting case when \( l \) becomes very small, i.e. this plane passes through \( P \).

Consider a slice of unit thickness, and isolate the prismatic body whose base is ABP. The tractions on the face AP are \( t_{22} \) and \( t_{21} \), whereas those on the face BP are \( t_{11} \) and \( t_{12} \), \( t_{12} \) being equal to \( t_{21} \) for sufficiently small \( l \). The tractions \( t_{(n)} \), transmitted across AB has a normal component \( t_{(n)} \) and a tangential component \( t_{(s)} \).

---

1 The superscript \( (n) \) signifies the fact that tractions on plane whose unit normal is \( n \), are considered.
The components of the exterior unit vector $n$ are $n_1 = \cos \varphi$ and $n_2 = \sin \varphi$; see Fig. 5.2d. If the length of $AB$ is $l$, then $BP$ and $PA$ will have the respective lengths of $l \cos \varphi$ and $l \sin \varphi$.

In addition to the surface tractions acting on $PAB$, body forces and inertial forces must also be considered. However, the resultant body forces and inertial forces, which are proportional to the mass contained within this prismatic body, are of the order of $l^2$, whereas the resultant forces acting on the boundary are of the order of $l$. Since the limiting case as $l$ tends to zero is considered, the body and inertial forces need not be included.

Consider now the equilibrium of $PAB$, in the same manner as discussed in connection with Fig. 5.1, and obtain

$$t_{(n)} = t_{11} \cos^2 \varphi + t_{22} \sin^2 \varphi + 2t_{12} \cos \varphi \sin \varphi$$

$$= \frac{1}{2} \left( t_{11} + t_{22} \right) + \frac{1}{2} \left( t_{11} - t_{22} \right) \cos 2\varphi + t_{12} \sin 2\varphi,$$

$$t_{(s)} = -t_{11} \cos \varphi \sin \varphi + t_{12} \cos^2 \varphi + t_{22} \sin \varphi \cos \varphi - t_{12} \sin^2 \varphi$$

$$= -\frac{1}{2} \left( t_{11} - t_{22} \right) \sin 2\varphi + t_{12} \cos 2\varphi.$$  \hspace{1cm} (5.7)

The student should compare these equations with Eqs. (7.18) and (7.21), Ch. 2.

From (5.7) it now follows that the normal and shear tractions transmitted at a point across a plane whose normal makes an angle $\varphi$ with the $X_1X_1$-axis, is completely defined in terms of the stress-components $t_{11}$, $t_{12}$, and $t_{22}$, at that point.

Consider now a unit vector $\mathbf{n}^*$ which makes an angle $\varphi^* = \frac{\pi}{2} + \varphi$ with
the $X_1 X_1$-axis, so that $n \cdot n^\ast = 0$. The components of $n^\ast$ are $n_1^\ast = \cos \left( \frac{n}{2} + \varphi \right) = -\sin \varphi$ and $n_2^\ast = \sin \left( \frac{n}{2} + \varphi \right) = \cos \varphi$. The normal traction $t_{(n)}$ may now be expressed by

$$t_{(n)} = t_{ab} n_a n_b$$
$$= t_{11} n_1^2 + t_{22} n_2^2 + 2t_{12} n_1 n_2,$$  \hspace{1cm} (5.8)

while the corresponding shear stress becomes

$$t_{(s)} = t_{ab} n_a n_b^\ast$$
$$= t_{11} n_1 n_1^\ast + t_{22} n_2 n_2^\ast + \left( n_1 n_2^\ast + n_2 n_1^\ast \right) t_{12}.$$  \hspace{1cm} (5.9)

If the components in the $X_1 X_1$- and $X_2 X_2$-directions of the traction vector $t_{(n)}$, are denoted by $t_{1}^{(n)}$ and $t_{2}^{(n)}$, it now readily follows that 1

$$t_{b}^{(n)} = n_a t_{ab}$$
$$= n_1 t_{1b} + n_2 t_{2b}, \hspace{1cm} b = 1, 2,$$  \hspace{1cm} (5.10)

so that

$$t_{(n)} = n_b t_{b}^{(n)}$$
$$= n_1 t_{1}^{(n)} + n_2 t_{2}^{(n)},$$  \hspace{1cm} (5.11)

and

$$t_{(s)} = n_b t_{b}^{\ast}$$
$$= n_1 t_{1}^{\ast} + n_2 t_{2}^{\ast}.$$  \hspace{1cm} (5.12)

1 Note that $t_{(n)}$ is the traction vector with normal and tangential components $t_{1}^{(n)}$ and $t_{2}^{(n)}$, respectively, and with components $t_{1}^{\ast}$ and $t_{2}^{\ast}$ along the respective coordinate axes.
Equations (5.10) state that the traction transmitted across a plane with unit normal $\mathbf{n}$, is a \textit{homogeneous, linear} function of the cosine directors $n_1$ and $n_2$, the coefficients in these linear relations being the corresponding stress components. These equations are commonly referred to as Cauchy's formulae, the above-stated result being referred to as Cauchy's theorem.

**PRINCIPAL STRESSES**

As in the case of the rate of deformation, Sec. 2.7, one may seek the orientation of planes across which the normal stress is an extremum. This can easily be done by maximizing with respect to $\varphi$ the right-hand side of (5.7)$_1$, arriving at

$$\tan 2\varphi = \frac{2t_{12}}{t_{11} - t_{22}}$$

(5.13)

which defines two orthogonal directions across which the corresponding normal tractions are

$$t_{1,II} = \frac{1}{2} \left( t_{11} + t_{22} \right) \pm \left[ t_{12}^2 + \frac{1}{4} \left( t_{11} - t_{22} \right)^2 \right]^{\frac{1}{2}},$$

(5.14)

where $t_1$ and $t_{II}$ are called the \textit{principal stresses}; the corresponding directions given by (5.13) are the \textit{principal directions} of stress.

Substitution from (5.13) into (5.7)$_2$ reveals that the shear stresses are zero on planes across which principal stresses $t_1$ and $t_{II}$ act.

The above results are in exact correspondance with those discussed in connection with planar deformation in Sec. 2.7. Hence, (5.14) can be viewed as the roots of the equation

$$t^2 - t_1 t + II_1 = 0,$$

(5.15)
where
\[ I_t = t_1 + t_{11} = t_{11} + t_{22}, \]
\[ II_t = t_1 t_{11} = t_{11} t_{22} - t_{12}^2, \]  
are the basic invariants of the stress tensor in planar motion.

**MOHR'S CIRCLE FOR STRESS**

Equations (5.7) which give the normal and shear stresses transmitted at a point across a plane whose normal makes the angle \( \varphi \) with the \( X_1X_1 \)-axis, define, as \( \varphi \) varies, points on a circle. Denote by \( n \) and \( s \) the abscissa and the ordinate of a typical point on this circle. At a given instant the stress components \( t_{11}, t_{12}, \) and \( t_{22} \), are fixed at a fixed material point. Hence, the coordinates of a point on this circle are
\[ n = \frac{1}{2} \left( t_{11} + t_{22} \right) + \frac{1}{2} \left( t_{11} - t_{22} \right) \cos 2\varphi + t_{12} \sin 2\varphi, \]
\[ s = -\frac{1}{2} \left( t_{11} - t_{22} \right) \sin 2\varphi + t_{12} \cos 2\varphi, \]  
so that the center of the circle is on the \( n \)-axis, having the abscissa \( \frac{1}{2} \left( t_{11} + t_{22} \right) \). To locate a point on the circle, the following sign-convention which is different from that stated before for shear stresses, is commonly adopted\(^1\). The shear stress transmitted across a face of a rectangular block is regarded positive if it turns clockwise with respect to an interior point. The normal stress is regarded positive if tensile. In Fig. 5.2b, therefore, \( t_{12} \) as shown is negative, whereas \( t_{21} \) is positive with respect to Mohr's circle.

\(^1\) This sign-convention is used only for the purpose of constructing Mohr's circle. For all other purposes the sign-convention stated before will be used.
To construct the Mohr circle for the state of stress shown in Fig. 5.3a, mark in the \( n_s \)-plane a point \( M \) with coordinates \( n = t_{11} \) and \( s = -t_{12} \), and another point \( M^* \) with coordinates \( n = t_{22} \) and \( s = t_{21} = t_{12} \). The line \( MM^* \) intersects the \( n \)-axis at the center of the circle, \( MM^* \) being the corresponding diameter. The radius of the circle is equal to \( \sqrt{t_{12}^2 + \frac{1}{4}(t_{11} - t_{22})^2} \). Points \( A_I \) and \( A_{II} \) on the \( n \)-axis have the abscissae given by Eq. (5.14) and, therefore, respectively correspond to the principal stresses \( t_I \) and \( t_{II} \), for which the corresponding shear-stress is zero; see Fig. 5.3b.

Suppose that the tractions transmitted across plane \( BB \) whose unit normal make angle \( \theta \) with the \( X_1X_1 \)-direction, Fig. 5.3a, are to be calculated. On the Mohr circle measure an angle \( MOQ \) equal to \( 2\theta \), to obtain point \( Q \). The coordinates of this point define the normal and shear stresses actions on plane \( BB \).

![Figure 5.3](image)

Mohr's Circle
EXAMPLE 5.1

Show that the state of stress on a plane whose normal makes the angle $\theta$ with the $X_1X_1$-axis is given by the coordinates of point $Q$ on Mohr's circle in Fig. 5.3b.

To this end denote the radius of the Mohr circle by $R$. Then the coordinates of point $Q$ are

$$n = \frac{1}{2} (t_{11} + t_{22}) + R \cos (2\phi' - 2\theta)$$

$$= \frac{1}{2} (t_{11} + t_{22}) + R (\cos 2\phi' \cos 2\theta + \sin 2\phi' \sin 2\theta)$$

$$= \frac{1}{2} (t_{11} + t_{22}) + \frac{1}{2} (t_{11} - t_{22}) \cos 2\theta + t_{12} \sin 2\theta ,$$

$$s = R \sin (2\theta - 2\phi')$$

$$= - R (\sin 2\phi' \cos 2\theta - \cos 2\phi' \sin 2\theta)$$

$$= - t_{11} \cos 2\theta + \frac{1}{2} (t_{11} - t_{22}) \sin 2\theta .$$

Comparison with Eq. (5.17) shows that the above expressions indeed represent the normal and shear stresses on a plane whose normal makes $\theta$ with the $X_1X_1$-axis.

EXAMPLE 5.2

The state of stress at a point is shown in Fig. 5.4a. Draw the corresponding Mohr circle, and find the shear and normal stresses transmitted on a plane whose normal makes a $15^\circ$ angle with the AB-direction. Also find the plane of maximum shear stress.

To construct the Mohr circle, mark in the $n,s$-plane points $M$ with coordinates $n = -10,000$ and $s = -5,000$, and $M'$ with coordinates $n = 2,000$.

1 The minus sign in the second equation is due to the sign convention which is adopted for the construction of Mohr's circle; p. 3-30.
and \( s = 5,000 \), the intersection of \( MM' \) with the \( n \)-axis defining the center of the circle. The radius of the circle is \( R = \left[ (5,000)^2 \right. \\
+ \frac{1}{4} \left( 2,000 + 10,000 \right)^2 \right]^{\frac{1}{2}} \approx 7,810 \).

The angle \( 2\psi' \) is given by

\[
2\psi' = \arctan \frac{2 \times 5,000}{-10,000 - 2,000} \approx -39.8^\circ.
\]

The normal and shear stresses on a plane whose normal makes a \( 15^\circ \) angle with the \( AB \)-direction, are

\[
n_Q = \frac{1}{2}(-10,000 + 2,000) + \frac{1}{2}(-10,000 - 2,000) \cos 30^\circ + 5,000 \sin 30^\circ
\]

\[
= -6,700
\]

\[
s_Q = 7,330
\]

The maximum shear stress occurs on a plane whose normal makes the angle \( 25.1^\circ \) with the \( AB \)-direction, the maximum shear being equal to 7,810. The corresponding normal stress is -4,000. Figures 5.4a,b,c show the above results graphically.
Figure 5.4
PROBLEMS FOR CHAPTER 3

2.1 What is the gravitational potential of Earth at 1000 miles altitude?

[Take the radius of Earth 4000 miles.]

2.2 An incompressible liquid flows down an inclined plane which makes an angle $\alpha$ with the horizontal plane. For the rectilinear motion along the $X_1X_4$-axis which lies on the inclined plane, find the potential of the corresponding body forces.

2.3 Consider a solid sphere of radius $c$ and a uniform mass distribution with mass-density $\rho$, in equilibrium under its own gravitation (self-gravitating sphere). A point-mass of unit mass placed at a radius $r \leq c$ from the center, will be attracted to the center by the gravitational forces of the mass contained within the sphere of radius $r$. The resultant of these gravitational forces can be obtained if one assumes that the entire mass of the sphere of radius $r$ is at the center.

(a) Show that the body force within the sphere (due to self-gravitation) is given by $f_r = -g \frac{r}{c}$, where $f_r$ is the body force in the radial direction, and $g$ is the gravitational intensity at the surface of the sphere.

(b) Assume that the material of this sphere behaves like a liquid, having a uniform mass-density of 11 slugs/ft$^3$. Find the pressure at the center when $c \approx 4,000$ miles, and $g \approx 32.2$ ft/sec$^2$.

3.1 A circular cylinder of 2 in diameter, is being pulled by forces $F$ uniformly distributed on its end sections.
(a) If it is known that the material of this cylinder fractures under a tensile stress of 20,000 lbf/in², find the maximum admissible value of F.

(b) If it is known that the material of this cylinder fails over any plane on which the value of shear stress exceeds 20,000 lbf/in², find the maximum allowable force F.

3.2 Prove the Archimedes principle which states that: a body immersed in a fluid is subjected to a lift (buoyant) force equal to the weight of the displaced fluid. Can this force act through any point other than the center of gravity of the displaced volume? [This center is called the center of buoyancy.]

3.3 In an isothermal case, the pressure and mass-density of a gas are related by \( p = K \rho \), where \( K \) is a constant, whereas for an adiabatic situation, one has \( p = k_0 \rho^\gamma \), where \( k_0 \) is a constant, and \( \gamma \) is the ratio of the specific heat of the gas at constant pressure to that at constant volume; for air \( \gamma \approx 1.4 \). Find the pressure-height relation in both isothermal and adiabatic situations of a clam (no motion) atmospheric condition.

4.1 A bar of a circular cross-section, consisting of an incompressible material, is extended to twice of its original length by means of forces that are uniformly distributed over its end sections.

(a) Find the final radius if initially the bar is 2 in in diameter.

(b) If the required force is 50 lbf, find the true (Cauchy's) and the nominal stresses.
(c) What are the longitudinal and lateral extensions.

(d) Find the Lagrangian strain $E_i$ and discuss whether linearization is acceptable in this case.

5.1 In a biaxial state of stress, $t_1 = 10,000$ lbf/in$^2$ and $t_2 = 5,000$ lbf/in$^2$.

(a) Find the plane on which the shear stress is 1,250 lbf/in$^2$.

(b) What is the normal stress on the plane in (a).

5.2 The state of stress at a point is shown in Fig. Prob. 5.2. If the material is such that it fails as soon as the shear stress reaches the value of 20,000 lbf/in$^2$, find the value of $F$ and the plane of failure.

5.3 The state of stress at a point is shown in Fig. Prob. 5.3.

(a) Draw the corresponding Mohr's circle.

(b) Find both graphically and analytically the shear and normal stresses that are transmitted across a plane whose normal makes the angle $15^\circ$ with the AB-direction, as shown.

(c) Find the direction of the normal to the plane of maximum shear stress.
CHAPTER 4

ON THERMO-MECHANICAL PROPERTIES OF MATTER: CONSTITUTIVE RELATIONS

4.1 INTRODUCTION

In the preceding two chapters certain kinematical and dynamical aspects of continua were discussed. These discussions apply to continuous bodies of all kinds. They do not reflect the particular constitution of matter. Similarly the basic conservation laws which will be discussed in Ch. 5, are valid for all continua, and hence these laws likewise do not reflect the constitution of the particular body. Whether the considered continuum is gaseous, or liquid, its motion must be such that the basic laws are not violated\(^1\). However, gaseous continua do not respond in the same manner as do liquids. Their differences must, therefore, be accounted for by additional relations which must reflect their basic constitution. These relations are called constitutive laws or constitutive relations. Actually, the term constitutive relations is preferred, since they are not laws with universal validity. They only apply to a given class of materials.

The constitutive relations are not "exact" in the sense that they do not predict with a great degree of accuracy the behaviour of actual materials. Rather, they represent ideal material behaviour, and, therefore, can only be used as mathematical models for certain classes of

\(^1\) Of course this excludes shock waves and other discontinuous occurrences, since it is assumed that all considered quantities are continuous, and possess continuous derivatives of any desired order.
materials; hence they should only be regarded as approximations to reality. The constitutive relations, however, cannot be acceptable, if they violate the basic conservation laws which are discussed in Ch. 5.

In this chapter some simple constitutive relations for fluids and solids are presented.
4.2 FLUIDS

Fluids can be divided into gases and liquids. Although this distinction is not quite clear-cut, the gases include those substances which fill the space within their container, whereas liquids usually undergo small volume changes under finite pressures.

In Sec. 1.2 the constitutive relation for perfect gases was mentioned. This constitutive equation is

\[ p\nu = R\theta \]  \hspace{1cm} (2.1)

where \( \nu \) is the specific volume, i.e. the volume of a unit mass of the gas; \( \nu = \frac{1}{\rho} \), where \( \rho \) is the mass-density measured as gram mole per unit volume. In (2.1) \( \theta \) is the temperature measured from absolute zero.

A gas is called calorically perfect, if its specific heats are constant. It is recalled that the specific heat of a substance is the amount of heat energy required to increase the temperature of one unit mass of that substance by one degree. Since gases can be heated under various circumstances, for example at constant volume, or at constant pressure, one can define, correspondingly, various specific heats. Denote by \( C_V \) and \( C_P \) the specific heat at constant volume and at constant pressure, respectively. For calorically perfect gases these are constant, otherwise they may depend on the temperature. Note that these definitions are equally applicable to liquids and solids. From Eq. (2.1) one obtains

\[ p\Delta\nu + \nu\Delta p = R\Delta\theta \]  \hspace{1cm} (2.2)

where \( \Delta \) preceding a quantity indicates its increment. At constant pressure, \( \Delta p = 0 \), the additional work done on the environment while the temperature of a unit mass of a gas is increased by one degree (\( \Delta\theta = 1 \)),
is \( p\Delta v = R \). Hence, \( C_p \) exceeds \( C_v \) by \( R \), i.e.

\[
C_p = C_v + R \quad .
\] (2.3)

Since at constant volume the entire heat energy used to increase the temperature of a perfect gas by \( \Delta \theta \), is stored in the gas as internal energy, the change in internal energy, \( \Delta \varepsilon \), per unit mass corresponding to the temperature change \( \Delta \theta \) is

\[
\Delta \varepsilon = C_v \Delta \theta \quad .
\] (2.4)

When the state of a perfect gas is slowly changed in such a manner that no heat is added to it, or subtracted from it, the process is called adiabatic. For a change \( dv \) in volume, the work done by the gas at pressure \( p \) is \( -pdv \); minus sign indicates that pressure is negative tension. The corresponding change in the internal heat energy then is \( d\varepsilon = C_v d\theta \), and it follows that

\[
C_v d\theta = -pdv \quad .
\]

Combining this with the thermal equation of state, Eq. (2.1), written incrementally as \( pdv + vdp = Rd\theta \), one obtains

\[
\frac{dp}{p} + \gamma \frac{dv}{v} = 0 \quad ,
\] (2.5)

where \( \gamma = \frac{C_p}{C_v} > 1 \). Upon integration, (2.5) yields

\[
p^{\gamma} = \text{constant} = k_o \quad ,
\] (2.6)

\[
p = k_o \rho^{\gamma} \quad .
\] (2.7)
which is valid for adiabatic processes in perfect gases.

Most gases are not perfect, i.e. they do not comply with the thermal
equation of state (2.1). At low pressure and high specific volume,
however, their behaviour approaches asymptotically that of perfect gases.
Also, most substances are not calorically perfect. For example, at
one atmosphere pressure, water has the specific heat of 1.0074 cal/\textdegree{}C
at 0\textdegree{}C, and 0.99828 cal/\textdegree{}C at 25\textdegree{}C.

Real substances can change phase. Gases can liquefy, and liquids
can solidify. A detailed study of the thermodynamics of these processes
is outside the scope of this book\textsuperscript{1}. Therefore, only a brief sketch of a
few relevant features is given in the sequel.

Figure 2.1 gives schematically the isotherms (curves of constant
temperatures) in the $p, v$-plane in various phases. At a temperature
$\theta_1 < \theta_c$, where the critical temperature $\theta_c$ corresponds to the isotherm
that passes through the critical point C, the real gas at low pressures
and high specific volumes behaves almost like a perfect gas; the isotherm
in this region is (almost) a parabola, as required by Eq. (2.1). As the
pressure is increased, the specific volume decreases until the isotherm meets
with the vapor line in Fig. 2.1, which marks the boundary between the
gaseous phase and the vapor-liquid state. At this point one has a saturated
vapor. Further decrease in specific volume at constant temperature $\theta_1$,
can be obtained by liquefaction which occurs at a constant pressure,
and continues until the entire vapor has liquefied. This corresponds to

\textsuperscript{1} See, for example, Kestin, op. cit. Vol. I, 1966, Ch. 7, Vol. II,
1968, Ch. 20.
Figure 2.1

The \( p, v \)-Diagram for a Normal Substance: At sufficiently small values of \( p \) and sufficiently large values of \( v \), most normal gases behave like a perfect gas. At a constant subcritical temperature \( \theta_1 < \theta_c \), the substance begins to liquefy at point A. For an intermediate state on AB the two phases, liquid and vapor, coexist. At point B the entire substance is liquefied. At point a the liquid begins to solidify, and at b it is entirely in the solid phase. Below the triple-point line, the gas begins to solidify when the isotherm reaches the vapor line at D; the solid and vapor coexist for intermediate states on line DE. At E the entire substance is in the solid state.
the liquid line of Fig. 2.1. At point B the entire vapor is liquefied, and further decrease in specific volume can be accomplished by a substantial increase in pressure, which corresponds to a sharp rise in the isotherm toward the left of the figure. Note that further increase in pressure usually results in solidification which also occurs at a constant pressure; this is shown in Fig. 2.1 by line ab.

As one repeats the above-mentioned process at higher temperatures, the reduction in specific volume taking place during liquefaction, decreases until it becomes zero at the critical temperature \( T_c \) for which the isotherm skirts the liquid-, vapor-curve in Fig. 2.1 at the critical point C, where this isotherm has an inflection point. On this isotherm the transition from gaseous to liquid states (phase change) is not clear-cut at all. The pressure and the specific volume corresponding to the critical point will be denoted by \( p_c \) and \( \nu_c \), respectively.

From the above remarks it is clear that the behaviour of real gases is quite complicated, and therefore, its description must involve a more complicated form than (2.1). Nevertheless, at low pressure and large specific volume, where intermolecular forces are negligible, the perfect gas-type behaviour must prevail. As a gas is compressed at a constant subcritical temperature, its molecules attain a smaller average distance, and, therefore, intermolecular forces become more and more important, resulting in deviation from a perfect gas-type behaviour. To account for this and other real gas effects, over one hundred analytical expressions have been proposed for the thermal equation of state of real gases, none of which, however, has even limited universal applicability\(^1\).

\(^1\) For a more detailed account, see Ibid.
The simplest equation of this kind which, although quantitatively inaccurate, gives a remarkably accurate qualitative picture of the situation, has been proposed by van der Waals\textsuperscript{1} in 1873.

\textbf{THE VAN DER WAALS EQUATION}

This equation is

\[ p = \frac{R\theta}{\nu - b} - \frac{a}{\nu^2} \quad (2.8) \]

where \( a \) and \( b \) are constants which are peculiar to a given substance. From a microscopic point of view, the additional terms in van der Waals' equation have simple interpretations. The constant \( b \) which has the physical unit of specific volume, corrects for the space that is actually occupied by the molecules of the gas. The term \( \frac{a}{\nu^2} \) corrects the perfect gas equation for the intermolecular forces (van der Waals forces) which tend to pull together the molecules inside of the container, away from the walls of the container, and in this manner alter the effective pressure on the walls.

Table 2.1 gives the van der Waals' constants, \( a \) and \( b \), for a few substances, when the pressure is measured in atmospheres and the volume in liters; thus \( R \) in the van der Waals' equation is 0.08206 liter atm/mole \( ^{0}\text{K} \).

\begin{table}[h]
\centering
\begin{tabular}{lll}
\hline
\textbf{SUBSTANCE} & \textbf{a (liter}\(^2\text{ atm/mole}\(^2\))} & \textbf{b (liter/mole)} \\
\hline
Ammonia & 4.170 & 0.03707 \\
Carbon Dioxide & 3.592 & 0.04267 \\
Helium & 0.03412 & 0.02370 \\
Hydrogen & 0.2444 & 0.02661 \\
Mercury & 8.093 & 0.01696 \\
Nitrogen & 1.390 & 0.03913 \\
Oxygen & 1.360 & 0.03183 \\
\hline
\end{tabular}
\caption{Van der Waals' Constants\textsuperscript{2}}
\end{table}

\textsuperscript{1} Van der Waals, Johannes, Diderik (1837-1923), Dutch physicist who received the Nobel prize for physics in 1910.

\textsuperscript{2} Data taken from Handbook of Chemistry and Physics, thirty-sixth edition, Chemical Rubber Publishing Co. 1954, p. 2124.
Typical isotherms for van der Waals' equation are sketched in Fig. 2.2. At subcritical temperatures the van der Waals isotherm admits a maximum and a minimum. The portion of the isotherm between the vapor and the liquid line represents unstable regimes. For states between points c and e, an increase in specific volume is accompanied by an increase in pressure, which causes an additional increase in the specific volume. The gas then continues to expand until equilibrium state f is reached. Between the vapor and the liquid lines, the isotherm is therefore represented by the horizontal line fb in Fig. 2.2. The area between the curve fed and the horizontal line fd, is equal to that between dcb and db. There are several ways that this can be shown, One simple argument is that if the two areas were not equal, then by taking the substance carefully and slowly through the states represented by the curve fedcb, and then returning on the straight line bdf, or conversely, one would have a perpetual energy-creator, since the algebraic sum of the above-mentioned areas represents the net work done by the substance. This of course is not admissible.

The critical point C of the van der Waals equation is given where the corresponding isotherm has a horizontal inflection point. This point is defined if one sets the first and second derivatives of p with respect to \( \nu \) equal to zero, i.e. \( \frac{dp}{d\nu} = \frac{d^2p}{d\nu^2} = 0 \), arriving at

\[
\nu_c = 3b, \quad p_c = \frac{a}{27b^2}, \quad \theta_c = \frac{8a}{27Rb} \tag{2.9}
\]

The equation of state at the critical point then is

\[
p_c \nu_c = \frac{3}{8} R \theta_c \tag{2.10}
\]
Figure 2.2

Typical Isotherms for van der Waals' Gas: On the isotherm $\theta_1 < \theta_c$, the substance is in liquid state at point a. On bf, the liquid and vapor coexist. At point f the entire substance is vaporized, whereas at point b, it is liquefied. For isotherms which pass through j and k, the van der Waals gas can take tension (negative pressure).
which shows that \( p \nu \) at the critical point falls substantially short of the value given by the perfect gas law. If one employs the critical values \( p_c, \nu_c, \) and \( \theta_c \), as units of pressure, specific volume, and temperature, the van der Waals equation reduces to

\[
(p + \frac{3}{\nu^2})(\nu - \frac{1}{3}) = \frac{8}{3}\theta
\]

(2.11)

which is referred to as a reduced equation of state. The form (2.10) permits a quantitative check on the accuracy of the van der Waals equation. For example, for carbon dioxide, the van der Waals equation predicts \( \nu_c = \frac{3R\theta_c}{8p_c} \approx 130 \text{ cm}^3/\text{mole} \), since for this substance, \( \theta_c = 304.3 \text{ ^oK} \), and \( p_c = 73 \text{ atm} \); note that \( R = 82.06 \text{ cm}^3 \text{ atm}/\text{mole} ^oK \). The observed value, however, is about 96 \( \text{ cm}^3/\text{mole} \); since one mole of carbon dioxide is 44 g, from Table 2.2 we obtain \( 44/0.46 \approx 96 \text{ cm}^3/\text{mole} \). Table 2.2 lists the critical values of temperature, pressure, and density of several substances.

**TABLE 2.2**

<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>( \theta_c (^oC) )</th>
<th>( p_c (\text{atm}) )</th>
<th>( \rho_c = \frac{1}{\nu_c} = (\text{g/cm}^3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMMONIA</td>
<td>132.4</td>
<td>111.5</td>
<td>0.235</td>
</tr>
<tr>
<td>CARBON DIOXIDE</td>
<td>31.1</td>
<td>73.0</td>
<td>0.460</td>
</tr>
<tr>
<td>HELIUM</td>
<td>-267.9</td>
<td>2.26</td>
<td>0.0693</td>
</tr>
<tr>
<td>HYDROGEN</td>
<td>-239.9</td>
<td>12.8</td>
<td>0.0310</td>
</tr>
<tr>
<td>NITROGEN</td>
<td>-147.1</td>
<td>33.5</td>
<td>0.3110</td>
</tr>
<tr>
<td>OXYGEN</td>
<td>-118.8</td>
<td>49.7</td>
<td>0.430</td>
</tr>
<tr>
<td>WATER</td>
<td>374.0</td>
<td>217.7</td>
<td>0.4</td>
</tr>
</tbody>
</table>

\footnote{\text{Data taken from *Ibid.*, p. 2121.}}
**VIRIAL EXPANSION**

Most real gases behave like perfect gases, at low pressure and high specific volume. This suggests that one may consider a series representation of the gas law in ascending powers of the pressure \( p \), as

\[
p \nu = A_0(\theta) + A_1(\theta)p + A_2(\theta)p^2 + \ldots
\]

(2.12)

with the restriction that it reduces to the perfect gas law (2.1), as \( p \) becomes small, which is satisfied if we set

\[
A_0(\theta) = R\theta
\]

(2.13)

Expansion (2.12) is commonly called a virial expansion. Another form of virial expansion is arrived at, if the right-hand side of (2.12) is expressed in negative powers of specific volume,

\[
p \nu = B_0(\theta) + B_1(\theta)\nu^{-1} + B_2(\theta)\nu^{-2} + \ldots
\]

(2.14)

The coefficients in the right-hand side of (2.12) and (2.14) are, of course, related. Since, as \( \nu \) becomes large, (2.14) must also reduce to (2.1), it follows that

\[
B_0(\theta) = A_0(\theta) = R\theta
\]

To establish the relationship between the other coefficients in these two equations, solve for \( p \) from (2.14), and substitute into the right-hand side of (2.12). Comparing now the coefficients of equal negative powers of \( \nu \) in the right-hand side of the resulting equation, with those of (2.14), arrive at the sought relations\(^1\).

If only the first two terms in the right-hand side of (2.12) are

\[^1\text{See Kestin, op. cit. Vol. II, p. 229.}\]
retained, this equation can be written as

\[
\frac{PD}{R\theta} = 1 + A_1(\theta) \frac{P_c}{R\theta}.
\] (2.15)

Using the critical values of temperature, pressure, and specific volume, one reduces (2.15) to

\[
\frac{PD}{R\theta} = 1 + \frac{P}{P_c} \frac{P_c}{\theta} \frac{\theta}{\nu_c} A_1(\theta). \tag{2.16}
\]

For gases such as oxygen (O_2), nitrogen (N_2), helium (He), and nitric oxide (NO), for example, the quantity \(\frac{P_c \nu_c}{\theta_c}\) is approximately equal to \(A_1(\theta)\); 0.295. The quantity \(\frac{\theta_c}{\nu_c}\) for these gases is almost a universal function of \(\frac{\theta}{\theta_c}\). Thus, (2.16) can be written as

\[
\frac{PD}{R\theta} = 1 + \frac{P}{P_c} \varphi_0\left(\frac{\theta}{\theta_c}\right). \tag{2.17}
\]

Table 2.3 gives a few typical values of \(\varphi_0\) for indicated values of \(\frac{\theta}{\theta_c}\).

<table>
<thead>
<tr>
<th>(\frac{\theta_c}{\theta})</th>
<th>0.1</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi_0(\frac{\theta_c}{\theta}))</td>
<td>0.009</td>
<td>0.015</td>
<td>-0.005</td>
<td>-0.067</td>
<td>-0.18</td>
</tr>
</tbody>
</table>

**TENSILE STRENGTH OF LIQUIDS**

Liquids can sustain high tension under carefully adjusted circumstances. For example, tension over 500 atm has been applied to liquids before failure. This kind of experiment is, however, very difficult to perform, since vapor bubbles usually form within the liquid, and also the liquid

---


2 These data are taken from *Ibid.*, p. 35.
is usually torn away from the walls of the container.

The van der Waals equation (2.11) gives a qualitative picture of the situation, which, however, is quantitatively far from experimental results. As is seen in Fig. 2.2, at sufficiently small temperatures the van der Waals isotherm has a minimum at which the pressure is negative. To obtain the maximum tension admitted by the van der Waals "gas", solve (2.11) for p, set \( \frac{dp}{d\nu} = 0 \), and arrive at

\[
(3\nu - 1)^2 = 4\nu^3 \theta .
\]

This equation shows that, as \( \theta \) goes to zero, \( \nu \) approaches the value of \( \frac{1}{3} \), and substitution back into (2.11) shows that \( p \) approaches the value 27. Recalling that \( p \) in (2.11) is expressed in terms of \( p_c \) as the unit of pressure, note that the van der Waals "gas" is supposed to take on 27\( p_c \) tension. For water, for example, this is more than 5,800 atm. Water has been stressed up to 277 atm of tension\(^1\).

**COMPRESSIBILITY OF LIQUIDS**

Liquids, like gases (also, like solids), are compressible, that is, their volume can be reduced by compressing them. However, in contrast to gases whose intermolecular distances are large, the intermolecular distances in liquids are usually of the order of molecular size, and this enables them to resist high pressures with small changes of volume.

A measure of compressibility is the so-called bulk modulus. To define the bulk modulus, consider a substance under pressure \( p_1 \), having specific volume \( \nu_1 \), and let the pressure be increased to \( p_2 \), resulting in the

decrease of the specific volume to \( \nu_2 \). The average bulk modulus then is defined by

\[
K_{av.} = \frac{p_2 - p_1}{\nu_2 - \nu_1} \nu_1
\]

(2.19)

If the change in pressure is made smaller and smaller, the bulk modulus can then be defined by

\[
K = -\nu \frac{dp}{d\nu}
\]

(2.20)

Since \( \nu = \frac{1}{\rho} \), this equation can also be written as

\[
K = \rho \frac{dp}{d\rho}
\]

(2.21)

The reciprocal of the bulk modulus is called the **compressibility**. Table 2.4 gives the compressibility of some liquids at indicated temperatures and pressures. As is evident from these results, the compressibility, and hence the bulk modulus vary with temperature and pressure.

**TABLE 2.4**

<table>
<thead>
<tr>
<th>LIQUID</th>
<th>TEMPERATURE (°C)</th>
<th>PRESSURE (atm)</th>
<th>COMPRESSIBILITY (Contraction in unit volume per atm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACETONE</td>
<td>0.</td>
<td>1 - 500</td>
<td>82 \times 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>0.</td>
<td>500 - 1000</td>
<td>59 \times 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>0.</td>
<td>1000 - 1500</td>
<td>47 \times 10^{-6}</td>
</tr>
<tr>
<td>GLYCERINE</td>
<td>14.9</td>
<td>1 - 10</td>
<td>22 \times 10^{-6}</td>
</tr>
<tr>
<td>MERCURY</td>
<td>0.</td>
<td>---</td>
<td>3.92 \times 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>15.</td>
<td>100 - 200</td>
<td>3.76 \times 10^{-6}</td>
</tr>
<tr>
<td>WATER</td>
<td>0.</td>
<td>1 - 25</td>
<td>52.5 \times 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>20.</td>
<td>1 - 25</td>
<td>49.1 \times 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>0.</td>
<td>2500 - 3000</td>
<td>26.1 \times 10^{-6}</td>
</tr>
</tbody>
</table>

The speed at which small pressure disturbances propagate in substances is called the sound speed. In Ch. 6 it is shown that the sound speed $c$ is given by

$$c = \sqrt{\frac{\Delta p}{\Delta \rho}} = \sqrt{\frac{K}{\rho}}.$$  \hspace{1cm} (2.22)

Thus, with the knowledge of the sound speed and the mass-density, the bulk modulus can be calculated for a given substance. Table 2.5 gives the sound speed for a few liquids and gases.

**TABLE 2.5**

<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>TEMPERATURE ($^\circ$C)</th>
<th>$c$ (m/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALCOHOL 95%</td>
<td>12.5</td>
<td>1241.</td>
</tr>
<tr>
<td>WATER, air-free</td>
<td>13.</td>
<td>1441.</td>
</tr>
<tr>
<td>WATER, air-free</td>
<td>19.</td>
<td>1461.</td>
</tr>
<tr>
<td>WATER, air-free</td>
<td>31.</td>
<td>1505.</td>
</tr>
<tr>
<td>AIR, dry, CO$_2$-free</td>
<td>0.</td>
<td>331.78</td>
</tr>
<tr>
<td>AIR, 25 atm</td>
<td>0.</td>
<td>332.0</td>
</tr>
<tr>
<td>AIR, 100 atm</td>
<td>0.</td>
<td>350.6</td>
</tr>
<tr>
<td>AIR</td>
<td>20.</td>
<td>344.</td>
</tr>
<tr>
<td>AIR</td>
<td>1000.</td>
<td>700.</td>
</tr>
<tr>
<td>HYDROGEN</td>
<td>0.</td>
<td>1269.5</td>
</tr>
<tr>
<td>WATER, vapor</td>
<td>130.</td>
<td>424.4</td>
</tr>
</tbody>
</table>

**COMRESSIBILITY OF WATER**

The compressibility of fresh water and sea water has been studied by P. G. Tait$^2$ who proposed empirical equations which, according to a

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$^1$ Data taken from Handbook of Chemistry and Physics op. cit. p. 2312.

$^2$ See his Scientific Papers, Cambridge: At the University Press, 1900, Vol. II, p. 1; Scottish mathematician and physicist (1831-1901) who made contributions to thermodynamics.
recent comparative study by Yuan-Hui Li\textsuperscript{1}, are in good accord with experimental data. For fresh water, Tait's empirical equation can be written as
\[ \nu(p) = \nu\textsuperscript{(1)}_o - C \log \frac{B + p}{B + 1}, \] (2.23)
where \( \nu(p) \) is the specific volume measured as ml/g, \( \nu\textsuperscript{(1)}_o \) is the specific volume at \( p = 1 \) bar, \( p \) is measured in bars\textsuperscript{2}, and \( C \) and \( B \) are given in the range of \( 0 \leq \theta \leq 45^\circ \text{C} \), and \( 1 \leq p \leq 1,000 \) bars, by
\[ C = 0.315 \nu\textsuperscript{(1)}_o, \]
\[ B = 2,668.0 + 19.867\theta - 0.3110\theta^2 + 1.778 \times 10^-3\theta^3. \] (2.24)
For sea water, the equation is
\[ \nu(p) = \nu\textsuperscript{(1)} - (1 - S \times 10^-3)C \log \frac{B^* + p}{B^* + 1}, \] (2.25)
where \( S \) is salinity in percent, varying between 30 and 40\%, and
\[ B^* = (2,670.8 + 6.89668) + (19.39 - 0.07038)\theta - 0.223\theta^2, \] (2.26)
where the range of \( p \) is as before, but \( 0 \leq \theta \leq 20^\circ \text{C} \).


\textsuperscript{2} 1 bar = 0.9869 atm.
4.3 **ELASTICITY AND PLASTICITY OF METALS**

When subjected to sufficiently small forces, most solids have a spring-like property, that is, the small deformation induced by the small force will disappear upon the removal of the force, and the solid recovers its original configuration. This property is called **elastic**. On the other hand, if the applied load is sufficiently large, most metals undergo some permanent deformation, so that, upon the removal of the load, they cannot regain their original dimensions. Such a deformation is called **plastic**. In order to give a quantitative account of this behavior, one uses certain standard tests, the most common of which is the tensile test which will be discussed below.

**TENSILE TEST**

The elasticity and the strength of solids are established quantitatively in standard tests performed on standard specimens. The tension test is most commonly used to this end. The test specimen is a cylindrical bar with a \( \frac{1}{2} \) inch diameter and 2 inch gauge length. The specimen is prepared with enlarged cross-sections at both ends, in order to prevent slipping in the grips of the testing machine. The length of the specimen between the two enlarged end portions is at least one diameter longer than the gauge length, \( L \). The testing machine is commonly equipped such that it can automatically prepare a diagram of load versus the corresponding elongation. In some modern testing machines the load or the elongation, together with their rates, can be controlled.

A typical load \( P \) versus elongation \( \delta \) is shown in Fig. 3.1. The elongation \( \delta \) is linearly related to the applied load \( P \) for the portion OA of
The Load-Elongation, P-δ, Diagram in Tension Test: OA corresponds to the elastic behaviour. Point A defines the proportionality limit. At B the specimen is unloaded, tracing curve BHCD. OD is the permanent set. The reloading takes place along DEHF. FG is almost the continuation of OAB. The loop HCDEH is called the hysteresis loop. In engineering practice the nominal stress (load divided by the initial area) at point Q is called the yield stress. This point corresponds to 0.2% permanent set; this point (almost) coincides with the point which is marked on the curve by a line parallel to OA with 0.2% offset.
the curve. The nominal stress corresponding to point A, which is obtained by dividing the corresponding load by the initial cross-sectional area of the rod, defines the stress below which the material response is elastic. This stress gives the proportionality limit. Upon further increase in the load, the load-elongation is no longer linear, and plastic deformation takes place. On portion AB of the diagram, additional elongation requires much smaller additional loads. If at point B the specimen is slowly unloaded, the curve BCD is traced, and the specimen does not recover its original length. The abscissa of point D is called the permanent set. If now the specimen is reloaded, the curve DEFG will be traced. The portion FG of this curve would be (almost) identical with that which would have been obtained, if the unloading had not been performed, and the specimen had been taken along OABFG. (This statement is correct, if one does not "age" the specimen at the state D.) Since the area under the P-δ diagram represents the mechanical work done to deform the body, the area inside of loop CDE represents the energy loss during the unloading and reloading. This loop is called the hysteresis loop.

Not all metals behave in the manner shown by Fig. 3.1. A few typical P-δ diagrams are shown in Fig. 3.2. Mild steel has a sharp proportionality limit which coincides with its upper yield limit, point A in Fig. 3.2. The upper yield limit is followed by a drop in load to the lower yield limit at point B. The portion BC where plastic deformation is taking place, is traced with almost no increase in load. If at any point on this portion the specimen is unloaded, a line (such as ED) parallel to OA is traced. At point C work-hardening takes place, and further elongation is
Figure 3.2

Typical P-δ Diagrams: Mild steel has an upper yield limit at A and a lower yield limit at B. Along BC plastic flow takes place, and at C the work-hardening begins. At E, unloading takes place along ED which is parallel to QA. In the right, the necked portion of the test specimen is sketched.
obtained by an additional increase in load, until point M is reached. At this stage the specimen begins to neck, i.e. at a certain small region close to the middle along its length, the specimen begins to develop a large decrease in its cross-section, and henceforth, additional elongation takes place with a decrease in the total load, to the point N at which the specimen breaks; see the sketch in the right-hand corner of Fig. 3.2. Note that, for portion MN, although the total load is decreasing, the true stress at the neck, which is obtained by dividing the load by the cross-sectional area of the neck, increases.

Certain materials such as aluminum do not exhibit a distinct proportional limit. To obtain uniformity in results, the International Congress for Testing Materials at Brussels in 1906, defined the proportional limit as the tensile stress at which the permanent set is 0.001\%; the tendency since then, however, has been to increase this to 0.01\%. In engineering practice it is common to consider the proportional limit and the yield stress, both at the point where upon unloading a permanent set of 0.2\% is attained. This is called the offset method; see Fig. 3.1. The yield stress obtained in this manner is quite arbitrary and does not represent a definite material characteristic.

ON ELASTICITY AND PLASTICITY OF METALS

Metals owe many of their properties to their crystalline structure. Crystals are formed by closely packed atoms in a well-defined pattern which,

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in a perfect crystal, repeats itself in all directions. The basic pattern is called a space lattice, and simple crystals have atoms at their lattice points and at symmetrical positions with respect to these points\(^1\). The face-centered cubic (copper, aluminum, lead, silver, gold), the body-centered cubic (\(\alpha\)-iron, vanadium, tungsten, and alkali metals), and hexagonal close packed (zinc, magnesium, cadmium), are common crystal structures in metals. In Fig. 3.3 the basic units of these three crystal structures are sketched; the dots in this figure show the location of only the centers of the particles (the dimension of each unit is of the same order as the atomic size).

In Sec. 3.3 the nature of interatomic forces in metals was discussed, see Fig. 3.1. According to this figure, the interatomic attractive or repulsive forces are almost linearly related to the increase or decrease in the equilibrium interatomic distances, on the BC portion of the curve. This fact accounts for the linearity of the load-elongation curve in the elastic range. The plasticity of the metal, on the other hand, relates to the sliding of atoms in groups with respect to other groups. A whole slab of crystal glides over an adjacent slab in such a manner that the crystalline structure is preserved\(^2\). The slip surface is usually cylindrical with its axis defining the slip direction. When the slip surface is planar, it is referred to as the slip plane. The intersection of the slip surface with the boundary of the crystal is called the slip band. In a perfect, single-crystal specimen the slip occurs on a crystallographic plane which

\(^1\) op. cit. Cottrell, Ch. 3. See also L. I. Van Vlack, Materials Science for Engineers, Addison-Wesley Publishing Co., 1970.

\(^2\) op. cit. Cottrell, Ch. 9.
Figure 3.3

(a) The face-centered cubic (f.c.c.). ABC defines one of the octahedral planes.

(b) The body-centered cubic (b.c.c.).

(c) The closed-packed hexagonal structure.
is closely packed with atoms. For example, in the aluminum specimen this
occurs parallel to one of the octahedral planes, such as plane ABC in Fig.
3.3a. The slip bands are about 100 to 10,000 atomic distances apart, and
between two slip bands the crystal structure is intact. Figure 3.4a sche-
matically shows the slip bands in a tensile specimen. The slip occurs
when the shear stress (see Sec. 3.3) in the most unfavorably oriented
crystallographic plane reaches a certain critical value. In Sec. 3.3 it
was observed that in simple tension the maximum shear occurs on a plane at
a $45^\circ$ angle with the direction of pull. This, however, is not necessarily
the slip plane. The actual slip plane depends on the orientation of the
crystal. Suppose that the rod in Sec. 3.3 is composed of a crystal having
a slip plane that forms angle $\alpha$ with the $X_1 X_1$-axis\(^1\), where $\alpha'$ is not necessarily
$45^\circ$. The shear stress on this plane is, Eq. (3.4), $s = t_{\parallel} \cos \alpha$
$\sin \alpha$. Let now the direction of slip on the slip plane make an angle $\beta$
with the direction of shear stress $S$. Then the component of shear, $\tau$,
that causes the slip, is $\tau = t_{\parallel} \cos \alpha \sin \alpha \cos \beta$; see Fig. 3.4b.

Plastic deformation also occurs by other mechanisms such as twinning,
where atoms slide layer by layer in a special way. We shall not discuss
this here, rather refer the reader to Cottrell's book.

A piece of ordinary metal consists of a compact aggregate of crystal
grains which have random orientation and are of varying shapes and sizes.
Crystal grain sizes are usually of the order $10^{-3}$ to $10^{-4}$ cm, each containing
rows of $10^5$ to $10^4$ atoms. In such a polycrystalline metal the crystal
continuity is terminated at the grain boundaries where two crystals border
each other. These discontinuities are the essential cause of viscous

\(^1\) The direction normal to the slip plane make an angle $\frac{\pi}{2} - \alpha$ with
the $X_1 X_1$-axis.
Figure 3.4

(a) A tension specimen deforms plastically when the shear stress on the slip plane reaches a critical value. An entire slab of crystal, consisting of some 100 to 10,000 layers of atoms, glides over other slabs.

(b) The slip plane makes an angle $\alpha$ with the $X_1 X_{11}$-axis. The slip direction is at an angle $\beta$ with the direction of shear stress $S$. The shear stress in the slip direction hence is $\tau = t_1 \cos \alpha \sin \alpha \cos \beta$, where $t_1 = F/2$. 
flow, intercrystalline fracture, and dislocation in metals.

The strength of a crystal is usually many orders of magnitude less than its "theoretical" strength. This is caused by the many impurities and faults that exist in crystals, and which initiate plastic deformation of the crystal at much lower stress levels. It is substantially for this reason that materials in the form of fine fibers or "whiskers" of diameter about $10^{-4}$ cm can undergo elastic strains of the order of 0.1, i.e. 10%, before yielding.

**DUCTILITY VERSUS BRITTLE FAILURE**

The plasticity of steel is responsible for its enormous ductility, so that a large plastic flow precedes the failure of the test specimen. In certain materials such as cast iron and fiberglass which undergo small elongation at high stress levels, the failure can be quite abrupt; the failure is brittle, see Fig. 3.5. If a test specimen of a brittle material is pulled while it is subjected to high pressure, substantial ductility is manifested. An experiment of this kind was performed by T. von Karman on marble specimens. Extensive experiments under large hydrostatic pressures have been performed by Bridgman.

The brittle fracture occurs by the growth of cracks across planes pulled apart by large tensile stresses. This should be contrasted to the ductile failure in which layers of atoms slide over other layers, leading to large plastic strains. Brittle materials made of thin fibers can with-

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stand high tension before fracture. For example, graphite whiskers and silica fibers have a tensile strength of up to $3.5 \times 10^6 \text{lb} / \text{in}^2$. These fibers (of approximately $10^{-4} \text{cm}$ diameter) do not readily admit flaws and internal cracks, and hence, are of substantial strength. Fiberglass tendons with high tensile strength are made of drawn threads of glass and silica which are bonded together by a resin. Table 3.1 gives typical tensile strengths of a number of metals.

Figure 3.5 compares a typical $P$-$\delta$ diagram of a high strength brittle material with that of a ductile material. The brittle material undergoes small strains at high stress levels, and fractures at point $M$. The area under the curve $OM$ represents the mechanical work that needs to be spent to elongate the specimen to fracture. This work represents the toughness of the material, and is commonly called that. The corresponding mechanical work which is spent to yield the ductile material to failure at point $N$ in Fig. 3.5, is substantially larger than that for the brittle material, although in the former case the stress level is substantially lower. Hence, the ductile material in this figure is tougher, while the brittle one is stronger.

**COMPRESSION TEST**

In compression tests, specimen of usually cylindrical shape, whose length is twice their diameter, are employed. The test is most commonly performed on brittle materials. For example, for concrete, a cylindrical specimen of length 12 in and diameter 6 in, is most commonly employed in the United States.
### TABLE 3.1

**TENSILE STRENGTH OF METALS**

<table>
<thead>
<tr>
<th>METAL</th>
<th>TENSILE STRENGTH (10^3 lbf/in^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALUMINUM, wire</td>
<td>30-40</td>
</tr>
<tr>
<td>BRASS, wire</td>
<td>50-150</td>
</tr>
<tr>
<td>BRONZE, wire, phosphor, hard-drawn</td>
<td>110-140</td>
</tr>
<tr>
<td>BRONZE</td>
<td>60-75</td>
</tr>
<tr>
<td>COPPER, wire, hard-drawn</td>
<td>60-70</td>
</tr>
<tr>
<td>GOLD, wire</td>
<td>20</td>
</tr>
<tr>
<td>IRON, cast</td>
<td>13-33</td>
</tr>
<tr>
<td>IRON, wire, hard-drawn</td>
<td>80-120</td>
</tr>
<tr>
<td>LEAD, cast or drawn</td>
<td>2.6-3.3</td>
</tr>
<tr>
<td>NICKEL, hard-drawn</td>
<td>155</td>
</tr>
<tr>
<td>PLATINUM, wire</td>
<td>50</td>
</tr>
<tr>
<td>SILVER, wire</td>
<td>42</td>
</tr>
<tr>
<td>STEEL</td>
<td>40-330</td>
</tr>
<tr>
<td>STEEL, specially treated</td>
<td>250</td>
</tr>
<tr>
<td>nickel steel</td>
<td></td>
</tr>
<tr>
<td>STEEL*, piano wire, 0.033 in diam.</td>
<td>357-390</td>
</tr>
<tr>
<td></td>
<td>0.051 in diam.</td>
</tr>
<tr>
<td>TUNGSTEN</td>
<td>590</td>
</tr>
<tr>
<td>ZINC, cast</td>
<td>7-13</td>
</tr>
<tr>
<td>ZINC, drawn</td>
<td>22-30</td>
</tr>
</tbody>
</table>

---

Ductility Versus Brittle Failure: Brittle materials fail after relatively small elongation, whereas ductile materials can sustain large plastic strains before failing. The area under the $P-\delta$ curve is the energy spent in order to break the specimen. Hence, even though brittle materials may fail under larger loads (they may be stronger), the mechanical work required for this is much less than that necessary to break a ductile specimen which fails under much smaller loads. The ductile material is then said to be tougher.
Ductile materials respond in compression first elastically, then, after the yield limit is reached, plastically. The mechanism of the plastic flow is similar to that which occurs in tensile tests. However, as the specimen shortens, it tends to expand laterally, and, therefore, the load-elongation curve becomes concave upward. Moreover, the end sections of the specimen, which are in contact with the heads of the testing machine, are, because of frictional forces, restrained from lateral expansion, and this adds to the resistance of the specimen. Usually the specimen tends to bulge out at the middle in a barrel shape, and it eventually deforms into a flat disk. To avoid this bulging effect which tends to make the stress nonuniformly distributed over various cross-sections, Taylor and Quinney\(^1\) in studying copper specimen, performed a test in stages, where, in each stage, the specimen was machined back to its original proportions; this resulted in a continual decrease in the absolute size of the specimen. A similar procedure was applied by Bridgman\(^2\). In practice the initial elastic and the following plastic response of ductile materials, for small values of extension, say, for less than 0.2 to 0.3 per cent, is assumed to be the same in compression and in tension. Therefore, a tensile test is used to obtain data which is then applied to both tension and compression.

For brittle materials, on the other hand, the mode of failure in compression is by the sliding of a part of the specimen over the other part, and crumbling along an inclined plane which makes a certain angle with the direction of the applied force. In the tensile test, a brittle material


\(^2\) *op. cit.*, Bridgman, p. 181.
suddenly breaks as a result of the growth of the internal cracks. Thus, the mode of failure of brittle materials in compression is quite different from that in tension. They can withstand large compressive stresses, whereas their tensile strength can be very low. For example, carbon and graphite products have a tensile strength ranging between 150 and 2,500 lbf/in$^2$, whereas their compressive strength can change from 1,700 to 10,000 lbf/in$^2$. Hard glass, on the other hand, has a tensile strength of 6,000 to 10,000 lbf/in$^2$, and a compressive strength of 60,000 to 175,000 lbf/in$^2$. Similarly, in contrast to its tensile strength, masonry, such as brick and stone, has substantial compressive strength. Stone, for example, has the compressive strength of about 10,000 lbf/in$^2$.

4.4 **LINEAR ELASTICITY**

**HOOKE'S LAW**

From the results of the previous section it is clear now that most metals exhibit a certain elasticity for sufficiently small values of extension $\varepsilon_1$ (or engineering strain). As was explained in Sec. 3.4, for such small strains a linearized theory can be used, where the Lagrangian strain, $E_1$, and the engineering strain or extension, $\varepsilon_1$, are almost identical, and where the true stress, $t_1$, and the nominal stress, $T_{1R}$, are also identical; the common value of these stresses will be denoted by $\sigma_1$. The proportionality of the stress and extension (the extension is loosely called strain) can be stated by

$$\sigma_1 = E\varepsilon_1$$  \hspace{1cm} (4.1)
which is called Hooke's law after Robert Hooke\(^1\) who first proposed it. The proportionality constant \(E\) is called the elastic modulus or Young’s\(^2\) modulus. This constant represents a property of materials, and changes from one material to another. For steel, for example, \(E\) is about \(30 \times 10^6 \text{ lbf/in}^2\). Table 4.1 gives strength properties of certain metals and their alloys. Some of the quantities in this table will be defined later on.

**POISSON'S RATIO**

It is recalled from Sec. 3.3 that a small extension \(\varepsilon_1\) of a prismatic bar is generally accompanied by small lateral contractions, \(\varepsilon_2\) and \(\varepsilon_3\). This is called Poisson's\(^3\) effect. The ratio between the absolute values of the lateral strain \(\varepsilon_2\) or \(\varepsilon_3\) and \(\varepsilon_1\), in the linear range, is a constant called Poisson's ratio, and is commonly denoted by \(\nu\), i.e.

\[
\left| \varepsilon_2 \right| = \left| \varepsilon_3 \right| = \nu \left| \varepsilon_1 \right| .
\]

(4.2)

Theoretically (as will be shown later on), \(\nu\) can range between \(-1\) and \(\frac{1}{2}\). However, observation shows that \(\nu\) is positive. For aluminum, for example, it is about 0.33.

For an incompressible material (its volume does not change) Poisson's ratio has the value \(\frac{1}{2}\). To show this consider the prismatic bar in Sec. 3.4 and observe that the volume after deformation in the linearized case is

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\(^{1}\) Hooke, Robert (1635-1703), English physicist who originated many ideas in optics, mechanics, and meteorology.

\(^{2}\) Young, Thomas (1773-1829), English physicist and physician who worked on optics and the wave theory of light.

\(^{3}\) Poisson, Siméon, Denis (1781-1840), French mathematician who contributed to definite integrals, Fourier series, and probability (Poisson's distribution law), and in the application of mathematics to physics, in particular to electrostatics and magnetism.
\[ LHW(1 + \epsilon_1)(1 - \nu \epsilon_1)^2 = LHW \left[ 1 + \epsilon_1(1 - 2\nu) \right] \] .

If the volume is not changed, then the quantity inside the brackets must equal 1, leading to \( \nu = \frac{1}{2} \).

**ELASTIC STRAIN-ENERGY**

As an elastic rod with initial length \( L \) and initial cross-sectional area \( A \), is extended, energy is stored in it in the form of **elastic strain-energy** which is released upon the removal of the load. At a given stress level \( \sigma_1 \), consider an incremental strain \( d\epsilon_1 \) which corresponds to a change of length, \( L \, d\epsilon_1 \). The work done by the force \( F = \sigma_1 A \) in this additional elongation is

\[
du = FL \, d\epsilon_1 = AL \sigma_1 \, d\epsilon_1 .
\]

Substitution for \( \sigma_1 \) from (4.1) into (4.3) yields

\[
du = AL \, E \epsilon_1 \, d\epsilon_1 .
\]

Now, if the rod is strained elastically from its unstrained state to a final strained configuration of final extension \( \epsilon_1 \), the strain-energy stored in it will be

\[
u = \int_0^{\epsilon_1} du = \frac{1}{2} V E \epsilon_1^2 = \frac{V}{2E} \sigma_1^2 ,
\]

where \( V = AL \) is the volume of the rod. The quantity \( \frac{1}{2} E \epsilon_1^2 = \frac{1}{2} \frac{\sigma_1^2}{E} \) is the elastic strain-energy per unit volume of the rod. It is called the **elastic strain-energy-density**. In the \( \sigma_1 - \epsilon_1 \) diagram it represents the shaded area under the curve, see Fig. 4.1. When the stress level \( \sigma_1 \) (and
the corresponding strain \( \varepsilon_1 \) corresponds to the proportionality limit, the elastic strain-energy-density is called the modulus of elastic resilience. COMPRRESSIBILITY OF SOLIDS

Solids, like fluids, are compressible. At sufficiently small pressures, the pressure is related to the dilatation \( \varepsilon \) (see Sec. 3.4) linearly. The proportionality constant \( K \) is the bulk modulus, so that, if the pressure is \( p \), one has

\[
\varepsilon = \frac{\Delta V}{V} = -\frac{p}{K},
\]

where \( \Delta V \) is the change in the volume induced by the hydrostatic pressure \( p \).

At sufficiently large (uniform) pressures, the linear relation (4.6) is no longer valid. Tests by P. W. Bridgman\(^1\) under large hydrostatic pressures indicate that the dilatation and pressure can be empirically related by

\[
\frac{\Delta V}{V} = -ap + bp^2,
\]

where \( a \) and \( b \) are constants depending on the given material; when pressure is measured\(^2\) in kgf/cm\(^2\), Table 4.2 gives some values of \( a \) and \( b \) at indicated temperatures. As may be expected, the values of \( a \) and \( b \) depend on the temperature, since the compressibility is temperature sensitive. Note also that, (4.7) has only a limited range of validity, since it is only an empirical relation.

---

\(^1\) P. W. Bridgman, The Physics of Hight Pressure, G. Bell and Sons, 1947.

\(^2\) 1 kgf/cm\(^2\) = \(14.22\) lbf/in\(^2\).
According to Hooke's law the stress $\sigma_1$ is linearly related to the extension $e_1$, so that $\sigma_1 = E e_1$, where $E$ is the elastic modulus. The shaded area then is the strain-energy $\frac{1}{2} E e_1^2$ stored per unit volume in the tensile bar.
<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>TEMP °C</th>
<th>$a \times 10^7$</th>
<th>$b \times 10^{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALUMINUM</td>
<td>30</td>
<td>13.43</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>13.76</td>
<td>5.1</td>
</tr>
<tr>
<td>IRON</td>
<td>30</td>
<td>5.87</td>
<td>2.1</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>5.93</td>
<td>2.1</td>
</tr>
<tr>
<td>COPPER</td>
<td>30</td>
<td>7.19</td>
<td>2.6</td>
</tr>
<tr>
<td></td>
<td>75</td>
<td>7.34</td>
<td>2.7</td>
</tr>
</tbody>
</table>

\[1\] Data taken from Ibid., p. 160.
EXAMPLES FOR CHAPTER 4

EXAMPLE 4.1

Two elastic wires of common length $L$ and area $A$ support a force $F$, as shown in Fig. 4.2a. Assume that the extension of each wire is very small, $\varepsilon_1 \ll 1$, and find the displacement $\Delta$ of point $A$.

If the tension in each wire is $S$, it follows from Fig. 4.2b that $S = F/2\cos \alpha$. Denote the deformed length by $l$ and set $\delta = l - L$.

Then $\sin \alpha = H/l = (H/L)[1/(1 + \delta/L)] = \sin \theta(1 - \varepsilon_1 + \varepsilon_1^2 + \ldots) \approx \sin \theta$,

where $\varepsilon_1 = \delta/L$. Hence $\alpha \approx \theta$. Moreover, to the same order of approximation, it follows from Fig. 4.2b that $\delta \approx L \cos \theta$. In this manner one obtains

$$\Delta = \delta/\cos \theta = \varepsilon_1 L/\cos \theta = (S/E)\Delta L/\cos \theta = LF/2E \Delta \cos^2 \theta \quad . \quad (4.8)$$

If $F = 6 \times 10^3$ lb, $A = 0.1$ in$^2$, $L = 10$ in, $\theta = 45^\circ$, and $E = 30 \times 10^6$ psi, Eq. (4.8) yields $\Delta = 0.02$ in.

EXAMPLE 4.2

Let the system in Fig. 4.2a be replaced by a three-wire system shown in Fig. 4.3a. Use the same linearization to obtain an expression for the displacement $\Delta$ of point $A$.

Denote the tension in wire $AD$ by $S'$, its initial and final lengths by $L'$ and $L''$, respectively, and from Fig. 4.3b obtain

$$2S \cos \theta + S' = F \quad , \quad (4.9)$$

where, because of the linearization, $\alpha = \theta$ is used. Since wire $AD$ is elongated by $\Delta$, its extension is $\varepsilon_1' = \Delta / L'$. Hence, $S' = A'E'\varepsilon_1' = A'E'\Delta / L'$, where $A'$ and $E'$ are, respectively, the area and the Young modulus of
wire AD. Similarly, one obtains \( S = AE \varepsilon_1 = AE \delta \) \( \frac{L}{L} = AE \Delta \cos \theta \)

Substitution into (4.9) now yields

\[
\Delta = \frac{FL'}{(A'E' + 2AE \cos^3 \theta)}.
\]

(4.10)

The tension in the wires then is

\[
S' = F \left(1 + 2 \frac{A}{E'A'} \cos^3 \theta\right), \quad S = F \cos^2 \theta \left(\frac{E'A'}{E A} + 2 \cos^3 \theta\right).
\]

(4.11)

**EXAMPLE 4.3**

Two rods of common uniform cross-sectional area \( A \) of length \( L \) and \( L' \) are welded together to form a composite rod of length \( L + L' \).

Find the total shortening of the composite rod under a compressive force \( F \) uniformly distributed over the end sections, if the Young moduli are \( E \) and \( E' \), respectively. Assume small strain.

Denote by \( \delta \) and \( \delta' \) the corresponding shortening of each rod, and note that the common value of the stress is \( \sigma_1 = -F/A \). Hence, \( \delta = \varepsilon_1 L = L\sigma_1/E = -LF/AE \). Similarly, \( \delta' = -L'F/AE' \). It now follows that

\[
\delta_t = \delta + \delta' = -\frac{F}{A} \left(\frac{L}{E} + \frac{L'}{E'}\right).
\]

(4.12)
4.5 SMALL STRAIN PLASTICITY

In the same manner that Hooke's law represents an idealization of the (approximately) linear portion of the load-elongation curve in the tensile test, one can develop various models which represent idealizations of the plastic portion of the curve. For most metals, before necking takes place, the strain \( \epsilon_1 \) is still small, and, therefore, one may consider neglecting its square in comparison with itself. This way one again obtains the linearized equations discussed in Sec. 3.4. However, in contrast with linear elasticity, the constitutive relations modelling plastic behaviour cannot be expressed in the simple linear form of Hooke's law. Hence, even in the case of small strain theory, plasticity is inherently nonlinear.

In the sequel a few idealized constitutive relations for plasticity are presented.

RIGID, PERFECTLY-PLASTIC SOLIDS

In certain solids for which the elastic strain at the proportionality limit is very small, one can neglect the elastic strain completely, and assume that plastic deformation takes place as soon as the stress in the tensile test reaches the value at the yield limit, \( Y \). In this idealization, therefore, the solid is regarded rigid for stresses below the yield value.

An additional idealization can be entertained for materials for which the plastic portion of the load-elongation curve is almost flat. In this case one may assume that plastic flow takes place if the stress level is maintained at the yield value.

The combination of the above two idealizations gives constitutive relations for the rigid, perfectly-plastic model whose stress-strain,
\( \sigma_1 - \varepsilon_1 \), diagram is shown in Fig. 5.1. It is further assumed that the material behaves in the same way during compression as during tension. The constitutive law can be expressed as

\[
\dot{\varepsilon}_1 = 0 \text{ if } |\sigma_1| < Y \text{ or } |\sigma_1| = Y \text{ and } \dot{\sigma}_1 \sigma_1 < 0 ,
\]

\[
\text{sgn} \dot{\varepsilon}_1 = \text{sgn} \sigma_1 \text{ if } |\sigma_1| = Y \text{ and } \dot{\sigma}_1 \sigma_1 = 0 ,
\]

(5.1)

where a superposed dot denotes differentiation with respect to time \( t \), and where \( \text{sgn} \) is the sign-function standing for the sign of its argument. The first line in (5.1) states that no straining occurs as long as the stress is below the yield-level, \( Y \), both in tension or compression, or if the stress is instantaneously at the yield-level, but its time rate of change is negative when \( \sigma_1 \) is positive, and positive, when \( \sigma_1 \) is negative; that is, unloading is taking place. The second line in (5.1), on the other hand, states that the strain is increasing with time (tensile strain being positive), when the stress is tensile and at the yield-level, and strain is decreasing (compressive strain being negative), when stress is compressive and at the yield-level. If at point \( A' \) in Fig. 5.1 the stress is slightly decreased, unloading takes place, and the material rigidifies with the permanent set given by \( OC \). The material remains rigid until point \( B' \) is reached. Thereafter, yielding occurs in compression along the path \( B'B' \).

**Elastic, Perfectly-Plastic Solids**

A slight generalization of the above model is obtained, if one includes the elastic strains. This is shown in Fig. 5.2. On portion \( AB \) in this figure, the material is elastic. At points \( A \) or \( B \), yielding takes place,
Figure 5.1

The Stress-Strain Diagram for Rigid, Perfectly-Plastic Model: The yield stress in tension is $Y$, and that in compression is $-Y$. For stress-level below the yield, the material remains rigid. Plastic flow occurs if the stress is maintained at $\pm Y$. 
Figure 5.2

The Stress-Strain Diagram for Elastic, Perfectly-Plastic Model: The initial loading or unloading occurs elastically along AB, A'B', etc. Plastic flow occurs if the stress is maintained at ±Y.
and thereafter the material behaves in a perfectly plastic manner as in the preceding case. At a point A' unloading can occur along line A'B' which is parallel to AB. At point B' the plastic flow begins again along the horizontal line B'B. A similar remark is valid for the unloading path B''A'.

The constitutive relation in this case becomes

\[ \dot{\sigma}_1 = E \dot{\varepsilon}_1 \text{ if } |\sigma_1| < Y \text{ or } |\sigma_1| = Y \text{ and } \dot{\sigma}_1 \sigma_1 < 0 \quad , \]

\[ \text{sgn } \dot{\varepsilon}_1 = \text{sgn } \sigma_1 \text{ if } |\sigma_1| = Y \text{ and } \dot{\sigma}_1 \sigma_1 = 0 \quad . \]

(5.2)

The second line in (5.2) is the same as that in (5.1), while the first line in (5.2) states that Hooke's law, Eq. (4.1) (differentiate both sides of (4.1) with respect to time), holds during elastic loading, and in unloading.

**ELASTIC-WORK-HARDENING SOLIDS**

A more realistic model can be constructed if one includes both the elasticity and the plasticity, but in the latter case takes into account the fact that an additional load (however small) is required to effect additional plastic flow. This latter phenomenon is called **work-hardening**. An idealized \( \sigma_1 - \varepsilon_1 \) diagram for this type of behaviour is shown in Fig. 5.3.

The portion AB is the elastic part, and plastic yielding occurs along AA' or BB''. At A' the unloading can occur along the straight line A'B' parallel to AB, until point B' is reached. Thereafter, plastic flow occurs along B'B''.

There are various ways that the work-hardening phenomenon can be taken into account. Here only a simple model is presented.

At each instant of loading the time rate of change of the strain, \( \dot{\varepsilon}_1 \), may be divided into the elastic part \( \dot{\varepsilon}_1^e \) and the plastic part \( \dot{\varepsilon}_1^p \), so that

---

Figure 5.3

The Stress-Strain Diagram for Elastic-Work-Hardening Model: The initial loading or unloading occurs elastically along AB, A'B', etc. Plastic flow occurs for states on AA' or B'BB'.
\[ \dot{\varepsilon}_1 = \dot{\varepsilon}_1^e + \dot{\varepsilon}_1^p \]

The elastic strain-rate \( \dot{\varepsilon}_1^e \) is given by Hooke's law as

\[ \dot{\varepsilon}_1^e = \frac{1}{E} \dot{\sigma}_1 \]

(5.3)

where \( E \) is the elastic modulus. The plastic strain-rate \( \dot{\varepsilon}_1^p \), on the other hand, is related to the stress-rate \( \dot{\sigma}_1 \) through the so-called flow rule given by

\[ \dot{\varepsilon}_1^p = \gamma \frac{\dot{\sigma}_1}{E} \]

(5.4)

where \( \gamma \) is dependent on the state of stress, the amount of plastic deformation that has occurred, and the stress-rate, such that

\[
\begin{align*}
\gamma &> 0 \text{ if } |\sigma_1| = Y_y \text{ and } \dot{\sigma}_1 \sigma_1 > 0 \\
\gamma &< 0 \text{ if } |\sigma_1| < Y_y \text{ or } \dot{\sigma}_1 \sigma_1 < 0
\end{align*}
\]

(5.5)

The first line states that the function \( \gamma \) is positive during plastic flow. The second line, on the other hand, states that this function is identically zero, if the stress level is below the instantaneous yield value, \( Y_y \), or if unloading is just taking place. \( Y_y \) has the initial value \( Y \) where the elastic material just begins to yield at point A or B in Fig. 5.3. At any other point on the plastic portion of the curve, such as at A', if the stress level is reduced below \( Y_y \) along A'B', \( \gamma \) will vanish, until point B', where yielding begins again, but in compression. Observe that the yield limit \( Y_y \) at B' is considerably lower than the absolute value of the stress at point B, whereas the yield limit at A' is considerably larger than that at A. Hence, plastic flow in tension has reduced the yield value in compression and increased it for tension. This
phenomenon is called the Bauschinger effect, after J. Bauschinger\(^1\) who first studied it.

A simplification can be obtained if it is assumed that \(\gamma\) is constant. In this case a bilinear stress-strain curve is obtained, as in Fig. 5.4. On portion AB, or when unloading occurs, the stress-increment \(\Delta \sigma_1\) is related to the strain-increment \(\Delta \varepsilon_1 = \Delta \varepsilon_1^e\) by Hooke's law, so that

\[\Delta \sigma_1 = E \Delta \varepsilon_1.\]  \hspace{1cm} (5.6)

On the other hand, on portion AA' or BB', the strain-increment \(\Delta \varepsilon_1\) is given by

\[\Delta \varepsilon_1 = \Delta \varepsilon_1^p + \Delta \varepsilon_1^e\]

\[= \gamma \frac{\Delta \sigma_1}{E} + \Delta \sigma_1 \left(\frac{\gamma + 1}{E}\right)\]  \hspace{1cm} (5.7)

so that

\[\Delta \sigma_1 = \frac{E}{\gamma + 1} \Delta \varepsilon_1.\]  \hspace{1cm} (5.8)

**TIME-SCALE IN PLASTICITY**

The process of plastic deformation, as discussed above, does not depend on time, although one speaks of "time rate of change". Equation (5.4), for example, is written in terms of the time rate of change of strain and stress. A close examination, however, shows that, since the equation is homogeneous with respect to time, the time-scale does not play any role, and hence, "time" is just a parameter which can be replaced by any other monotonically increasing (or decreasing) parameter. For example, if one writes \(\varphi = f(t)\) where \(\varphi\) is a new parameter, and where \(f\) is a monotonic function (either increasing with increasing \(t\), or decreasing

---

\(^1\) Bauschinger, Johann (1833-1893), German material scientist who studied experimentally many engineering materials.
Figure 5.4

The Stress-Strain Diagram for Bilinear Elastic-Plastic Model: The initial loading, or unloading takes place elastically. Plastic flow occurs on AA' or B'B'B''.
with increasing \( t \), but not both), then it follows that

\[
\frac{d}{dt} = \frac{df}{dt} \frac{d}{d\beta}, \quad \frac{df}{dt} \neq 0,
\]

and hence (5.5) can be written as

\[
\frac{\varepsilon L^{P}}{d\beta} = \gamma \frac{d\sigma}{d\beta} E.
\]

Now this equation is expressed in terms of \( \beta \)-rate of change of the corresponding quantities.

From the physical point of view, whether the time-scale should or should not enter the considered models, depends on the material in question and on the loading situation. Most solids (for that matter, most materials) more or less "flow" under sustained loads. This relates to their "viscous" property which is discussed later on. Experience shows that the plasticity of most metals can be approximated with reasonable accuracy without involving the viscous effects. However, when one considers very low rates of loading or very high ones such as those in explosive loading, then one may have to include the rate effects, and deal with viscoplastic models.
4.6 LINEAR VISCO-ELASTICITY

It is an experimental fact that solids creep under large sustained loads, although the rate at which this deformation takes place may be slow. The viscous flow and the viscosity associated with it under shearing stresses in gases, liquids, and solids, will be discussed in Secs. 3.8 and 3.9, in some detail. Here the visco-elasticity of solids in uniaxial tension or compression is considered; for example, in the tension test discussed before. Attention will be focused on cases where strains are so small that the second and higher order terms in \( \varepsilon_1 \) are negligible compared to \( \varepsilon_1 \). Moreover, in contrast with plasticity theory which is inherently nonlinear even for the small strain case, here constitutive relations for linear visco-elastic models are presented.

Basic features which characterize a visco-elastic material are (1) the instantaneous elasticity and (2) the subsequent viscous flow. There are two tests which can be used to characterize visco-elastic materials. One is the creep test, and the other, the relaxation test. In the first case the specimen is subjected at the initial time \( t = t^0 \), to a suddenly applied stress whose magnitude is thereafter held fixed at a constant level. The associated strain is then measured as a function of time. In the second case, at initial time \( t^0 \) the specimen is subjected to a suddenly imposed strain whose magnitude is thereafter held fixed at a constant level. The induced stress is then measured as a function of time.

UNIT FUNCTIONS

To give analytic expression to the two above-mentioned tests, it is
convenient to introduce the unit-step function, or Heaviside's unit function \( H(t - t^0) \) defined by

\[
H(t - t^0) = \begin{cases} 
0 & \text{for } t < t^0 \\
1 & \text{for } t \geq t^0
\end{cases}
\]  \( (6.1) \)

Fig. 6.1a gives the graph of this function. Another function which will be useful in this study, is the unit-impulse function, or the Dirac\(^1\) delta function \( \delta(t - t^0) \). This is a singularity function with unbounded magnitude at \( t = t^0 \) over a vanishingly small interval about this point, in such a manner that, for every real number \( \tau > 0 \) one has

\[
\int_{t^0 - \tau}^{t^0 + \tau} \delta(t - t^0) dt = 1
\]  \( (6.2) \)

For example, \( \delta(t) \) may be defined in terms of \( H(t) \) with the following limiting process:

\[
\lim_{\tau \to 0} \frac{H(t) - H(t - \tau)}{\tau} = \frac{d}{dt} H(t)
\]  \( (6.3) \)

see Fig. 6.1b.

From the definition of the delta function one has, for, say, a continuous function \( f(t) \),

\[\]

\(^1\) Dirac, Paul Adrien Maurice (1902- ), British theoretical physicist, winner of the Nobel Prize (jointly with Erwin Schrödinger) in 1933 for his work in quantum mechanics.
\[
\int_{t^0 - \tau}^{t^0 + \tau} f(t) \delta(t - t^0) \, dt = f(t^0) \quad (6.4)
\]
for all finite, positive \( \tau \).

With these preliminaries out of the way, let us proceed to discuss various simple constitutive relations for linear visco-elasticity.

---

Figure 6.1

(a) The unit-step function, or Heaviside's step function \( H(t - t^0) \) is identically zero for \( t < t^0 \), and is equal to the unity for all \( t \geq t^0 \).

(b) The unit-impulse, or the Dirac-delta function \( \delta(t) \) may be viewed to represent a rectangular pulse of height \( \frac{1}{\tau} \) and width \( \tau \), as \( \tau \) approaches zero. This yields \( \delta(t) = \lim_{\tau \to 0} \frac{H(t) - H(t - \tau)}{\tau} = \frac{d}{dt} H(t) \).
CREEP TEST

Let at time $t = 0$ the test specimen be subjected to a step-load such that the corresponding stress is defined by

$$
\sigma_1(t) = \sigma_0 H(t) ;
$$

(6.5)

see Fig. 6.2a. For visco-elastic materials the instantaneous response is elastic, corresponding to a sudden increase of the strain from zero to $\frac{\sigma^o}{E}$. This is shown in Fig. 6.2b by the ordinate of point A. Since the stress level is maintained at the constant value $\sigma^o$, the strain $\varepsilon_1(t)$ will slowly increase as the material creeps. This is shown by the curve ABC. The analytic representation (approximation) of this curve, for $\sigma^o = 1$, is called the creep function of the visco-elastic material. The creep function will be denoted, by $c(t)$. Observe that the creep function is identically zero for the value of its argument less than zero, and at $t = 0^+$ one has $c(0^+) = \frac{1}{E}$. If the test begins at time $t^o$, the creep function becomes $c(t - t^o)$, so that for $t < t^o, c(t) \equiv 0$, and for $t = t^0^+$, $c(t^0^+-t^o) = c(0^+) = \frac{1}{E}$.

Suppose that at time $t^o$ the load is suddenly removed. Experiment shows that the visco-elastic material responds in the manner shown by curve BD so that, as time goes on, there is a certain amount of recovery (this is often called the delayed elasticity or elastic after-effect), and the material usually settles down with a residual strain $\varepsilon_1^r$. Note that for certain materials such as rubber, there may be an instantaneous

---

1 $t = 0^+$ if point 0 is approached along the $t$-axis from its right. Similarly, $t = t^0^+$ if $t^o$ is approached from its right. On the other hand, $t = t^0^-$ if $t^o$ is approached from its left.
Creep Test: (a) At $t = 0$ a step-stress of magnitude $\sigma^0$ is applied and maintained up to $t = t^0$, at which time the load is removed.

(b) The visco-elastic specimen has an instantaneous elastic response at $t = 0$, given by $\frac{\sigma^0}{E}$ (point A). This is followed by creeping along AB. After the removal of the load at $t = t^0$, there is a certain amount of recovery, curve BD, and the material may settle down with a residual strain $\varepsilon_1^r$. In certain materials, the unloading may result in some instantaneous elastic recovery, shown by the dashed curve.
recovery that is then followed by a slow recovery; the former is called the true elasticity.

**RELAXATION TEST**

Here the specimen is suddenly deformed by imposing a constant strain \( \varepsilon^0 \) at time \( t = 0 \), so that

\[
\varepsilon_1(t) = \varepsilon^0 H(t); \tag{6.6}
\]

see Fig. 6.3a. The instantaneous response is elastic, and hence, the stress is suddenly brought to the level of \( E \varepsilon^0 \) which corresponds to point A in Fig. 6.3b. Subsequently to this the material relaxes, and hence, the stress level decreases along the curve ABC. The analytic representation (approximation) of this curve, for \( \varepsilon^0 = 1 \), is called the relaxation function of the considered material. The relaxation function will be denoted by \( r(t) \).

If at time \( t^0 \) a strain of \(-\varepsilon^0\) is imposed on the material, there will occur an instantaneous elastic response, so that the stress would jump from point B to point \( B' \) in Fig. 6.3b. Thereafter the material relaxes along the curve \( B'D \).

It should be noted that the relaxation function \( r(t) \), similarly to the creep function, is identically zero for the value of its argument less than zero. When its argument is equal to \( 0^+ \), it has the value \( E \). If, instead of \( t = 0 \), the test is done at the instant \( t = t^0 \), the relaxation function becomes \( r(t - t^0) \) which has nonzero values for \( t \geq t^0^+ \).

**BOLTZMANN’S SUPERPOSITION PRINCIPLE**

Although the strains in the creep or relaxation test discussed above may be so small that linearized equations can be used, the response may not necessarily be linear in the sense that the superposition of two causes
Figure 6.3

Relaxation Test: (a) At \( t = 0 \) a step-strain of magnitude \( \varepsilon^0 \) is imposed and maintained up to \( t = t^0 \), at which time a strain of \(-\varepsilon^0\) is added.

(b) The visco-elastic specimen has an instantaneous elastic response \( E\varepsilon^0 \) (point A). This is followed by relaxation along AB. After the addition of strain \(-\varepsilon^0\), there is an instantaneous elastic response to point B', which then is followed by the relaxation along B'D.
would imply the superposition of their effects. Suppose that in the creep test the stress level \( \sigma^0 \) is replaced by \( k\sigma^0 \), where \( k \) is a constant. The corresponding strain \( \varepsilon_1(t) \) may not necessarily just change to \( k\varepsilon_1(t) \). In general it will be a totally different function of time. Similar remarks apply in the case of the relaxation test.

Suppose a test is performed in which the stress \( \sigma_1^{(1)}(t) \) is prescribed as a function of time, and the response \( \varepsilon_1^{(1)}(t) \) is measured. Moreover, suppose that the response to another imposed stress \( \sigma_1^{(2)}(t) \) is \( \varepsilon_1^{(2)}(t) \). The visco-elastic material is said to follow Boltzmann's superposition principle if, for a combined loading defined by

\[
\alpha \sigma_1^{(1)}(t) + \beta \sigma_1^{(2)}(t), \quad \alpha \text{ and } \beta = \text{ constants},
\]

the response is given by

\[
\alpha \varepsilon_1^{(1)}(t) + \beta \varepsilon_1^{(2)}(t).
\]

Similar statements apply when the relaxation test is considered.

The material shall be referred to as linearly visco-elastic if to within the needed accuracy it follows Boltzmann's superposition principle.

The linearly visco-elastic material is completely characterized by its creep and relaxation functions which can be used to give analytic expression to any other sequence of loading or deformation. As an example, consider in the tension test a stress of magnitude \( \Delta \sigma_1^{(1)} \) applied at time \( t = \tau_1 \), and maintained thereafter; thus the stress history corresponding to this loading is \( \Delta \sigma_1^{(1)} \mathbf{H}(t - \tau_1) \). The resulting strain is then given by

\[
\Delta \varepsilon_1^{(1)}(t) = \Delta \sigma_1^{(1)} c(t - \tau_1).
\]

(6.7)
Suppose that at time \( t = \tau_2 \) an additional loading in the form of \( \Delta \sigma_1^{(2)} H(t - \tau_2) \) is applied. If this were applied independently, the resulting strain would be

\[
\Delta \varepsilon_1^{(2)}(t) = \Delta \sigma_1^{(2)} c(t - \tau_2) .
\] (6.8)

According to Boltzmann's superposition principle, the application of both loadings results in a strain history equal to the sum of the right-hand sides of (6.7) and (6.8).

Consider now a sequence of incremental loading \( \Delta \sigma_1^{(1)}, \Delta \sigma_1^{(2)}, \Delta \sigma_1^{(3)}, \ldots, \Delta \sigma_1^{(n)} \), applied, respectively, at times \( \tau_1, \tau_2, \tau_3, \ldots, \tau_n \). The resulting strain at time \( t \) then will be

\[
\varepsilon_1(t) = \sum_{i=1}^{n} \Delta \sigma_1^{(i)} c(t - \tau_i) .
\] (6.9)

When the applied stress varies continuously with time, beginning, say, at \( t = 0 \) as shown in Fig. 6.4, it may be approximated by a sequence of step-functions with amplitude \( \Delta \sigma_1^{(1)}, \Delta \sigma_1^{(2)}, \ldots \), which, respectively, begin at times \( \tau_1, \tau_2, \ldots \). As the amplitude of these steps is reduced, the time intervals \( \tau_1, \tau_2 - \tau_1, \ldots \), approach zero, and instead of the summation on the right-hand side of (6.9), the following integral representing strain at time \( t \), will result:

\[
\varepsilon_1(t) = \int_{0^-}^{t} \frac{d\sigma_1(\tau)}{d\tau} c(t - \tau) d\tau .
\] (6.10)

Observe that in this equation the incremental stress-step at time \( \tau \) is represented by \( \frac{d\sigma_1(\tau)}{d\tau} d\tau \). Corresponding to this stress-increment, the

\[\text{Note that this integral must be taken from } 0^- \text{ (rather than } 0 \text{) to } t \text{ so that the instantaneous elastic response is included.}\]
Figure 6.4

A continuous loading defined by AB is divided into a sequence of step-loads. As the step-length is made smaller and smaller, by Boltzmann's super position principle, one obtains the response integral (6.10) for linearly visco-elastic materials.
increment of strain at time \( t \), \( d\varepsilon_1 \), is \( \frac{d\sigma_1(t)}{dt} c(t - \tau) d\tau \), as is evident by comparison with (6.7) or (6.8). Integration of \( d\varepsilon_1 \) from \( t = 0^- \) to time \( t \) now yields (6.10).

A similar remark applies to the case where the strain \( \varepsilon_1(t) \) is prescribed as a function of time together with the relaxation function. In this case the stress-increment \( d\sigma_1 \) at time \( t \) corresponding to a strain-increment \( \frac{d\varepsilon_1(\tau)}{d\tau} \) \( d\tau \) is given by

\[
d\sigma_1 = \frac{d\varepsilon_1(\tau)}{d\tau} r(t - \tau) d\tau .
\]

The integration of both sides from \( t = 0^- \) to \( t \) then gives

\[
\sigma_1(t) = \int_{0^-}^{t} \frac{d\varepsilon_1(\tau)}{d\tau} r(t - \tau) d\tau . \tag{6.11}
\]

Observe that, since both the relaxation and the creep functions are identically zero when their arguments take on negative values, the lower integration limit in (6.11) and (6.10) can be replaced by \( -\infty \).

There are various simple models (constitutive relations) that are often discussed as idealizations for certain visco-elastic behavior. These models lend themselves to simple analytic expressions for the creep and relaxation functions. In the sequel a brief discussion of some of these constitutive relations is presented.

**MAXWELL'S 1 MODEL**

Perhaps the simplest visco-elastic model would be the one whose creep function is linear in time, so that

\[
c(t) = \frac{1}{E} \left(1 + \frac{t}{\tau}\right) H(t) , \quad t = \text{constant.} \tag{6.12}
\]
Figure 6.5

(a) Maxwell's model for visco-elasticity can be visualized as a linear spring with spring constant $E$ and a dashpot with viscosity constant $\eta$, which are placed in series.

(b) The Creep Function for Maxwell Model: The applied stress is defined by $\sigma_1 = H(t) - H(t - t^0)$.

(c) The Relaxation Function for Maxwell's Model: The imposed strain is $\varepsilon_1 = H(t)$. 
The graph of this function is shown in Fig. 6.5b\(^1\). In (6.12) the constant \(\tau\) is called the relaxation time. At time \(t = 0^+\) the strain according to (6.12) is \(\frac{1}{E}\) which corresponds to the instantaneous elastic response to the stress \(\sigma_\parallel = 1\) that is applied then, and maintained thereafter. Subsequently to this, the material flows at a constant rate. The strain history for \(\sigma_\parallel = \sigma^0 H(t)\) is then given by

\[
\varepsilon_\parallel(t) = \frac{\sigma^0}{E} \left( 1 + \frac{t}{\tau} \right) H(t).
\]  

(6.13)

Equation (6.13) is the solution of the following linear ordinary differential equation:

\[
\dot{\varepsilon}_\parallel = \frac{\sigma_\parallel}{E} + \frac{\sigma_\perp}{\eta}, \quad \varepsilon_\parallel(0^+) = \frac{\sigma^0}{E},
\]  

(6.14)

where \(\eta = E\tau\) is, like \(E\), a material constant so that the relaxation time is given by

\[
\tau = \frac{\eta}{E},
\]  

(6.15)

and where the applied stress is

\[
\sigma_\parallel(t) = \sigma^0 H(t).
\]

Hence \(\dot{\varepsilon}_\parallel = \sigma^0 \frac{dH(t)}{dt} = \sigma^0 \delta(t)\), and (6.14) yields

\[
\dot{\varepsilon}_\parallel = \frac{\sigma^0}{E} \left( \delta(t) + \frac{H(t)}{\tau} \right).
\]  

(6.16)

which upon integration with respect to time reduces to (6.13), provided the initial condition \(6.14\) is noted.

Equation (6.14) characterizes the constitutive relation for Maxwell's material. It states that at each instant \(t\), after the application of stress, the strain-rate will consist of two parts: one due to the elastic

\(^1\) In this figure the stress is removed at \(t = t^0\).
response of the material, given by
\begin{equation}
\varepsilon_1^e = \frac{\dot{\sigma}}{E}, \tag{6.17}
\end{equation}
the other due to its viscous response, given by
\begin{equation}
\varepsilon_1^v = \frac{\sigma}{\eta}, \tag{6.18}
\end{equation}
where the constant \( \eta \) is the coefficient of viscosity. As is discussed in Sec. 4.8, Eq. (6.18) is similar to the constitutive relation for a viscous fluid which is commonly referred to as the Newtonian fluid after Newton who first studied it. In the case of the Newtonian fluid, however, it is the shear stress and the corresponding shear strain-rate that are related linearly in the manner defined by (6.18).

In literature various mechanical models are often discussed as visual and pedagogical aids for understanding the mechanics of visco-elastic phenomena. A spring with its spring constant equal to the elastic modulus \( E \), for example, is used to represent the elasticity of the material, and a dashpot with its coefficient equal to the coefficient of viscosity \( \eta \), is employed to represent the viscous effect of the material, Fig. 6.5a. The spring, being perfectly elastic, has an instantaneous elastic response so that upon the application of \( \sigma^0 \), it springs to the instantaneous strain \( \varepsilon_1^e = \frac{\sigma^0}{E} \). The dashpot, on the other hand, locks (is rigid) when subjected to an instantaneous load, but under a sustained load flows at the rate of \( \varepsilon_1^v = \frac{\sigma^0}{\eta} \). The Maxwell visco-elastic material, therefore, may be visualized by a spring and a dashpot in series, as shown in Fig. 6.5a.

In the above discussion, the word model is used in two distinctly different senses, one in connection with a constitutive relation modelling (approximately) the mechanical property of a certain class of materials,
and the other in connection with a certain visual aid (a mechanical model consisting of a spring and a dashpot) representing exactly the differential equation which defines the constitutive relation of the ideal material. While the latter mechanical model is a purely pedagogic device having nothing to do with the actual material behavior, the former model represents one's knowledge of the actual material properties, and hence is an essential part of the study. The student therefore should keep this distinction in mind, since it proves convenient to consider various mechanical models (consisting of springs and dashpots) in order to visualize more clearly the mechanical behavior of idealized materials which are used to model the actual material properties.

The relaxation function of Maxwell's material can be obtained from (6.14), if one sets

\[ e_1(t) = H(t) \quad (6.19) \]

and obtains

\[ \frac{r}{E} + \frac{r}{\eta} = \delta(t) \quad , \quad r(0^+) = E \quad . \quad (6.20) \]

Upon integration, (6.20) now yields

\[ r(t) = E e^{-\frac{r}{E} H(t)} \quad (6.21) \]

which is an exponentially decaying function of time; \( e \) in this equation is the Euler number, \( e \approx 2.7183 \), so that \( \ln e = 1 \), where \( \ln \) stands for the natural logarithm.

Note that, for a given, say of class \( C^1 \), stress history \( \sigma_1(t) \), the

\[ \text{A function is said to be of class } C^n \text{ in a given domain if the function together with its first } n \text{ derivatives, including the } n\text{th derivative, is continuous in that domain.} \]
corresponding strain history is defined in terms of the creep function by

\[ \varepsilon_1(t) = \frac{1}{E} \int_0^t \left( 1 + \frac{t - \tau}{t} \right) \frac{d\sigma_1(\tau)}{d\tau} \, d\tau \]

\[ = \frac{1}{E} \left[ \sigma_1(t) - \left( 1 + \frac{t}{\ell} \right) \sigma_1(0^+) \right] + \frac{1}{\eta} \int_0^t \sigma_1(\tau) \, d\tau . \]  \hspace{1cm} (6.22)

Here (6.18) and the integration by parts are used; the latter is defined by

\[ \int v \, du = uv - \int u \, dv \]  \hspace{1cm} (6.23)

which is obtained by integrating \( d(vu) = vdu + udv \). Figure 6.5c is the graph of the relaxation function (6.21).

**Voigt's Model**

In terms of the spring-dashpot representation, Voigt's model which is also commonly referred to as Kelvin's model, can be characterized by a spring and a dashpot placed parallel to each other, as in Fig. 6.6a. To obtain the differential equation of the constitutive relation of the corresponding ideal material, observe that the total stress \( \sigma_1 \) at each instant consists of an elastic part \( E \varepsilon_1 \) carried by the spring, and a viscous part \( \eta \dot{\varepsilon}_1 \) carried by the dashpot, the total strain at each instant being the same for the spring and the dashpot. It follows that

\[ \sigma_1(t) = E \varepsilon_1(t) + \eta \dot{\varepsilon}_1(t) , \hspace{0.5cm} \varepsilon_1(0^+) = 0 . \]  \hspace{1cm} (6.24)
(a) Voigt's model for visco-elasticity can be visualized as a linear spring with spring constant $E$ and a dashpot with viscosity constant $\eta$, which are placed parallel to each other.

(b) The Creep Function for Voigt's Model: The applied stress is defined by $\sigma_1 = H(t) - H(t - t^0)$.

(c) The Relaxation Function for Maxwell's Model: The imposed strain is $\epsilon_1 = H(t)$. 

Figure 6.6
Observe that, since the dashpot responds rigidly to an instantaneously applied stress, the initial strain for Voigt's model is zero, even though the initial stress may be nonzero.

The creep function for Voigt's model is obtained if $\sigma_1(t)$ is set equal to $H(t)$, arriving at

$$c(t) = \frac{1}{E} \left( 1 - e^{-\frac{t}{E}} \right) H(t)$$

(6.25)

which is the solution of (6.24) with $\sigma_1$ replaced by $H(t)$. The graph of this function is shown in Fig. 6.6b. In Fig. 6.6b the load is removed at time $t = t^0$. There is then a continuous recovery, as the spring tends to its original undeformed configuration, pulling the dashpot with itself.

In Fig. 6.6b this is shown by curve AB. If this material is then left alone, the curve AB eventually touches the horizontal t-axis.

The relaxation function for Voigt's model is

$$r(t) = EH(t) + \eta \delta(t)$$

(6.26)

which is deduced from (6.24) by substitution of $H(t)$ and $\delta(t)$ for $\dot{\varepsilon}_1$ and $\varepsilon_1$, respectively. The graph of (6.26) is shown in Fig. 6.6c. The instantaneous response here is a sharp spike characterizing the $\delta$-function, and the subsequent constant value $E$ stemming from the step-function.

**STANDARD SOLID**

The previous two models each have a deficiency. The Maxwell model represents materials which flow indefinitely, whereas Voigt's model has no instantaneous elastic response. Most solids respond elastically under instantaneous loading, and may flow (creep) under sustained forces.

To represent these phenomena ideally, consider the mechanical model
(a) The standard solid.

(b) The Creep Function for the Standard Solid: The stress is
\[ \sigma_1 = H(t) - H(t - t^0). \]

(c) The relaxation function for the standard solid.
shown in Fig. 6.7a, relating to the so-called standard solid with the following constitutive relation:

\[
\sigma_1 + t^* \dot{\varepsilon}_1 = E(\varepsilon_1 + t \dot{\varepsilon}_1), \quad t^* \sigma_1(0^+) = E t \varepsilon_1(0^+), \tag{6.27}
\]

where \( t^* \) and \( t \) are two material constants. To obtain (6.27) note that the mechanical model in Fig. 6.7a consists of a Maxwell model (at the bottom) placed parallel to a spring (at the top). Hence, the total stress, \( \sigma_1 \), is equal to the sum of the part, \( \sigma^* \), carried by the Maxwell model which, from (6.14), is defined by \( \varepsilon_1 = \sigma^* \varepsilon_1^* + \sigma^* \varepsilon_1^* \), and the other part, \( \sigma = E \varepsilon_1 \), arising by the spring at the top. From the fact that the total stress \( \sigma_1 \) must be equal to \( \sigma^* + \sigma \), now obtain

\[
\sigma_1 + \frac{\eta}{E^*} \dot{\varepsilon}_1 = E \left[ \varepsilon_1 + \left( \frac{\eta}{E} + \frac{\eta}{E^*} \right) \dot{\varepsilon}_1 \right].
\]

Hence, set \( t^* = \frac{\eta}{E^*} \) and \( t = \frac{\eta}{E} + \frac{\eta}{E^*} \), and readily arrive at (6.27).

Following the same procedure as in the case of the preceding two models, one obtains the creep and relaxation curves for the standard solid; they are defined by

\[
c(t) = \frac{1}{E} \left[ 1 - \left( 1 - \frac{t}{t^*} \right) e^{-\frac{t}{t^*}} \right] H(t), \tag{6.28}
\]

\[
r(t) = E \left[ 1 - \left( 1 - \frac{t}{t^*} \right) e^{-\frac{t}{t^*}} \right] H(t). \tag{6.29}
\]

The graphs of these functions are sketched in Figs. 6.7b and 6.7c, respectively.

From the above discussion it should now be clear that more complicated models can readily be constructed by the combination of springs and dashpots connected appropriately. In this manner one can construct constitutive relations for ideal linearly visco-elastic materials, which can fit a given set of actual experimental data in creep or relaxation tests.
EXAMPLE 6.1

Assume that the force $F$ in Example 4.1 is applied instantaneously, and that the wires consist of (a) the Maxwell model; (b) the Voigt model; and (c) the standard solid model materials. Find the corresponding displacement $\Delta$ as a function of time.

$\Delta$ is related to the common strain $\varepsilon_1$ in the wires by (see Eq. (4.8))

$$\Delta = \frac{\delta}{\cos \theta} = \varepsilon_1 \frac{L}{\cos \theta}.$$  \hspace{1cm} (6.30)

(a) For the Maxwell model, from (6.12), we obtain

$$\Delta(t) = \frac{L}{\cos \theta} \cdot \frac{F}{2A\cos \theta} \cdot \frac{1}{E} \left(1 + \frac{t}{\tau}\right) H(t)$$

$$= \frac{LF}{2A\cos^2 \theta} \cdot \frac{1}{E} \left(1 + \frac{t}{\tau}\right) H(t),$$  \hspace{1cm} (6.31)

where $t = \eta/E$.

(b) For the Voigt model, from Eq. (6.25), we have

$$\Delta(t) = \frac{LF}{2A\cos \theta} \cdot \frac{1}{E} \left(1 - e^{-\frac{t}{\tau}}\right) H(t).$$  \hspace{1cm} (6.32)

(c) For the standard solid (6.28) yields

$$\Delta(t) = \frac{LF}{2A\cos^2 \theta} \cdot \frac{1}{E} \left[1 - \left(1 - \frac{t^*}{\tau}\right)e^{-\frac{t}{\tau}}\right] H(t),$$  \hspace{1cm} (6.33)

where $t^* = \frac{\eta}{E^*} + \frac{\eta}{E^*}.$

EXAMPLE 6.2

Consider Example 6.2, Fig. 4.3a, and when the wires consist of (a) the Maxwell material; (b) the Voigt material; and (c) the standard solid material, find the displacement $\Delta$ for $F(t) = F H(t)$. Assume all three wires have the same creep function, $c(t)$.

Equation (4.9) remains valid because it represents the force-equilibrium condition. For $\Delta$, we have
\[ \Delta = \frac{\varepsilon_1 L}{\cos \theta} = \frac{\varepsilon_1 L'}{\cos \theta'} = \varepsilon_1' L'. \quad (6.34) \]

In terms of the creep function \( c(t) \), assumed to be the same for all three wires, we have

\[ \varepsilon_1(t) = \frac{S}{A} c(t), \quad \varepsilon_1'(t) = \frac{S'}{A'} c(t). \quad (6.35) \]

Hence, from (4.9) and (6.34), we obtain

\[ \Delta(t) = FL' c(t) / \left[ 2A\cos \theta + A' \right]. \quad (6.36) \]

To find \( \Delta(t) \) associated with each model, we simply substitute for \( c(t) \) the corresponding creep function; i.e., from Eqs. (6.12), (6.25), and (6.28), respectively.

**EXAMPLE 6.3**

In an actual test, the strain cannot be imposed on the test specimen instantaneously. Suppose that the actual strain-history is as shown in Fig. 6.8. Find the corresponding response for the Maxwell, Voigt, and the standard solid models.

For the Maxwell model,

\[ \dot{\varepsilon}_1 = \frac{\sigma_1}{E} + \frac{\sigma_1}{\eta} , \quad \varepsilon_1(0^+) = 0 , \quad 0 \leq t \leq t_1 , \quad (6.37) \]

where \( \dot{\varepsilon}_1 = \varepsilon_0 / t_1 \). Hence we obtain

\[ \sigma_1(t) = \frac{\eta \varepsilon_0}{t_1} \left( 1 - e^{-\frac{t}{t_1}} \right) \quad \text{for} \ 0 \leq t \leq t_1. \quad (6.38) \]

At time \( t = t_1 \), \( \varepsilon_1(t_1) = \varepsilon_0 \), and \( \dot{\varepsilon}_1(t) = 0 \) for \( t_1 < t \leq t_2 \). Integration of (6.37) for these conditions now yields
\[ \sigma_1(t) = \frac{n\varepsilon_0}{t_1} \left[ 1 - e^{-\frac{t_1}{t}} \right] e^{-\frac{t-t_1}{t}} \text{ for } t_1 < t \leq t_2. \] (6.39)

At \( t = t_2 \), the strain-rate becomes \( \dot{\varepsilon}_1 = -\frac{\varepsilon_0}{t_3-t_2} \). With \( \varepsilon_1(t_2^+) = \varepsilon_0 \) we obtain

\[ \sigma_1(t) = n \varepsilon_0 \left[ \frac{1}{t_1} (1 - e^{-\frac{t_1}{t}}) e^{-\frac{t_2-t_1}{t}} + \frac{1}{t_3-t_2} \right] e^{-\frac{t-t_2}{t}} - \frac{n\varepsilon_0}{t_3-t_2} \text{ for } t_2 < t \leq t_3. \] (6.40)
4.7 **FINITE ELASTICITY**

Certain metals in the form of thin fibers ($10^{-4}$ cm in diameter) can be strained **elastically** although non-linearly, up to almost 10%. However, nonlinear elasticity manifests itself most remarkably in polymers such as rubber, where the microscopic structure of the material consists of molecules in long chains\(^1\).

Figure 7.1 gives a typical stress-extension diagram of vulcanized (see below for definition) rubber, where the nominal stress $T^R_1$ (force divided by the initial area of the specimen) is plotted against extension $\varepsilon_1$. The extensibility of this material can be as high as 1,000%. The initial slope of the stress-extension curve is of the order of 100 to 500 lbf/in\(^2\), as compared to, say, that of steel, which is about $30 \times 10^6$ lbf/in\(^2\), and for which the elastic limit usually corresponds to about 0.1% extension.

The term rubber is usually used to designate both **natural** rubber which is polyisoprene with two distinct molecular structures, and the various synthetic polymeric products that have rubber-like properties.

The basic unit or **monomer** of a chain in a polymer, links (has chemical bonds with) at least two of its neighbors on either side, and this structure may be repeated 100 to 10,000 times. The molecular weight, hence, depends on the chain lengths, admitting wide variation. Natural rubber, for example, consists of flexible long chains (average molecular weight of about 350,000) with the following two structures:

---

Figure 7.1

A typical nominal stress versus extension diagram for rubber-like material (not an actual test result).
The rubber with the first structure is commonly referred to by the name *Hevea*, and has a spiral, whereas the second one which is called *gutta percha*, has a zig-zag molecular form. The chains are occasionally cross-linked, but are mostly tangled loosely together in an irregular manner. The double bond in carbon readily admits chemical bonding with sulfur, yielding **vulcanized rubber** with cross-links whose density can be controlled, which results in the products with different mechanical properties.

Polymers without cross-links such as rubber **latex** are viscous fluids. The viscosity stems from the sliding of chains past each other, and decreases with increasing temperature. At sufficiently low temperatures, the chains slide so sluggishly that the substance becomes a glass, whereas at elevated temperatures it flows like a Newtonian fluid (see next section).

Other polymeric materials may have numerous cross-links which give them hard form even at high temperatures, until they chemically disintegrate.

The long chains whether cross-linked or not, give the polymeric material substantial structural integrity. In a long-chain molecule the
individual elements have vibrational kinetic energy (thermal motion), but they are constrained in their motion along the chain by the other constituents (interatomic forces keeping atoms the integral part of the chain) in the chain, while there is a great deal of flexibility (easy rotation at each bond) with respect to lateral motion which is constrained only by the weak intermolecular (secondary) forces like in ordinary liquids. The weak intermolecular forces permit random rotation at various links of the chains. Thus, the material has certain liquid-like properties. For example, most rubbers are highly resistant to a change in volume. In fact, compared with their extensibility which is manifested by their very low elastic modulus mentioned before, they can be regarded as **incompressible**.

When loaded, these materials usually manifest a "true" instantaneous elasticity which corresponds to the interatomic forces developed in the same manner as discussed before, and a **high** elasticity which corresponds to the straightening or crumpling of the chains. This elasticity is usually followed by the viscous flow of the material, which pertains to the sliding of the chains past one another. In highly cross-linked polymers, this latter viscous effect can be negligibly small.

In many useful applications the elasticity of rubbery materials can be discussed separately from its viscous behaviour, either because this latter is negligibly small and manifests itself only under sustained loads, or because short-time response is of interest. In the following, therefore, constitutive relations which can be used as ideal models for elasticity at large strains are considered. To make the discussion
clear, however, small strain, but nonlinear elasticity is first briefly discussed.

**SMALL STRAIN NONLINEAR ELASTICITY**

For certain metals in the form of fine fibers, for example, while the strain (extension $\varepsilon_1$) remains sufficiently small (for example, less than 0.03 to 0.05), the stress-strain, $\sigma_1 - \varepsilon_1$, curve may be nonlinear; Fig. 7.2. Hence, one may continue neglecting second and higher order terms in $\varepsilon_1$, but deal with a nonlinear constitutive relation, i.e. nonlinear relation between $\sigma_1$ and $\varepsilon_1$. If the material is indeed elastic, then at point A in Fig. 7.2, the unloading would correspond to retracing the curve OA. This means that the entire strain-energy that is stored in the specimen at the deformed state corresponding to point A, is retrievable by unloading. As was discussed in Section 4.4, the area under the curve in this figure represents the mechanical work done per unit initial volume (which, for small strains, is almost the same as the final one), to deform the specimen to a given state.

The stress-strain relation may be written as

$$\sigma_1 = \sigma_1(\varepsilon_1)$$  \hspace{1cm} (7.1)

which states that $\sigma_1$ is a function of $\varepsilon_1$. It will be assumed that this is a one-to-one continuous relationship. Denote by $\Phi_o(\varepsilon_1)$ the work required per unit volume, to deform the specimen to strain $\varepsilon_1$, and write

$$\int_{\varepsilon_1}^{\varepsilon_1} \sigma_1(\varepsilon_1) d\varepsilon_1 = \Phi_o(\varepsilon_1) - \Phi_o(0)$$

$$= \Phi_o(\varepsilon_1)$$  \hspace{1cm} (7.2)
The $\sigma_1 - \epsilon_1$ Diagram for a Nonlinearly Elastic Material at Small Strains: The loading and unloading takes place along the curve OA. The area under the curve is the energy stored in the specimen of unit volume. Since the material is elastic, all of this energy can be retrieved upon unloading.
where \( \Phi_o(0) \) is set equal to zero without a loss in generality. Note in this equation that an increment in strain-energy-density, \( d\Phi_o \), corresponding to the stress level \( \sigma_1(\varepsilon_1) \), is given by \( d\Phi_o = \sigma_1(\varepsilon_1)d\varepsilon_1 \). Upon integration, this leads to Eq. (7.2). On the other hand from the definition of \( \Phi_o \) it follows that

\[
d\Phi_o = \frac{d\Phi_o}{d\varepsilon_1} d\varepsilon_1
\]

\[
= \sigma_1 d\varepsilon_1 ,
\]

so that

\[
\sigma_1 = \frac{d\Phi_o}{d\varepsilon_1} .
\] (7.4)

Thus, if the material is elastic (although nonlinearly), that is, if the work done to deform the specimen is completely recoverable\(^1\), then there exists a work function called the strain-energy-density, \( \Phi_o(\varepsilon_1) \), from which the stress can be derived in exactly the same manner (except for a minus sign) as conservative body forces are derivable from a potential, see Eq. (3.3) of Ch. 3. In linear elasticity the strain-energy-density becomes

\[
\Phi_o(\varepsilon_1) = \frac{1}{2} E\varepsilon_1^2 ;
\] (7.5)

see Sec. 4.4. Then the stress is given by

\[
\frac{d\Phi_o}{d\varepsilon_1} = E\varepsilon_1
\] (7.6)

which is Hooke's law.

---

\(^1\) This actually can be used as the definition of elasticity.
Consider now the question of nonlinear elasticity with finite strains.

**FINITE STRAIN NONLINEAR ELASTICITY: STRING**

When one deals with finite strains, a clear distinction must be made between initial and final length, area, and volume.

Let us begin with a simple case of an elastic string consisting of a very thin and long prismatic body whose resistance to bending and compression is negligibly small. Let the initial length of the string be $L$, and assume that upon the application of tensile forces $F$ at both its ends, the length increases to $l$. For an incremental change $dl$ of length, the work done by $F$ is $F \, dl$. Denoting by $\Lambda$ the stretch, so that

$$\Lambda = \frac{l}{L} \quad \text{,}$$

(7.7)

and observe that $dl = L \, d\Lambda$. Hence, obtain

$$F \, dl = L \, Fd\Lambda \quad \text{.}$$

(7.8)

The quantity $Fd\Lambda$, therefore, represents the incremental work per unit initial length of the string.

Suppose now that the string is elastic. This means that the energy corresponding to the work done to extend this string from its initial length $L$ to a final length $l$, is stored within the material of the string, and can be completely recovered upon unloading. The quantity $Fd\Lambda$ must, hence, be a complete differential, say, $d\psi = \frac{d\psi}{d\Lambda} \, d\Lambda$, so that

$$\int_{1}^{\Lambda} Fd\Lambda = \int_{1}^{\Lambda} \frac{d\psi}{d\Lambda} \, d\Lambda = \psi(\Lambda) - \psi(1) \quad \text{.}$$

(7.9)
The scalar-valued function \( \psi(\Lambda) \) is the strain-energy-density, and can be assumed, without a loss in generality, to have a zero value at the initial state where \( \Lambda = 1 \).

In the elastic string the force \( F \) which is transmitted across each element of the string along its length, is derivable from a function \( \psi(\Lambda) \), so that

\[
F = \frac{d\psi}{d\Lambda}.
\]  

(7.10)
The form of this function depends on the particular material constitution, and must be established experimentally.

For rubbers, which can be regarded incompressible, the following strain-energy density corresponds to a force-stretch relationship which is in fair agreement with experimental results up to a stretch of about 2 to 2.5:

\[
\psi(\Lambda) = A \frac{G}{2} \left( \Lambda^2 + \frac{2}{\Lambda} - 3 \right),
\]

(7.11)
where \( A \) is the initial cross sectional area of the string, and \( G \) is a material constant. The corresponding force-stretch relation is

\[
F(\Lambda) = A G \left( \Lambda - \Lambda^{-2} \right).
\]  

(7.12)
As \( \Lambda \) becomes large, this equation indicates that \( F \) increases approximately linearly with \( \Lambda \), while from Fig. 7.1 it follows that for most rubbers the force increases sharply with \( \Lambda \), for \( \Lambda \) greater than 4 or 5.

The constitutive relation (7.11) has its basis in a statistical theory for rubbers, in which it is assumed that the long, randomly linked chains of the constituent molecules form a cross-linked network. Using
simple assumptions regarding the statistical distribution of the "chains", which are then defined by the segments of molecules between successive points of cross-linkages, one can show that G in Eq. (7.11) is expressible as

\[ G = Nk^0 \]  
(7.13)

where \( N \) is the number of cross-linking (when only four chains meet at a cross-link) per unit volume, \( k \) is the Boltzmann constant equal to \( 1.380 \times 10^{-23} \text{ J/molecule } ^0\text{K} \), and \( \theta \) is the absolute temperature in \( ^0\text{K} \).

This equation shows that, at least to a first order of approximation, \( G \) is independent of the chemical nature of the molecules as long as these molecules are long enough so that the basic assumptions of the statistical theory are satisfied.

The ideal material with a constitutive relation defined by (7.11) or (7.12) is commonly referred to as \textit{neo-Hookian}.

**FINITE STRAIN NONLINEAR ELASTICITY: TRIAXIAL TEST**

Referring to Sec. 3.4, note that in the simple tension of the prismatic bar of Fig. 4.1, at a given deformed length \( l \) corresponding to the total force \( F_1 \), an increment \( dl \) of length corresponds to the incremental work \( F_1 dl \). The work done to deform this bar from its initial length \( L \) to a final length \( l \), hence becomes

\[ u_1 = \int_0^l F_1 \, dl \]  
(7.14)

\( ^1 \) \textit{op. cit.} Treloar, Ch. IV.
If one observes that the relative stress is defined by

$$T^R_I = \frac{F_1}{A},$$

and that \( dl = L \ d\Lambda_I \), since \( \Lambda_I = \frac{L}{l} \), one readily reduces (7.14) to

$$\mu_1 = AL \int_1^L T^R_I \ d\Lambda_I,$$  \hspace{1cm} (7.15)

where in the undeformed state \( l = L \) and hence \( \Lambda_I = 1 \).

It will be assumed that the material is isotropic. This means that its mechanical properties are the same in all directions. For example, the same nominal stress versus stretch curve would result, if one considers pulling the specimen in the, say, \( X_3X_3 \)-direction by uniformly distributed tensile forces \( F_3 \) applied on the lateral faces of the specimen that are orthogonal to this direction. The student should not be confused by the fact that the prismatic bar in Fig. 4.1 of Sec. 3.4, is shown to be very long. Notwithstanding the sketch in this figure, one could conceive of another specimen shown in Fig. 7.3, which is subjected to uniformly distributed tensile forces in the \( X_3X_3 \)-direction. The nominal stress \( T^R_{III} \) acting on a plane perpendicular to the \( X_3X_3 \)-direction is then given by

$$T^R_{III} = \frac{F_3}{LW},$$  \hspace{1cm} (7.16)

and the corresponding stretch is

$$\Lambda_{III} = \frac{h}{H},$$

so that the increment in height \( dh \) is

$$dh = H \ d\Lambda_{III}.$$  \hspace{1cm} (7.17)

---

1 The reason for the Roman numeral subscript \( I \) is that \( T^R_I \) and \( \Lambda_I \) are the principal stress- and stretch-components.
Figure 7.3

A Triaxial Test of a Rectangular Block with Initial Length $L$, Height $H$, and Width $W$: At the top the block is shown in its initial undeformed state, together with the forces that will act on its lateral faces. At the bottom, only the tractions in the $X_3X_3$-direction are applied.
When only the Force $F_3$ is acting and the other faces of the specimen are free from any applied force (they are traction-free), the work done in increasing the height of the specimen from $h$ to $\bar{h}$ is given by

$$u_3 = \frac{L}{hW} \int_1^\Lambda \frac{R}{III} d\Lambda_{III}.$$  \hspace{1cm} (7.18)

In a similar manner, if the specimen is extended only along its width by uniformly distributed forces $F_2$ that are applied on the lateral faces of the specimen which are perpendicular to the $X_2\bar{X}_2$-direction, the corresponding work becomes

$$u_2 = \frac{L}{hW} \int_1^\Lambda \frac{R}{II} d\Lambda_{II},$$  \hspace{1cm} (7.19)

where $\frac{R}{II} = \frac{F_2}{LH}$, $d\omega = W d\Lambda_{II}$, and where $\omega$ is the final width of the specimen.

Suppose now that we consider a triaxial test in which one applies simultaneously on all three directions the forces $F_1$, $F_2$, and $F_3$, each in the manner defined above, varying their magnitudes from zero to their final values, and hence stretching the specimen in three directions to final values of stretches $\Lambda_1$, $\Lambda_{II}$, and $\Lambda_{III}$, in the $X_1\bar{X}_1$, $X_2\bar{X}_2$, and $X_3\bar{X}_3$-direction, respectively. It is evident that the total work spent to this end will be

$$u = u_1 + u_2 + u_3$$

$$= \frac{L}{hW} \sum_{i=1}^{III} \int_1^{\Lambda_i} \frac{R}{i} d\Lambda_i.$$  \hspace{1cm} (7.20)

Assume now that the material is elastic so that the entire work $u$ can be retrieved upon unloading. Since the specimen may be deformed to its final shape in various ways, for example, one may apply the forces in different order, the work $u$ can only depend on the final values.
of stretches $\Lambda_1^I$, $\Lambda_1^II$, $\Lambda_1^III$. Similarly the integral in the right-hand side of (7.20), which represents the work per unit initial volume, must only depend on the final values of the stretches, and not on the integration paths, that is $\sum_{i=1}^{III} T_i^R d\Lambda_i$ must be an exact differential $d\Psi$, so that

$$\sum_{i=1}^{III} T_i^R d\Lambda_i = d\Psi(\Lambda_1^I, \Lambda_1^II, \Lambda_1^III) = \frac{\partial \Psi}{\partial \Lambda} d\Lambda_1 + \frac{\partial \Psi}{\partial \Lambda_1^II} d\Lambda_1^II + \frac{\partial \Psi}{\partial \Lambda_1^III} d\Lambda_1^III. \quad (7.21)$$

At each stage of loading one may consider an incremental change in one of the stretches, while keeping the values of the other two stretches fixed. Equation (7.21) must remain valid, and

$$T_i^R = \frac{\partial \Psi}{\partial \Lambda_i}, \quad T_1^R = \frac{\partial \Psi}{\partial \Lambda_1^II}, \quad T_1^R = \frac{\partial \Psi}{\partial \Lambda_1^III}. \quad (7.22)$$

This equation states that, if the material is elastic, then the nominal stresses induced in a triaxial test of a rectangular specimen subjected on its lateral surfaces to uniform tractions, are derivable from a strain-energy-density function $\Psi = \Psi(\Lambda_1^I, \Lambda_1^II, \Lambda_1^III)$. The mechanical property of the elastic body is thus characterized by this function whose form must, of course, be established experimentally.

Instead of the nominal stresses, one may deal with the true stresses which are defined as forces measured per unit current area of the faces of the specimen in its current deformed state. If these true stresses are denoted by $T_i^I$, $T_1^II$, and $T_1^III$, acting respectively on faces perpendicular to the $X_1X_1^-$, $X_2X_2^-$, $X_3X_3^-$ directions, from definition (7.15) (and similar
definitions for other stresses) it follows that

\[
\begin{align*}
T^{R}_{I} &= \frac{F}{HW} = \frac{F}{\pi a} = \frac{k}{H} \frac{\alpha}{\beta} \\
T^{R}_{II} &= T^{R}_{I} \Lambda_{I} \Lambda_{II} \Lambda_{III} \\
T^{R}_{III} &= T^{R}_{I} \Lambda_{I} \Lambda_{II} \Lambda_{III} \\
\end{align*}
\]

(7.23)

From (7.23) and (7.22) one deduces that

\[
\begin{align*}
T^{L}_{I} &= \frac{1}{\Lambda_{I}} \frac{\partial \Psi}{\partial \Lambda_{I}} \\
T^{L}_{II} &= \frac{1}{\Lambda_{II}} \frac{\partial \Psi}{\partial \Lambda_{II}} \\
T^{L}_{III} &= \frac{1}{\Lambda_{III}} \frac{\partial \Psi}{\partial \Lambda_{III}} \\
\end{align*}
\]

(7.24)

To reduce these equations to a simple form, multiply the first one by \( \frac{1}{\Lambda_{I}} \), the second by \( \frac{1}{\Lambda_{II}} \), the third one by \( \frac{1}{\Lambda_{III}} \), and note that

\[
J = \Lambda_{I} \Lambda_{II} \Lambda_{III} \\
\]

arriving at

\[
J T_{i} = \frac{\partial \Psi}{\partial \Lambda_{i}} \Lambda_{I}, \quad i = I, II, III, \quad (i \text{ not summed}) \quad (7.25)
\]

Note that this equation stands for three separate ones which are obtained by replacing \( i \) in both sides of (7.25) respectively by \( I, II, \) and \( III \).

As mentioned before, compared with their extensibility, most rubber-like materials can be assumed with reasonable accuracy to be incompressible. Since in this case the volume does not change, from Eqs. (4.2) and (4.3) of Sec. 3.4, it follows that \( J = 1 \). Since in this case no deformation takes place if an additional hydrostatic pressure is applied on the specimen, the stress cannot be uniquely defined in terms of the deformation alone. That is, for given stretches of the prismatic specimen in Fig. 7.3, one may have numerous (infinitely many) states of stress which differ from each other by the hydrostatic pressure \( p \). Hence, for an incompressible elastic material, instead of (7.24) one writes
\[ T_i = -p + \frac{\partial \Psi}{\partial \Lambda_i} \Lambda_i, \quad i = I, II, III, \quad (i \text{ not summed}) \]  

(7.26)

Observe that, since in this incompressible case \( \Lambda_1^\Lambda_1^\Lambda_3 = 1 \), one of the stretches, say, \( \Lambda_3 \), can be expressed in terms of the other two by

\[ \Lambda_3 = \frac{1}{\Lambda_1^\Lambda_2} \]  

(7.27)

and the result substituted into the expression for \( \Psi \). In this manner one obtains

\[ \Psi = \Psi \left( \Lambda_1^\Lambda_2^\Lambda_3, \frac{1}{\Lambda_1^\Lambda_2} \right) \]

\[ = \Psi \left( \Lambda_1^\Lambda_2 \right) \]  

(7.28)

In the literature there are various constitutive relations proposed for finite elasticity. For the so-called neo-Hookean materials which was mentioned before, the strain-energy-density is

\[ \Psi = \frac{G}{2} \left( \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2 - 3 \right) \]  

(7.29)

where, as before, the material is assumed incompressible. Another constitutive relation for rubber-like incompressible materials is the Mooney\(^1\) form, given by

\[ \Psi = K_1 \left( I_C - 3 \right) + K_2 \left( II_C - 3 \right) \]  

(7.30)

where \( I_C = \sum_{i=1}^{I} \Lambda_i^2 \), \( II_C = \sum_{i=I}^{III} \Lambda_i^2 \), and \( K_1 \) and \( K_2 \) are material constants.

A more general form of the strain-energy-density function for an incompressible isotropic elastic material may be defined by the following series:

---

\(^1\) M. J., Mooney, J. Appl. Phys. 11 (1940) 582
\[ \psi = \sum_{i=0}^{n} K_{ij} (I_C - 3)^i (II_C - 3)^j, \]  
(7.31)

where \( I_C \) and \( II_C \) are defined above, \( K_{ij} \)'s are material constants, and \( n \) is to be chosen so that the experimental results can be fitted with a desired accuracy.

In certain situations where the hydrostatic pressure is very high, one may have to take into account the compressibility of the material. For rubber-like materials, a constitutive relation of the form

\[ \psi = \frac{1}{2} G (I_C - 3) + \left( K - \frac{2}{3} G \right) (J - 1) - \left( K + \frac{1}{3} G \right) \ln J \]  
(7.32)

has been considered by some authors. Here \( G \) and \( K \) are material constants; \( K \) may be associated with the bulk modulus for small values of extension. The first term on the right is the same as that for the neo-Hookean material and the last two terms relate to the material compressibility.
4.8 SIMPLE SHEAR-FLOW AND NEWTONIAN FLUID

Consider two parallel plates, containing between them a fluid; see Fig. 8.1a. Let the lower plate be kept stationary, and the upper one be moved at a constant velocity \( V^o \) parallel to itself, maintaining the constant distance \( h \) from the lower plate. Experience shows that the fluid particles in contact with the upper and lower plates adhere to them, so that those immediately adjacent to the upper plate move with velocity \( V^o \), and those immediately next to the lower plate remain stationary. This fact which has been ascertained (at least in the range of moderate speeds) by numerous experiments, is commonly referred to as the no-slip condition.

The no-slip condition is a characteristic of the real fluid-solid interface relation. Of course, a similar no-slip condition exists at the interface between two different fluids.

The no-slip condition is a macroscopic phenomenon, as is discussed in connection with the question of viscosity later on in this section.

It is evident that in the steady motion of the upper plate relative to the lower plate, the fluid particles move in the \( X_1X_1 \)-direction only, and have a velocity which depends linearly on the \( x_2 \)-coordinate, so that

\[
v_1(x_2) = \frac{V^o}{h} x_2
\]  

(8.1)

Referring to Sec. 2.7, observe that the rate of stretch \( d_{11} \) defined by Eq. (2-7.16) is identically zero in the present case, so that the material line elements of the fluid parallel to the \( X_1X_1 \)-axis do not suffer any stretching. However, the gradient of the velocity \( v_1 \), in the \( X_2X_2 \)-direction is nonzero, given by
(a) The upper plates move relative to the lower one at a constant velocity $V^0$, as shown. The particle velocity is linear in the $x_2$-coordinate. Particles next to the upper plate move with this plate, and those adjacent to the lower plate remain stationary. This is called the no-slip condition. $\gamma_{21} = \frac{V^0}{h}$ represents the rate at which the line element $PP^{(2)}$ rotates towards the element $PP^{(1)}$, and hence is equal to the rate of decrease in the instantaneously right angle formed by these elements.

(b) To maintain the constant velocity $V^0$, shear stresses of magnitude $t_{21}$ must be applied per unit area of the plates, as shown. The balance of linear and angular momenta for the fluid within the cylindrical region $ABCD$, $A'B'C'D'$ then requires that $t_{12} = t_{21}$, as shown.
\[
\frac{\partial v_1}{\partial x_2} = \frac{v^0}{h}.
\] (8.2)

This velocity-gradient gives the rate at which a material line element instantaneously parallel to the \(X_2\)-direction rotates with respect to an axis perpendicular to the \(X_1\) and \(X_2\)-plane (the plane of the paper in Fig. 8.1) in the clockwise direction (toward the \(X_1\) axis). Since material line elements parallel to the \(X_1\) axis simply \textit{translate} in that direction with a constant velocity which is proportional to their common distance from the \(X_1\) axis, the velocity-gradient (8.2) defines the rate of decrease in the angle between two instantaneously orthogonal material line elements which emanate from a given particle, parallel to the \(X_1\) and \(X_2\)-axes, respectively. To show this, proceed as follows.

At an instant \(t\) consider two orthogonal material line elements at a generic particle \(P\), as shown in Fig. 8.1a. At a later time \(t + dt\) the material line element \(PP^{(1)}\) has translated in the \(X_1\) direction to a new position \(QQ^{(1)}\). The material line element \(PP^{(2)}\), on the other hand, has moved to the new position \(QQ^{(2)}\) which makes an angle

\[d\gamma_{21} = \tan d\gamma_{21} = \frac{v^0}{h} \, dt\]

with the \(X_2\)-direction. The rate of the decrease in the instantaneously right angle of the directions \(PP^{(1)}\) and \(PP^{(2)}\), therefore is

\[
\frac{D\gamma_{21}}{Dt} = \dot{\gamma}_{21} = \frac{v^0}{h} = \frac{\partial v_1}{\partial x_2}.
\] (8.3)
In most engineering literature in fluid mechanics $\dot{\gamma}_{21}$ (which is the total decrease in the right angle formed by instantaneously orthogonal material directions, e.g. $PP^{(2)}$- and $PP^{(1)}$-directions) is called the shear-rate. It is, however, mathematically more convenient to refer to half of this quantity as the shear-rate. To eliminate possible confusion, the notation $\dot{\gamma}_{21}$ is used for the velocity-gradient in the present section; compare with Sec. 2.7.

**Shear Stress**

Experience shows that in order to maintain the constant velocity $v^0$ of the upper plate in Fig. 8.1a, one must apply a constant pull (force) per unit area of the plate, in the $X_1X_1$-direction. A tangential force, therefore, is transmitted from the plate to the fluid particles immediately adjacent to it, and whence throughout the fluid. Consider an imaginary plane perpendicular to the $X_2X_2$-direction. The fluid particles immediately on the upper face of this imaginary plane tend to pull to the right the particles on the lower face, and this way, a shear stress $t_{21}$ is transmitted across this plane. This shear stress represents the intermolecular forces (measured per unit area) that are created within the fluid, as the particles located in an upper layer tend to move horizontally, with a slightly larger speed relative to those in a lower layer.

In the present simple shear flow problem, the shear stress $t_{21}$ is constant. Consider at the instant $t$ the portion of fluid contained within a cylindrical region whose axis is perpendicular to the plane of
Fig. 8.1b, and whose base is the square ABCD. If this portion of fluid is removed, one must apply to the lateral faces of the cylinder the shear stresses $t_{21}$ and $t_{12}$ as shown in Fig. 8.1b. Since the fluid particles within this cylindrical region have zero acceleration, the sum of the total forces which act on the lateral surface of the cylinder, which represent the action of fluid particles outside of this region upon those within it, must be equal to zero. Summing the forces in the $X_1X_1$-direction, one concludes that shear stress $t_{21}$ acting on face DCC'D' must be equal to and point oppositely to that acting on face ABB'A'. Similar remarks apply to shear stress $t_{12}$ acting on faces ADD'A' and BCC'B', as shown in Fig. 8.1b. Moreover, the sum of the moments of these forces about any axis perpendicular to the plane of the figure must vanish. Taking the moment about the axis DD', for example, one obtains

$$t_{12} = -t_{21}.$$  

(8.4)

The state of stress in the fluid is therefore that of simple shear discussed in Sec. 3.3.

NEWTONIAN FLUID

Newtonian fluid represents a model of an ideal material which is characterized by the assumption that the shear stress $t_{21}$ is proportional to the velocity-gradient $\dot{\gamma}_{21}$, i.e.

$$t_{21} = \mu \dot{\gamma}_{21}.$$  

(8.5)

1 For a slice of unit thickness, the total tangential force on face ABB'A' is $t_{21}d$, and that acting on face BCC'B' is $t_{12}d$, where $d$ is the length of the edge of the square. The moment of these forces about DD' is $t_{21}d^2 - t_{12}d^2 = 0$. 


The coefficient $\mu$ is called the *coefficient of viscosity*, or, for short, *viscosity*.\textsuperscript{1} It represents the resistance of the fluid to shearing, so that the more viscous the fluid, the larger the coefficient of viscosity. Constitutive relation (8.5) is often referred to as Newton's law of viscosity.

Most real fluids, whether gaseous or liquid, are Newtonian at small shear-rates. The exception occurs in the case of polymer solutions, where the existence of long-chain molecules changes the viscosity characteristic of the fluid.

Table 8.1 gives the coefficient of viscosity for air and water at indicated temperatures. As can be expected, viscosity is temperature-sensitive. In this table the quantity

$$\nu = \frac{\mu}{\rho} \quad (8.6)$$

is the so-called *kinematic viscosity*, where $\rho$ is the mass-density of the fluid. [The same notation, namely $\nu$, was used to denote Poisson's ratio in Section 3.4, p. 4-33. This, however, should not cause any confusion, since from the context it will be clear which quantity is being considered.]

The physical unit of viscosity $\mu$ is lbm/ft sec in the English system, and g/cm sec in the CGS metric system. In this latter system the unit of viscosity is called poise after J. L. M. Poiseuille.\textsuperscript{2} The physical unit

\textsuperscript{1} Some authors also call this *dynamic viscosity*.

\textsuperscript{2}
of the kinematic viscosity, on the other hand, is \( \text{ft}^2/\text{sec} \) in the English system, and \( \text{cm}^2/\text{sec} \) in the CGS system, respectively; the latter is called Stokes. The inverse of the viscosity is commonly called the fluidity.

### TABLE 1.1

**VISCOSITY OF WATER AND AIR AT 1 atm**

<table>
<thead>
<tr>
<th>TEMP. (°C)</th>
<th>VISCOSITY ( \mu ) (g/cm sec)</th>
<th>KINEMATIC VISCOSITY ( \nu ) (cm²/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WATER at 0</td>
<td>( 1.787 \times 10^{-2} )</td>
<td>( 1.787 \times 10^{-2} )</td>
</tr>
<tr>
<td>10</td>
<td>( 1.304 \times 10^{-2} )</td>
<td>( 1.304 \times 10^{-2} )</td>
</tr>
<tr>
<td>20</td>
<td>( 1.002 \times 10^{-2} )</td>
<td>( 1.004 \times 10^{-2} )</td>
</tr>
<tr>
<td>30</td>
<td>( 0.798 \times 10^{-2} )</td>
<td>( 0.802 \times 10^{-2} )</td>
</tr>
<tr>
<td>40</td>
<td>( 0.654 \times 10^{-2} )</td>
<td>( 0.659 \times 10^{-2} )</td>
</tr>
<tr>
<td>50</td>
<td>( 0.548 \times 10^{-2} )</td>
<td>( 0.554 \times 10^{-2} )</td>
</tr>
<tr>
<td>80</td>
<td>( 0.355 \times 10^{-2} )</td>
<td>( 0.366 \times 10^{-2} )</td>
</tr>
<tr>
<td>100</td>
<td>( 0.283 \times 10^{-2} )</td>
<td>( 0.295 \times 10^{-2} )</td>
</tr>
<tr>
<td>AIR at 0</td>
<td>( 1.71 \times 10^{-4} )</td>
<td>0.132</td>
</tr>
<tr>
<td>10</td>
<td>( 1.76 \times 10^{-4} )</td>
<td>0.141</td>
</tr>
<tr>
<td>20</td>
<td>( 1.81 \times 10^{-4} )</td>
<td>0.150</td>
</tr>
<tr>
<td>30</td>
<td>( 1.86 \times 10^{-4} )</td>
<td>0.160</td>
</tr>
<tr>
<td>40</td>
<td>( 1.90 \times 10^{-4} )</td>
<td>0.169</td>
</tr>
<tr>
<td>100</td>
<td>( 2.18 \times 10^{-4} )</td>
<td>0.230</td>
</tr>
<tr>
<td>500</td>
<td>( 3.58 \times 10^{-4} )</td>
<td>0.785</td>
</tr>
<tr>
<td>1000</td>
<td>( 4.82 \times 10^{-4} )</td>
<td>1.74</td>
</tr>
</tbody>
</table>

ON THE NATURE OF VISCOSITY

While the macroscopic manifestation of viscosity has common features in gases, liquids, and even crystalline solids, the involved microscopic mechanisms are quite different.

In gases, viscosity relates to the transport of momentum by gas molecules from regions with larger bulk velocities to contiguous regions with smaller ones, and vice versa. In fact, the coefficient of viscosity \( \mu \) is proportional to the heat conductivity \( \kappa \), as will be shown later on.

MEAN FREE PATH

The molecules of a gas are in constant random motion having frequent "collisions"\(^1\) among themselves and with the walls of their container. The average distance \( \ell \) that a molecule travels before colliding with another particle is called the mean free path. To estimate the mean free path, observe that a particle of diameter \( d \) will collide with another

Figure 8.2

One collision occurs if a particle of diameter \( d \) meets another particle, as it sweeps a volume

\[
\frac{\pi d^2}{4} \ell.
\]

The same result is obtained if it is assumed that the first particle has a diameter of \( 2d \), and the second one is a point-mass. For one collision, \( \pi d^2 \ell = 1 \).

---

\(^1\) In the classical kinetic theory of gases, gas particles are viewed as elastic balls that have elastic collisions, i.e. no energy is lost because of collision. It is in this sense that the word "collision" is used here.
particle, if the center of the latter comes to within a distance $d$ from the center of the former; assume the particles have the same average diameter. In a distance $\ell$, one particle sweeps a volume of $\frac{\pi d^2}{4} \ell$, and collides with the particle which is partly within this volume. Let there be, on the average, $n$ particles per unit volume. Then one collision occurs within a distance $\ell$ when $n\pi d^2 \ell = 1$ which gives the following estimate for the mean free path:

$$\ell = \frac{1}{n\pi d^2}.$$  \hspace{1cm} (8.7)

This equation can also be obtained if one molecule is regarded as having a diameter $2d$, and, hence, sweeping a volume $\pi d^2 \ell$ in a distance $\ell$. One collision occurs if this molecule comes in contact with another molecule which can then be regarded as a point-mass, see Fig. 8.2.

Equation (8.7) is an approximation, but it does give the right order of magnitude. A more accurate estimate is obtained if one accounts for the fact that the particles are moving relative to each other, and, therefore, are not stationary targets; in estimate (8.7), it is assumed that one particle moves, and the other is a stationary target. When the relative motion of particles is accounted for, one obtains the following estimate:

$$\ell = \frac{\sqrt{2}}{2} \frac{1}{n\pi d^2}.$$  \hspace{1cm} (8.8)

As an example, consider a gas at 0°C and 1 atm, and assume that the average diameter of the molecules is $2 \times 10^{-8}$ cm. Since there are about $3 \times 10^{19}$ molecules per cm$^3$, from (8.7) obtain $\ell \approx 3 \times 10^{-5}$ cm. If these are air molecules, their speed is of the order of $10^5$ cm/sec. Thus, there will be about $3 \times 10^9$ collisions in each second.
VISCOSITY IN CASES

Referring to Fig. 8.1a, consider a plane perpendicular to the $X_2X_2$-axis at $x_2 = x_2$, and let us estimate the flux of linear momentum transported by the gas molecules across this plane which will be referred to as plane I I in the sequel. Assume that the gas is macroscopically homogeneous. The gas particles move randomly in all directions, colliding with each other. In their random motion, particles cross back and forth through plane I I. Since the gas is homogeneous, the net mass flow across I I is zero. However, since particles which cross I I from above are coming from a higher velocity region, they carry with them more linear momentum. On the other hand, particles moving up from below carry less momentum, since they come from a lower velocity region. The net effect, therefore, is that momentum is transferred from the high to the low velocity region. To make this quantitative proceed as follows.

Let the average speed of a particle be $v$. Since there is an equal number of particles which move downward along the $X_2X_2$-axis, as particles moving upward, and since there is, on the average, an equal number of particles moving back and forth along the three mutually orthogonal $X_1X_1$, $X_2X_2$, and $X_3X_3$-axes, the average flux of particles downwards along the $X_2X_2$-axis is $\frac{1}{6}nv$, where $n$ is the number of particles per unit volume. The factor $\frac{1}{6}$ stems from the fact that there are three equally possible directions for particle movement, and, on the average, the number of particles moving to the right along, say, the $X_1X_1$-axis is equal to that moving to the left.

From the above remarks it follows that $\frac{1}{6}$ of the molecules situated in a layer of thickness $\ell$ above the considered plane cross this plane
downward, where \( l \) is the mean free path. These particles carry with them an extra linear momentum of \( \frac{\partial v_1}{\partial x_2} l \) per particle, because the average molecular velocity at a distance \( l \) above this plane exceeds by \( \Delta v_1 = \frac{\partial v_1}{\partial x_2} l \) that at this plane. Since there are \( n \) particles per unit volume with the average microscopic speed \( v \), the extra linear momentum transported across the considered plane per unit time is \( \frac{1}{6} n v m \frac{\partial v_1}{\partial x_2} l \), and since there is an equal flux of particles upward across this plane from regions of lower velocity, the total time rate of change in the linear momentum is twice of the above quantity. This change of linear momentum equals the corresponding force which particles above the considered plane exert tangentially across and per unit area of this plane, to those particles below, i.e., the shear stress. It thus follows that

\[
\tau_{12} = 2\left( \frac{1}{6} n v m \frac{\partial v_1}{\partial x_2} l \right) = \frac{1}{3} \rho v l \frac{\partial v_1}{\partial x_2} = \mu \gamma_{12}, \tag{8.9}
\]

where \( \rho = nm \) is the mass-density. Hence, the dynamic viscosity is given by

\[
\mu = \frac{1}{3} \rho v l, \tag{8.10}
\]

which, since \( \rho \) is proportional to \( n \) (and hence to the pressure \( p \)), and \( l \) is inversely proportional to \( n \), does not depend on \( n \) and hence on the pressure of the gas at constant temperature. This seemingly unexpected result has been borne out by experiment, and represents one of the triumphs of the classical kinetic theory of gases.
In Sec. 4.11 the heat conductivity $\kappa$ for gases is estimated as
\[ \frac{1}{3} \rho C_v v^2. \]
Hence $\kappa$ and $\mu$ are related by
\[ \kappa = \rho \mu. \]  
(8.11)

Although this estimate is rough, it shows the order of magnitude involved. More accurate calculations\(^1\) yield
\[ \kappa = \alpha \mu C_v, \]  
(8.12)
where the numerical factor $\alpha$ has the value 2 for common diatomic gases, and 2.5 for mono-atomic ones.

It is interesting to note that, according to the above development, the viscosity of gases should increase with temperature, since at higher temperatures the particles have more random kinetic energy, which results in a greater momentum exchange between contiguous gas layers. This conclusion has been borne out by experimental results, as is also evident from Table 8.1, p. 4-91. The situation for liquids and solids is the opposite, so that their viscosity in general decreases with temperature (see Table 8.1), revealing that a different microscopic mechanism is responsible for viscosity in these substances.

**VISCOSITY IN LIQUIDS AND SOLIDS**

The viscosity of gases is usually of the order of $10^{-4}$ poise at ordinary temperatures. As temperature is decreased, the viscosity first decreases, but then sharply increases upon liquefaction, attaining values

\(^1\) op. cit., Cottrell, p. 27.
of the order $10^{-3}$ to $10^{-2}$ poise at the condensation point. It then rises as the temperature is decreased, increasing sharply upon solidification. If, for example, crystallization takes place, the value of viscosity reaches the order of $10^{18}$ poise, and increases rapidly as the solid is cooled. Some solids do not crystallize, i.e., they are amorphous or glass. These substances remain fluid but with very high viscosity, so that they do not flow noticeably under ordinary conditions. The distinction between solids and fluids in such situations is made at the viscosity of $10^{15}$ poises; below this the substance is regarded as fluid, and above it, as solid.

In contrast to gases which have negligible interatomic forces, and owe their viscosity to the momentum transfer between gas layers, the viscosity in liquids may, to a large extent, be attributed to the intermolecular forces which tend to retard relative movement of the contiguous layers of the liquid. The situation is not very well understood, and at the present time there is no satisfactory microscopic description of this phenomenon. Qualitatively, the viscosity may be roughly discussed in terms of the intermolecular forces, as stated by Bachelor$^1$: "It seems likely that coherent groups of molecules in a liquid resist deformation in some manner involving the direct action of intermolecular forces, and that the primary effect of a simple shearing motion (for instance) of the liquid is to tear apart some of the existing groups against this resistance. Coherent groups constantly reform in the liquid with consequent release of energy of molecular motion, and in this way some of the energy of the ordered bulk motion of the liquid is converted (or 'dissipated') to disordered molecular motion, or heat."

---

1 op. cit., p. 59.
Consider the relaxation time $t$ introduced in connection with the phenomena of linear visco-elasticity, see p. 4-60. Measurement shows that the relaxation time for liquids are of the order\(^1\) of $10^{-11}$ sec. This relaxation time is so short that during which only 10 to 100 atomic vibrations can take place. The molecules, therefore, quickly relax into new positions under a shearing stress. Upon cooling, this relaxation time $t$ increases, and when $t$ exceeds $10^4$ sec, the substance becomes solid.

It is remarkable that, for even very large values of shear-rate, most ordinary fluids are Newtonian, i.e., the shear stress is linearly related to the corresponding shear strain-rate. This is essentially because of the very short relaxation time that is involved for these substances. For materials such as waxes, asphalt, and putty, on the other hand, the relaxation time $t$ can vary between $10^{-4}$ to $10^4$ sec. For large values of relaxation time the Newtonian linear viscosity law is no longer valid and one must consider other constitutive models. Fluids for which Newton's law of viscosity, Eq. (8.5), is not valid, are commonly called non-Newtonian.

An important class of non-Newtonian fluids are the polymer solutions where the existence of long-chain molecules substantially alters the viscous and other characteristics of the fluid. In general, the character of these fluids is affected by the history of their past deformation, i.e., they possess a memory.

The phenomenon of viscosity in crystalline solids is very complicated, and involves many factors. A detailed discussion, therefore, would carry us far from the scope of this book.\(^2\)

\(^1\) op. cit., Cottrell, p. 198.

\(^2\) For references, see Mechanical Behavior of Materials at Elevated Temperatures, ed. by John E. Dorn, McGraw-Hill, 1961.
ON THE NO-SLIP CONDITION

The phenomenon of no-slip of real fluids over solid boundaries can be given an explanation in a microscopic sense.

The surfaces of solids are quite irregular and rough when viewed microscopically. At that level the surface of even a very finely polished, say, steel, consists of hills and valleys with differential elevation of more than several thousand atomic distances. This is schematically shown in Fig. 8.3. As a gas flows with the average bulk velocity \( V \) over such a surface, the gas molecules within a layer of thickness equal to their mean free path collide with the solid wall as well as with each other. There are more collisions with the back face of the wall irregularity, as there are with its front face, because of the forward additional velocity \( V \) which is superimposed on the average random translational velocity of the molecules. Hence, there are, on the average, more molecules reflected backward against the flow, than otherwise, and, consequently, the gas particles adjacent to the solid wall manifest an average velocity equal to that of the solid boundary.

In the case of the liquid-solid interface relation, the microscopic situation is quite different, although macroscopically the liquid adheres to the solid boundary. There, coherent groups of liquid molecules are trapped within the valleys of the solid's surface irregularity, and remain attracted there, since these groups of molecules can relax into the shape of the irregularities. Therefore, a layer of liquid with thickness of the order of the dimension of the solid's
Figure 8.3

(a) Flow of Gas Over Solid Boundaries: Surfaces of solids consist of hills and valleys with differential elevation of more than several thousand atomic distances. As a gas with average bulk velocity $V$ moves over such a surface, there are more molecules colliding with the back faces of irregularities than with the front faces, and, hence, there are on the average more molecules reflected backward against the flow than otherwise. Hence the average bulk velocity of the gas at the solid boundary reduces to that of the solid. This is the no-slip condition.
surface irregularities, remains in (almost) constant contact with the solid, attaining the solid's velocity. Of course, there is a movement of molecules within this layer, and an exchange of molecules with the bulk part of the liquid. These phenomena, however, relate to the self diffusion of molecules and to their motion due to random kinetic activity (their heat energy), which would also occur when the fluid does not flow over the solid, but is stationary.

[It is interesting to compare the liquid-solid interface relation with that of solid-solid interface relation. In contrast with the liquid which "fills" the depressions of the solid's surface, the solid-solid contact occurs at scattered points. The actual contact area in this latter case is much less than the macroscopic area of the two solids which appear in contact. When the two solids are compressed, the contact points deform plastically and weld to each other, the actual contact area being proportional to the magnitude of the compressive force. The "dry" friction, therefore, is the resistance to separation at contact points. When solids in contact are moved relative to each other, the contact points are ruptured, and new contacts are therefore made continuously. In this process a great deal of energy is lost into heat energy, and the contact surface may actually melt.]

The student without a doubt has noticed the contradiction between the no-slip condition which, as mentioned, exists at all real fluid-solid interfaces, and the rectilinear motion of continua in a tube
in Sec. 2.5 and thereafter. There it was assumed that the velocity distribution is uniform over each cross-section. The contradiction is avoided, if one does not view the tube in Sec. 2.5 as an actual solid boundary, but view it rather as the stream tube which marks the considered part of the material in a large body of fluid that moves in the \( X_1X_2 \)-direction, having uniform velocity over planes perpendicular to this direction.

The stream tube is an imaginary surface consisting of stream lines which intersect a simple closed curve (circuit) drawn in the fluid. Fluid inside of a stream tube cannot escape through the walls of this tube, since at each point the velocity of the particle instantaneously there, is tangent to the stream line passing through that point. For the rectilinear motion, the stream lines are all parallel to the direction of flow, i.e. the \( X_1X_2 \)-axis.

On the other hand, if the tube in Sec. 2.5 is, indeed, a rigid one, the analysis presented is strictly valid only for the so-called inviscid (perfect) fluid which is assumed not to adhere to the solid boundary. No real fluid is inviscid. Nevertheless, the results can be used with good accuracy, when one deals with turbulent flow. This will be discussed further in Ch. 6. Here it is mentioned that in a turbulent flow, the fluid particles, in addition to their bulk velocity, have secondary velocities which tend to equalize the bulk velocity over each cross-section. Close to the solid boundary, however, there is a thin layer of the fluid, in which the velocity changes sharply, reducing to zero at the boundary. In most ordinary flows this layer is very thin, and the assumption of a uniform velocity profile usually leads to good results, except for the energy loss due to the viscous effects at the boundary, which must be accounted for separately.
4.9 **SIMPLE SHEAR DEFORMATION**

**SMALL STRAIN**

Parallel to the simple shear-flow of fluids now consider the small shear deformation of solids. Whereas, under a constant shear stress, fluids maintain a constant shear-flow, solids undergo only a shear deformation. Of course, if the shear stress is sufficiently large and is sustained for a sufficiently long time, all solids would undergo a slow shear-flow, i.e. they creep. The kinematics of shear deformation can be studied without a reference to the forces that cause it. This is first briefly discussed and then the corresponding stresses and constitutive relations are examined in the sequel.

**SIMPLE SHEAR**

Consider the cube in Fig. 9.1a, and assume that it is deformed into the shape shown, where material planes parallel to plane ABCD are translated parallel to themselves, so that planes OA'D'G and EB'C'F are inclined upon deformation, making a small angle $\gamma_{21} \ll 1$ with the $X_2X_2$-direction.

In the small strain theory (linearized theory) the angle $\gamma_{21}$ is so small that its square and higher powers are negligible compared with itself.

In the engineering literature $\gamma_{21}$ is called the **shear strain** corresponding to the $X_2X_2$, $X_1X_1$-directions. It is, however, mathematically more convenient to refer to half of this quantity by the term shear strain. This is shown in Fig. 9.1b. The shear strains $\epsilon_{21} = \epsilon_{12}$ are,
(a) The cube $ABCD$ - $OEFG$ is deformed to $A'B'C'D'$ - $OEFG$. The angle between $OA'$ and the $X_2X_2$-axis is $\gamma_{21}$.

(b) Refer to $\epsilon_{12} = \epsilon_{21} = \frac{1}{2} \gamma_{21}$ as the shear strain of initially orthogonal material directions $OA$ and $OE$. The deformed block is simply rotated counterclockwise about the axis $OG$ by the angle $\frac{1}{2} \gamma_{21}$. 

Figure 9.1
therefore, defined by

\[ \varepsilon_{12} = \varepsilon_{21} = \frac{1}{2} \gamma_{21} \quad (9.1) \]

In this figure the final position of face OA'D'G is obtained by a counterclockwise rotation of magnitude \( \frac{1}{2} \gamma_{21} \), about the axis OG. Thus, the simple shear deformation in Fig. 9.1b corresponds to a clockwise rotation with magnitude \( \varepsilon_{21} = \frac{1}{2} \gamma_{21} \) of element OA, and the counterclockwise rotation of magnitude \( \varepsilon_{12} = \frac{1}{2} \gamma_{21} \) of element OE. The total decrease in the initially right angle formed by elements OA and OE is, of course, given by \( \varepsilon_{12} + \varepsilon_{21} = \gamma_{21} \).

**LINEAR ELASTICITY**

The constitutive relation of a linearly elastic ideal material for simple shear deformation states that the shear stress is proportional to the corresponding shear strain. If the shear stress which has produced the shear deformation in Fig. 9.1b is \( t_{12} = t_{21} \), the constitutive relation for linearly elastic solids then becomes

\[ t_{21} = G \gamma_{21} \quad (9.2) \]

The coefficient \( G \) represents a material constant called the shear modulus. Typical values of shear modulus are listed in Table 4.1. As is shown later on, the shear modulus \( G \) for isotropic materials can be expressed in terms of the elastic modulus \( E \) and the Poisson ratio \( \nu \) as

\[ G = \frac{E}{2(1 + \nu)} \quad (9.3) \]

Since for most materials \( 0 < \nu \leq \frac{1}{2} \), it follows that

\[ \frac{1}{3} E \leq G < \frac{1}{2} E \quad (9.4) \]
where \( G = \frac{1}{3} \frac{E}{\nu} \) corresponds to incompressible materials for which \( \nu = \frac{1}{2} \); see Sec. 4.4. Note that in terms of the shear strains \( \varepsilon_{12} = \varepsilon_{21} \), Eq. (9.2) becomes

\[
\tau_{21} = 2G \varepsilon_{21}.
\]  

(9.5)

Whereas, for small strains (linearized theory) considered in this section, no distinction needs to be made between shearing stresses, as referred to the deformed and undeformed interior surfaces, one must note that, in general (for large deformations), such a distinction may become necessary. It is recalled that in the case of simple tension or compression the symbol \( \sigma^1 \) was used to denote in the linearized case the common value of the Cauchy \( T^1 \), and the nominal \( T^R \) stresses. This section is exclusively concerned with linearized theory, and therefore the linearized shear stress will be denoted by \( \tau_{21} \). Observe that results for the biaxial state of stress discussed in Sec. 3.5 remain valid for the large, as well as the small strain theory. The approximation is made only when one relates the true shear stress \( \tau_{21} \) to the linearized strain by a constitutive equation such as (9.2).

**Small Strain Plasticity**

As was discussed in Section 4.3, the plastic flow, even in the simple tension test, is the manifestation of the shear deformation of the material, so that layers of atoms glide over other layers on slip planes, as soon as the shear stress reaches a critical value.

Similarly to the case of simple tension (or compression), one may introduce constitutive relations which may represent ideal models for the plastic deformation of metals in shear. These constitutive relations are easily obtained, if one replaces in the corresponding relation for simple
tension in Sec. 4.5, the (linear) normal stress \( \sigma_1 \) by the shear stress \( t_{12} \), the (linear) normal strain-rate \( \dot{\varepsilon}_1 \) by the shear strain-rate \( \dot{\gamma}_{12} \), and use for the yield stress \( Y \) the corresponding yield stress in simple shear. For example, the constitutive relation for rigid, perfectly-plastic solids in shear is obtained in this manner from Eqs. (5.1), i.e.

\[
\dot{\gamma}_{12} = \begin{cases} 
0 & \text{if } |t_{12}| < Y \text{ or } |t_{12}| = Y \text{ and } t_{12} t_{12} < 0 , \\
\text{sgn} \ t_{12} & \text{if } |t_{12}| = Y \text{ and } t_{12} t_{12} = 0 .
\end{cases}
\]

Similarly, the counterpart of Eqs. (5.2) are

\[
\dot{t}_{12} = G \dot{\gamma}_{12} \text{ if } |t_{12}| < Y \text{ or } |t_{12}| = Y \text{ and } t_{12} t_{12} < 0 ,
\]

\[
\text{sgn} \ \dot{\gamma}_{12} = \text{sgn} \ t_{12} \text{ if } |t_{12}| = Y \text{ and } t_{12} t_{12} = 0 ,
\]

which corresponds to the elastic, perfectly-plastic constitutive model.

In the same manner, the constitutive relation for the elastic-work-hardening model can be written down. For example, for the bilinear elastic-plastic model in shear one obtains

\[
\Delta t_{12} = \frac{G}{\lambda + 1} \Delta \dot{\gamma}_{12} ,
\]

where \( G \) is the shear modulus, \( \lambda \) is zero for elastic loading and for unloading, and has a given positive value when plastic flow takes place; see Fig. 5.4. In (9.8) \( \Delta \) represents the increment of the quantity that follows it.

**LINEAR VISCO-ELASTICITY**

As in the case of plasticity above, the results of Sec. 4.6 for the simple tension case, can readily be extended to yield constitutive models
of linear visco-elasticity in shear. To this end one needs only to introduce the corresponding creep and relaxation functions, $c_{12}(t)$ and $r_{12}(t)$, which have exactly the same general features as their counterparts in the simple tension test. Hence, for a given shear stress history $t_{12}(t)$, the corresponding shear strain $\gamma_{12}(t)$ at time $t$ is obtained by superposition as in Eq. (6.10), yielding

$$\gamma_{12}(t) = \int_{0}^{t} \frac{dt_{12}(\tau)}{d\tau} c_{12}(t - \tau) d\tau.$$  \hspace{1cm} (9.9)

In the same manner, for a given strain history $\gamma_{12}(t)$, the stress $t_{12}(t)$ at time $t$ is defined by

$$t_{12}(t) = \int_{0}^{t} \frac{d\gamma_{12}(\tau)}{d\tau} r_{12}(t - \tau) d\tau.$$  \hspace{1cm} (9.10)

which is the counterpart in shear of Eq. (6.11).

It is clear now that by specialization of creep and relaxation functions in Eqs. (9.9) and (9.10), one may arrive at various models for linear visco-elasticity in shear, such as Maxwell's, Voigt's, and the standard solid's, in the same manner as discussed in Sec. 4.6.
4.10 THERMAL EXPANSION AND THERMAL STRAIN

Most materials expand as their temperature is increased at constant pressure. For gases the explanation of this phenomenon in terms of the classical kinetic theory is quite simple. As the thermal energy of the gas is increased, the linear momentum transmitted in each collision of a gas particle with the boundary of the container increases, and this in turn raises the pressure if the volume is kept constant. On the other hand if the pressure is to be kept constant, the volume must increase, so that the average linear momentum transmitted to the container's walls (i.e. the pressure) remain constant. Of course, the thermal equation of state for perfect gases, Eq. (2-2.1), exactly reflects this phenomenon.

For solids and liquids, the situation is much more complicated, and for a microscopic explanation quantum mechanical effects need to be considered\(^1\). Here attention is confined to a purely phenomenological approach.

Experiment shows that when the temperature of, say, a steel rod is increased, its linear dimensions increase accordingly. To characterize this, one defines a coefficient of thermal expansion $\alpha$ which is the change in length per unit initial length induced by changing the temperature by one degree, so that

$$\alpha = \frac{1}{l} \frac{dl}{dT} \text{ ,}$$

(10.1)

\(^1\) For a simple account, see C. Kittel, *Introduction to Solid State Physics*, John Wiley and Sons, 3rd Ed. 1968. See also *op. cit.* Cottrell, Section 6.8.
where \( l \) is the original length of the specimen. Table 10.1 gives the coefficient of thermal expansion for a few solids.

**TABLE 10.1**

**COEFFICIENT OF THERMAL EXPANSION**

<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>TEMP. °C</th>
<th>COEFFICIENT PER °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALUMINUM</td>
<td>20</td>
<td>25.5 × 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>31.50 × 10^{-6}</td>
</tr>
<tr>
<td>BRASS, wire</td>
<td>0-100</td>
<td>19.30 × 10^{-6}</td>
</tr>
<tr>
<td>BRICK</td>
<td>-</td>
<td>9.5 × 10^{-6}</td>
</tr>
<tr>
<td>CADMIUM</td>
<td>20</td>
<td>28.8 × 10^{-6}</td>
</tr>
<tr>
<td>CARBON, diamond</td>
<td>40</td>
<td>1.18 × 10^{-6}</td>
</tr>
<tr>
<td>graphite</td>
<td>40</td>
<td>7.86 × 10^{-6}</td>
</tr>
<tr>
<td>CEMENT and CONCRETE</td>
<td>-</td>
<td>10.14 × 10^{-6}</td>
</tr>
<tr>
<td>GLASS</td>
<td>-</td>
<td>5.8-9.7 × 10^{-6}</td>
</tr>
<tr>
<td>GOLD</td>
<td>0-100</td>
<td>15.52 × 10^{-6}</td>
</tr>
<tr>
<td>IRON, cast</td>
<td>40</td>
<td>10.61 × 10^{-6}</td>
</tr>
<tr>
<td>steel</td>
<td>40</td>
<td>13.22 × 10^{-6}</td>
</tr>
<tr>
<td>ZINC</td>
<td>300-400</td>
<td>15 × 10^{-6}</td>
</tr>
</tbody>
</table>

The coefficient of thermal expansion of solids may depend on the considered direction, since the solid may not be isotropic in its response. For isotropic solids the coefficient of thermal expansion is the same in all directions. For example, a cube made of an isotropic solid would expand equally in all three directions when its uniform temperature is increased. The isotropy depends on the crystal structure of the solid. The thermal expansion is isotropic in cubic crystals, but for other crystals it is, in general, anisotropic. For instance, for the close packed hexagonal

---

1 Data taken from op. cit., Handbook of Physics and Chemistry, pp. 2059-2064.
structure, see Fig. 4.3, the coefficient of thermal expansion for the direction parallel to the hexagonal axis is different from that for directions which lie in the hexagonal plane. In the polycrystalline structure, however, the solid consists of numerous crystals which are randomly oriented. This usually leads to a macroscopically isotropic thermal expansion.

Consider a cube of a solid whose coefficients of thermal expansion along its three axes are given by $\alpha_1$, $\alpha_2$, and $\alpha_3$, which would be equal if the solid were isotropic. When the temperature is increased by $\Delta\theta$, the length $l$ of the edges of the cube increases to $l(1 + \alpha_1 \Delta\theta)$, $l(1 + \alpha_2 \Delta\theta)$, and $l(1 + \alpha_3 \Delta\theta)$. The change in the volume of the cube, therefore, is

$$l^3(\alpha_1 + \alpha_2 + \alpha_3)\Delta\theta + l^3(\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_3 \alpha_1)\Delta\theta^2 + l^3\alpha_1 \alpha_2 \alpha_3 \Delta\theta^3.$$ 

The last two terms are very small and can be neglected since, as is seen from Table 10.1, $\alpha$ is of the order of $10^{-5}$ per °C. Thus, the change in volume per unit initial volume is

$$\frac{\Delta V}{V} \approx (\alpha_1 + \alpha_2 + \alpha_3)\Delta\theta \quad .$$  \hspace{1cm} (10.2)

If the solid is isotropic, this equation becomes

$$\frac{\Delta V}{V} \approx 3\alpha \Delta\theta \quad .$$  \hspace{1cm} (10.3)

Hence, the coefficient of cubical thermal expansion $\beta$, defined as the change in volume per unit initial volume per degree temperature, in isotropic solids, is three times the corresponding coefficient of linear thermal expansion $\alpha$.

As is seen from Table 10.1, the coefficient of linear thermal expansion
varies with the temperature. There are empirical equations which may be used for various substances to account for this effect. For example, one may consider the following quadratic equation

\[ l = l_o (1 + a\theta + b\theta^2) \quad \ldots (10.4) \]

where \( \theta \) is measured in degrees centigrade, and where \( l_o \) is the length at \( 0^\circ C \). Table 10.2 gives typical values of \( a \) and \( b \) for a few substances.

<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>TEMP. LIMITS</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALUMINUM</td>
<td>10-90</td>
<td>0.221 x 10^{-4}</td>
<td>0.114 x 10^{-7}</td>
</tr>
<tr>
<td>BRASS</td>
<td>10-90</td>
<td>0.1781 x 10^{-4}</td>
<td>0.098 x 10^{-7}</td>
</tr>
<tr>
<td>COPPER</td>
<td>10-90</td>
<td>0.1596 x 10^{-4}</td>
<td>0.102 x 10^{-7}</td>
</tr>
<tr>
<td>GOLD</td>
<td>10-90</td>
<td>0.1410 x 10^{-4}</td>
<td>0.042 x 10^{-7}</td>
</tr>
<tr>
<td>IRON, pure</td>
<td>0-38</td>
<td>0.1145 x 10^{-4}</td>
<td>0.071 x 10^{-7}</td>
</tr>
<tr>
<td>LEAD</td>
<td>10-90</td>
<td>0.2829 x 10^{-4}</td>
<td>0.120 x 10^{-7}</td>
</tr>
<tr>
<td>NICKEL</td>
<td>0-38</td>
<td>0.1255 x 10^{-4}</td>
<td>0.057 x 10^{-7}</td>
</tr>
<tr>
<td>PLATINUM</td>
<td>0-1,000</td>
<td>0.0868 x 10^{-4}</td>
<td>0.013 x 10^{-7}</td>
</tr>
<tr>
<td>SILVER</td>
<td>10-90</td>
<td>0.1862 x 10^{-4}</td>
<td>0.074 x 10^{-7}</td>
</tr>
<tr>
<td>TIN</td>
<td>10-90</td>
<td>0.2094 x 10^{-4}</td>
<td>0.175 x 10^{-7}</td>
</tr>
<tr>
<td>ZINC</td>
<td>10-90</td>
<td>0.2969 x 10^{-4}</td>
<td>-0.0635 x 10^{-7}</td>
</tr>
</tbody>
</table>

Similarly to solids, empirical equations for the coefficient of cubical thermal expansion for liquids may be defined as follows:

\[ V = V_o (1 + a\theta + b\theta^2 + c\theta^3) \quad \ldots (10.5) \]

where \( \theta \) again is the change of temperature from \( 0^\circ C \). For water, for instance, \( a = -0.06427 \times 10^{-3}, b = 8.5053 \times 10^{-6}, c = 6.7900 \times 10^{-8} \), in the range of \( 0 \leq \theta \leq 33^\circ C \).

\[ \quad ^1 \text{Data Taken from Ibid., p. 2065} \]

\[ \quad ^2 \text{From Ibid., p. 2067. See this reference for values of } a, b, \text{ and } c, \text{ for other liquids.} \]
THERMAL STRESSES

From the preceding discussion it is seen that a change in length is induced in solids by a change in temperature. If this change of length is prevented by constraining the solid, then stresses will be developed in the solid. These are called thermal stresses. For example, if a linearly elastic bar whose coefficient of thermal expansion is \( \alpha \), is constrained at both ends so that its length remains constant while its temperature is changed by \( \Delta \theta \), a thermal stress of magnitude

\[
\sigma_1 = -E \alpha \Delta \theta \tag{10.6}
\]

is produced in the solid. To see this, assume that the solid is not constrained. The change of length per unit initial length (i.e. the extension \( \varepsilon_1 \)) then would be

\[
\varepsilon_1 = \alpha \Delta \theta \tag{10.7}
\]

Since \( \alpha \) is very small, one is dealing in the linear range. Now, to return the solid to its initial length one must apply forces which correspond to the stress given by (10.6). The minus sign in this equation indicates that when \( \Delta \theta \) is positive, the bar tends to expand, but since it is constrained, compressive thermal stresses are developed.

In the case of linear elasticity the total longitudinal extension \( \varepsilon_1 \) may be regarded as consisting of two parts, one, due to thermal expansion given by \( \varepsilon_1^0 = \alpha \Delta \theta \), the other, due to the elasticity given by \( \varepsilon_1^e = \frac{\sigma_1}{E} \), where \( \sigma_1 \) is the stress in the bar. The total strain then is

\[
\varepsilon_1 = \varepsilon_1^0 + \varepsilon_1^e
\]

\[
= \alpha \Delta \theta + \frac{\sigma_1}{E} \tag{10.8}
\]
Solve this equation for $\sigma_1$, and obtain

$$\sigma_1 = E \varepsilon_1 - E \alpha \Delta \theta . \quad (10.9)$$

Note that in the example mentioned above, the total strain $\varepsilon_1$ of the rod is zero, since the rod is constrained at both ends. Thus, setting $\varepsilon_1 = 0$ in (10.9), obtain (10.6).

It should be noted that the above discussion has validity only for the linear case. For finite elasticity the situation is much more complicated.

4.11 HEAT TRANSFER

Under a temperature-gradient, matter transfers heat energy. For example, if one end of a copper rod is maintained at a higher temperature than the other end, heat energy "flows" from the hot end to the cold one.

In general, the transport of heat energy may occur: (1) by convection (in fluids) where matter is moved from one point to another, carrying with it heat energy; (2) by radiation in transparent substances, where electromagnetic waves are transmitted through material (for example, sun rays passing through window glass); (3) by the movement of electrons in metals; and (4) by the vibration of atoms in the lattice structure of crystalline solids, or by the vibration of molecules and atoms in non-crystalline substances. The last two modes of heat transportation are commonly referred to as heat conduction.

The transport of heat in gaseous material has essentially a different microscopic mechanism than that occurring in liquids and solids. Although the explanation of the microscopic mechanism involved in heat conduction in liquids and solids is outside of the scope of this dis-
cussion, it may be instructive to briefly examine some aspects of heat conduction in gases.

HEAT CONDUCTION IN GASES

It is not easy to maintain a temperature-gradient in gases without creating convective processes. Imagine, however, that one could do so, and let the temperature \( \theta \) change in a given direction, say, \( X_1X_1 \), so that \( \frac{d\theta}{dX_1} \) is nonzero. Assume further that the gas is homogeneous. The gas particles move randomly in all directions, colliding with each other. Consider the plane \( \Pi \Pi \), Fig. 11.1, and let us calculate the average heat energy transferred across this plane. In their random motion, particles cross back and forth through plane \( \Pi \Pi \). Since the gas is homogeneous, the net mass flow across \( \Pi \Pi \) is zero. However, since particles which cross \( \Pi \Pi \) from the left are coming from a higher temperature region, they carry with them more heat energy. On the other hand, particles moving to the left from the right carry less energy. The net effect, therefore, is that heat energy is transferred from the high to the low temperature region.

Figure 11.1

The temperature \( \theta \) is decreasing, along the \( X_1X_1 \)-axis, from left to right. On the average, \( \frac{1}{6} \) of the particles within the shaded area \( \Pi \) cross plane \( \Pi \Pi \) to the right. Similarly, \( \frac{1}{6} \) of those in \( \Pi \Pi \) cross \( \Pi \Pi \) to the left. The heat flux then is 

\[ q_1 = -2\left(\frac{1}{6}nv\right)(c\frac{d\theta}{dX_1} \ell) \],

where \( n \) is the average number of particles per unit volume, \( v \) is the average particle speed, \( c \) is the heat capacity of each particle, and \( \ell \) is the mean free path.
Let the heat capacity of each particle be \( c \); i.e. the heat carried by each particle per degree temperature is \( c \). By the same reasoning used in Sec. 4.8, it follows that \( \frac{1}{6} \) of the particles located within a distance \( \ell \) (mean free path) of plane I I, cross this plane from each side. The temperature change \( \Delta \theta \) in this distance is \( \Delta \theta = \frac{d\theta}{dx_1} \ell \).

Thus, the total heat flux \( q_1 \) across plane I I is

\[
q_1 = -2\left( \frac{1}{6}nv \right)(c\Delta \theta)
= -\frac{1}{3}(nc)(\nu \ell)\frac{d\theta}{dx_1} ,
\]

where \( q_1 \) is heat flux per unit area of plane I I, per unit time. Since there are \( n \) particles per unit volume, \( nc \) is the heat capacity of particles in a unit volume. If the mass of a particle is \( m \), then

\[
nc = \left( \frac{nm}{m} \right) c = \rho C_v ,
\]

where \( \rho \) is the mass-density, and \( C_v \) is the specific heat (per unit mass) at constant volume. Equation (11.1) then becomes

\[
q_1 = -\frac{1}{3} \rho C_v \nu \ell \frac{d\theta}{dx_1}
= -\chi \frac{d\theta}{dx_1} ,
\]

where the coefficient of thermal conductivity \( \chi \) is given by

\[
\chi = \frac{1}{3} \rho C_v \nu \ell .
\]

Table 11.1 gives the (approximate) coefficient of thermal conductivity for a few gases and liquids.
TABLE 11.1
HEAT CONDUCTIVITY

Approximate Values

<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>TEMPERATURE °C</th>
<th>CONDUCTIVITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIR</td>
<td>0</td>
<td>0.0000568</td>
</tr>
<tr>
<td>ARGON</td>
<td>0</td>
<td>0.0000389</td>
</tr>
<tr>
<td>CARBON DIOXIDE</td>
<td>0</td>
<td>0.0000307</td>
</tr>
<tr>
<td>HELIUM</td>
<td>0</td>
<td>0.000339</td>
</tr>
<tr>
<td>HYDROGEN</td>
<td>0</td>
<td>0.000327</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.000369</td>
</tr>
<tr>
<td>OXYGEN</td>
<td>7-8</td>
<td>0.0000563</td>
</tr>
<tr>
<td>WATER</td>
<td>0</td>
<td>0.00139</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0.00138</td>
</tr>
<tr>
<td></td>
<td>15</td>
<td>0.00144</td>
</tr>
</tbody>
</table>

HEAT CONDUCTION IN SOLIDS

In solids, heat energy is transferred by the movement of free electrons (in metals), and by the thermal vibration of atoms and molecules. The vibration of atoms is usually at sufficiently large amplitudes, so that the repulsive and attractive interatomic forces (see Fig. 3-3.1, p. 3-6) are not linearly related to the corresponding change in the interatomic distances; this amplitude of vibration is about 10% of this distance at low temperatures and increases with the temperature, and hence goes beyond the straight BC part of the curve in Fig. 3-3. This and other nonlinear effects are an essential cause of the diffusive mode of heat conduction in solids, so that the heat energy (the mechanical energy of atomic vibration) is not transmitted through the solid in the form of, for example, pressure waves which move through the solid with the

1 Data taken from Ibid. 2252
speed of sound (Ch. 7). A detailed explanation of heat conduction in solids requires the discussion of the interaction of energy quanta called phonons, which is beyond the scope of our consideration. A phenomenological discussion, however, may be presented as follows.

Consider, say, a copper rod which is thermally insulated on its cylindrical surface, but not at its ends. Suppose that the rod is initially of a constant uniform temperature \( \theta^0 \), and that at a certain time \( t^0 \) a temperature \( \theta > \theta^0 \) is applied and maintained at one end. Experience shows that the temperature at the other end of the rod (almost) immediately but slowly increases from \( \theta^0 \) as time goes on, until a steady state situation is reached. That is, the heat energy diffuses along the rod from one end to the other.

The heat diffusion in solids (and liquids) may be described phenomenologically by Fourier's law which states that the heat flux is proportional to the temperature-gradient. In the one-dimensional case where heat flux occurs across planes perpendicular to the \( X_1 \) axis, one has

\[
q_1 = - \kappa \frac{\partial \theta}{\partial X_1},
\]

(11.4)

where \( \kappa \) is the coefficient of thermal conductivity. When the material is nonhomogeneous (the material properties vary from point to point) \( \kappa \) may be a function of position varying from point to point, otherwise it is a constant which, however, depends on temperature. Table 11.2 gives the heat conductivity of a few solids. In this table \( \kappa \) is the heat quantity in calories transmitted per second through a plate 1 cm thick, across a 1 cm \(^2\) area, when the temperature difference is 1\(^\circ\)C.
TABLE 11.2
HEAT CONDUCTIVITY

<table>
<thead>
<tr>
<th>SUBSTANCE</th>
<th>TEMPERATURE °C</th>
<th>CONDUCTIVITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>ALUMINUM</td>
<td>-160</td>
<td>0.514</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>0.504</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.49</td>
</tr>
<tr>
<td></td>
<td>600</td>
<td>1.01</td>
</tr>
<tr>
<td>COPPER, pure</td>
<td>-160</td>
<td>1.097</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>0.918</td>
</tr>
<tr>
<td></td>
<td>100-837</td>
<td>0.858</td>
</tr>
<tr>
<td>IRON, pure</td>
<td>18</td>
<td>0.161</td>
</tr>
<tr>
<td></td>
<td>100-1,245</td>
<td>0.191</td>
</tr>
<tr>
<td>STEEL</td>
<td>-160</td>
<td>0.113</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>0.108</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>0.107</td>
</tr>
<tr>
<td>ASBESTOS fiber</td>
<td>500</td>
<td>0.00019</td>
</tr>
<tr>
<td>GLASS (window)</td>
<td>-</td>
<td>0.0025</td>
</tr>
<tr>
<td>GRANITE</td>
<td>100</td>
<td>0.0045-0.0050</td>
</tr>
<tr>
<td>GRAPHITE</td>
<td>-</td>
<td>0.012</td>
</tr>
<tr>
<td>RUBBER</td>
<td>(gutta percha)</td>
<td>-</td>
</tr>
<tr>
<td>SILK</td>
<td>-</td>
<td>0.000095</td>
</tr>
</tbody>
</table>

In many problems of heat conduction in solids, the deformation may be neglected. Consider a bar of uniform cross-section insulated on its lateral surface, and assume that heat flows into the bar from one end and flows out from the other; see Fig. 11.2. Consider at point $X_1$ an element of rod with length $\Delta X_1$, and let the heat-flux at the left end

---

1 Data taken from Ibid. pp. 2247-2251.
be \( q_1(X_1,t) \), and that at the right end be \( q_1(X_1 + \Delta X_1, t) = q_1(X_1, t) \)
\[
q_1(X_1, t) \rightarrow q_1(X_1 + \Delta X_1, t)
\]
\[
X_1 \quad X_1 + \Delta X_1
\]

\[
\frac{\partial q_1(X_1, t)}{\partial X_1} \Delta X_1 + \ldots
\]

, where it is assumed that the heat-flux vector \( q_1 \) is a smooth function of its arguments. If there are no heat sources within the body, the net heat flow into this element is
\[
- \frac{\partial q_1}{\partial X_1} \Delta X_1 A,
\]
where \( A \) is the cross-sectional area.

If the rate of change of temperature is \( \frac{\partial \theta}{\partial t} \), then the rate at which the heat energy per unit volume of the rod is increasing is \( \rho_o C_v \frac{\partial \theta}{\partial t} \)
where \( C_v \) is the heat capacity at constant volume. It therefore follows that
\[
\rho_o C_v \frac{\partial \theta}{\partial t} = - \frac{\partial q_1}{\partial X_1}.
\]

(11.5)

Substitution from (11.4) into (11.5) now yields
\[
\rho_o C_v \frac{\partial \theta}{\partial t} = \frac{\partial}{\partial X_1} \left( \kappa \frac{\partial \theta}{\partial X_1} \right).
\]

(11.6)
If the thermal conductivity is constant, (11.6) reduces to

$$\frac{\partial \theta}{\partial t} = g \frac{\partial^2 \theta}{\partial x_1^2} , \quad g = \frac{x}{\rho_0 C_v} ,$$  \hspace{1cm} (11.7)

which must be solved with appropriate initial and boundary conditions.

The initial condition defines the temperature distribution at the initial time $t^0$. For example, one may have at time $t = 0$

$$\theta(x_1,0) = G(x_1) ,$$  \hspace{1cm} (11.8)

where $G(x_1)$ is a given function of $x_1$ along the rod. If the rod is finite, one must prescribe the temperature at its ends as functions of time. For example, one may have

$$\theta(0,t) = H(t) , \quad \theta(L,t) = K(t) ,$$  \hspace{1cm} (11.9)

where $H(t)$ and $K(t)$ are given functions of time, and where it is assumed that the left end of the rod is at the origin 0 along the $x_1 x_1$-axis, and the right end at $x_1 = L$. Note that these functions may be specified to be constant.

**STEADY-STATE SOLUTION**

When the cylindrical boundary of the rod is insulated, while heat flows steadily from one end to the other in a manner that the temperature at each point does not change in time, a steady-state heat flow situation prevails. Since temperature is independent of time, the left-hand side of (11.7) is zero, yielding

$$\frac{d^2 \theta}{dx_1^2} = 0 .$$  \hspace{1cm} (11.10)
Upon integration, this gives

$$\theta = AX_1 + B \quad , \quad (11.11)$$

where $A$ and $B$ are integration constants. Since (11.10) does not involve
time, there is no initial condition, as should be the case because of
the steady-state requirement. The constants of integration in (11.11),
therefore, must be fixed by the boundary conditions which also cannot
contain time, since otherwise they would violate the steady-state con-
dition. Thus, if the temperature at the left end is kept at a constant
$\theta_1$, and that at the right end at a constant $\theta_2$, one obtains

$$\theta = (\theta_2 - \theta_1) \frac{X_1}{L} + \theta_1 \quad . \quad (11.12)$$

This equation states that in a steady-state one-dimensional heat con-
duction in a homogeneous rod, the temperature distribution is linear
along the length of the rod. Note that, if the rod is not homogeneous,
then instead of (11.12) one must use

$$\frac{d}{dX_1} \left[ \kappa(X_1) \frac{d\theta}{dX_1} \right] = 0 \quad . \quad (11.13)$$

The solution of this equation is

$$\theta = A \int \frac{dX_1}{\kappa(X_1)} + B \quad . \quad (11.14)$$

Again there are two integration constants which must be fixed by the
boundary conditions. Here, however, the temperature distribution is
not linear.
TRANSIENT SOLUTION

When either the initial temperature distribution is arbitrary, or the boundary conditions (11.9) are given as functions of time, there will occur a nonsteady flow of heat through the rod, so that the temperature at each point varies with time.

The differential equation (11.7) can be solved by the method of the separation of variables, as follows. Let \( \theta = \theta(X_1, t) \) be defined as the product of two functions, one depending only on \( X_1 \), the other depending only on \( t \), and write

\[
\theta = X(X_1) \ T(t) \ .
\]  

(11.15)

Substitute into (11.7), to obtain

\[
T' \ X = g \ T \ X'' \ ,
\]

where primes denote differentiation with respect to the corresponding variable. Upon dividing by \( X \ T \), express this equation as

\[
\frac{T'}{T} = g \ \frac{X''}{X} \ .
\]  

(11.16)

The right-hand side of (11.16) is a function of \( X_1 \) only, whereas the left-hand side is a function of \( t \) alone. In order that the equality in this equation hold for all admissible values of \( t \) and \( X_1 \), both sides of (11.16) must be equal to a parameter which is independent of \( X_1 \) and \( t \). Depending on whether one chooses a positive parameter, or a negative one, one obtains different solutions. Consider the case where this parameter is negative, which corresponds to an exponentially decaying (with time) solution.
Equating both sides of (11.16) to \( -a^2 \), obtain

\[
\frac{dT}{dt} = -a^2 T, \quad g \frac{d^2 X}{dx_1^2} = -a^2 X. \tag{11.17}
\]

Integration of the first equation yields

\[
T = T^0 e^{-a^2 t}, \tag{11.18}
\]

where \( T^0 \) is the integration constant. The second equation in (11.17), on the other hand, gives

\[
\frac{d^2 X}{dx_1^2} + \frac{a^2}{g} X = 0
\]

whose solution is

\[
X = A \cos \frac{a}{\sqrt{g}} x_1 + B \sin \frac{a}{\sqrt{g}} x_1, \tag{11.19}
\]

as can be verified by direct substitution.

As an example, consider a rod of length \( L \) with the initial temperature distribution of

\[
\theta(x_1,0) = \theta^0 \sin \frac{\pi x_1}{L}, \quad \theta^0 = \text{constant}, \tag{11.20}
\]

and the boundary conditions of

\[
\theta(x_1,0) = \theta(L,t) = 0. \tag{11.21}
\]

From (11.19), (11.18), and (11.15), obtain with the aid of (11.20) and (11.21)

\[
\theta(x_1,t) = \theta^0 e^{-\frac{\pi^2 g}{L^2}} \sin \frac{\pi x_1}{L}. \tag{11.22}
\]
Students who are familiar with the Fourier expansion of sufficiently smooth functions defined on a finite region, should observe that when the initial condition (11.8) can be expanded in Fourier, sine, and cosine series, the above method can easily be used to generate the corresponding solutions by superposition.

For an infinitely extended rod one can generate a solution for any given (sufficiently smooth) initial temperature distribution by first obtaining the so-called fundamental solution of the heat equation, as follows.

Since \( \theta = e^{-a^2 t} \cos \frac{a}{\sqrt{4g}} X_1 \) is a solution of (11.7), and since (11.7) is a linear equation, if one integrates or differentiates this solution with respect to \( a \), one obtains expressions which are also solutions of (11.7). Consider

\[
\theta = \int_{-\infty}^{+\infty} e^{-a^2 t} \cos \frac{a}{\sqrt{4g}} X_1 \, da
\]

\[
= \sqrt{\frac{\pi}{t}} e^{-\frac{X_1^2}{4gt}}
\]

(11.23)

which is also a solution of (11.7), as can be verified by direct substitution. This is the fundamental solution of the heat equation (11.7).

To use (11.23), first normalize it in such a manner that

\[
K \int_{-\infty}^{+\infty} e^{-\frac{X_1^2}{4gt}} \, dX_1 = 1
\]

\[\text{Ibid.} \ p. \ 276, \ Eq. \ 432.\]
To find the factor $K$, note that

$$\int_{-\infty}^{+\infty} e^{-\alpha^2 x^2} \, dx = \frac{1}{\alpha \sqrt{\pi}},$$

and hence deduce that

$$u(x_1, t) = \frac{1}{2\sqrt{\pi}gt} e^{-\frac{x_1^2}{4gt}}$$

is the solution of the heat conduction in an infinitely extended rod with zero initial temperature everywhere, except at the origin, where a local initial temperature of unity is applied. If this local temperature of unity is applied at point $x_1 = \xi$, rather than at the origin, solution (11.24) must be replaced by

$$u(x_1, t; \xi) = \frac{1}{2\sqrt{\pi}gt} e^{-\frac{(x_1 - \xi)^2}{4gt}}.$$  

(11.25)

To obtain the solution corresponding to the initial condition (11.8), one can view (11.8) as a series of local temperatures of magnitude $G(\xi)$ applied at point $\xi$. The final solution is then obtained by superposition, as follows:

$$\theta(x_1, t) = \frac{1}{2\sqrt{\pi}gt} \int_{-\infty}^{+\infty} G(\xi) e^{-\frac{(x_1 - \xi)^2}{4gt}} \, d\xi.$$  

(11.26)

\footnote{\textit{Ibid.}, p. 275, Eq. 423.}
PROBLEMS FOR CHAPTER 4

2.1 The pressure of a 3 ft$^3$ gas, is doubled while the temperature is kept constant. Find the final volume, assuming Eq. (2.1) applies.

2.2 A pound of helium is to be moved while at 60°F. If the pressure in the tank should not exceed 100 psi (lbf/in$^2$), what should the volume of the tank be?

2.3 A 10 ft$^3$ tank contains hydrogen at 20 psia (psia = the absolute pressure in lbf/in$^2$), and 60°F. Hydrogen is pumped into the tank until the pressure is 100 psia, and the temperature 150°F. How much hydrogen was added?

2.4 Find the work required to compress 20 pounds of hydrogen initially at 60°F, from 10 ft$^3$ to 5 ft$^3$ volume by
(a) an isothermal process,
(b) an adiabatic process; $\gamma = 1.40$.

2.5 Find the heat energy needed to raise the temperature of 10 pounds of air from 70°F to 2000°F
(a) at constant volume, assuming that $C_v = 0.17$ Btu/lb °R,
(b) at constant pressure, assuming that $C_p = 0.24$ Btu/lb °R,
(c) at constant pressure, assuming that $C_p = 6.36 + 9.92 \times 10^{-4} \theta$
- $8.52 \times 10^{-8} \theta^2$ Btu/lb mole °R.

2.6 Express the bulk modulus of an ideal gas in terms of pressure for
(a) an adiabatic process,
(b) an isothermal process
2.7 Use the van der Waals equation of state, Eq. (2.8), to obtain the expression for the speed of sound
(a) in an isothermal process,
(b) in an adiabatic process; assume that $a = 0$.

2.8 Use the van der Waals equation of state, Eq. (2.8), to obtain the expression for the work needed to compress a gas from an initial volume $V_1$ to a final volume $V_2$, isothermally.

4.1 A steel rod of circular cross-section $\frac{1}{2}$ in in radius is pulled by a 10,000 lbf.
(a) Find the elongation, if the initial length is 10 in and Young's modulus is $30 \times 10^6$ psi.
(b) Find the percent reduction of the cross-sectional area, if the Poisson ratio is 0.3.

4.2 Find the force required to move point A in Fig.
(1) vertically by 0.01 in, where the common area, length, and elastic modulus of the two wires are 0.1 in$^2$, 20 in, and $30 \times 10^6$ psi, respectively.

Fig. (1)
4.3 The elastic structure

in Fig. (2) consists

of two rods of length,

area, and elastic mod-

ulus, as shown.

(a) Find the expres-

sion for the force

$F$ which is needed to

displace point $A$

only vertically up-

ward by an amount $\Delta_v$.

(b) Find the force $F$ re-

quired to move $A$ only

horizontally by an

amount $\Delta_h$.

(c) If $E = 30 \times 10^6$ psi, $E' = 10 \times 10^6$ psi, $L = 10$ in,

$A = A' = 1$ in$^2$, $\theta = 45^\circ$, and $\theta' = 30^\circ$, find the total

force needed when $A$ is moved both horizontally and

vertically such that $\Delta_v = \Delta_h = 0.001$ in.

Figure (2)
4.4 Find the load carried by each elastic wire in Fig. (3) and the corresponding elongation, if \( F = 20,000 \) lbf.

![Diagram](image)

**Figure (3)**

4.5 A structure rests at each of its four corners on an eight-inch-long column made of a steel tube with 8 in inside diameter and 0.2 in thickness, and filled with concrete. Each column carries 100,000 lbf. The elastic modulus of steel is \( E_s = 30 \times 10^6 \) psi, and that of concrete is approximately \( 3 \times 10^6 \) psi.

(a) Find the load carried by steel and concrete, separately.

(b) Find the total shortening of each column.
5.1 Find the maximum load that can be carried by the structure of Fig. (1) in Prob. 4.2, if the wires are rigid, perfectly-plastic, with yield-stress \( Y = 30,000 \) psi.

5.2 The structure shown in Fig. (4) consists of three wires which are elastic, perfectly-plastic, with

\[ E = 30 \times 10^6 \text{ psi} \]

\[ Y = 30,000 \text{ psi} \]

If the common area is \( A = 0.1 \text{ in}^2 \)

(a) find which wire reaches the yield limit first, and

(b) at what value of \( F \), the maximum value of \( F \) that the structure can carry.

Figure (4)

5.3 (a) Find the maximum value of \( F \) in Fig. (3) if the wires are made of a rigid, perfectly-plastic material with \( Y = 20,000 \) psi for the middle wire and \( Y = 30,000 \) psi for the other two.

(b) Let the middle wire consist of an elastic, perfectly-plastic material with \( E = 10 \times 10^6 \text{ psi} \) and \( Y = 20,000 \) psi, while the material of the other two wires is rigid, perfectly-plastic with \( Y = 30,000 \) psi. Find the maximum value of \( F \) that the structure can carry.
PROBLEMS

6.1 Two viscoelastic bars $B_1$ and $B_2$ are connected to a rigid plate as shown. The structure is being strained at a constant rate $\varepsilon = a_0$, starting at $t = 0$. Find the force $F(t)$ if $B_1$ follows the Voigt and $B_2$ the Maxwell model constitutive behavior.

Assume that $B_1$ and $B_2$ have cross sections $A_1$ and $A_2$, elastic constants $E_1$ and $E_2$, and coefficients of viscosity $\gamma_1$ and $\gamma_2$, respectively.

6.2 A bar is subjected to a stress which varies with time according to

$$\sigma_1 = \sigma_0 t$$

Find the ensuing strain assuming a

(a) Maxwell model,
(b) Voigt model, and
(c) standard solid model.

6.3 The strain of a bar is given by

$$\varepsilon_1 = \varepsilon_0 t$$

Find the corresponding stress $\sigma_1(t)$ if the bar consists of a

(a) Maxwell material,
(b) Voigt material, and
(c) standard solid material.
PROBLEM

8.1 The velocity field for the flow of a Newtonian fluid down an inclined plane is given by

\[ v_1 = A_0 \times_2 \left[ 2h - x_2 \right], \quad v_2 = 0, \]

where \( h \) is the thickness of the fluid; see Fig. 8.1.

Assume uniform mass-density, and constant gravitational force. For steady motion, balance the forces applied on an element of length \( dx_1 \) in the \( x_1 \)-direction, and obtain the constant \( A_0 \).

Find the flow rate. Find the vorticity.  

Figure 8.1
9.1 A circular cylindrical thin tube of thickness \( t \), internal radius \( a \), and length \( L \), is made of an elastic material and is subjected to a constant torque \( T \). Find the shear stress, shear strain, and the relative rotation of its two ends when the shear modulus is \( G \).

[Hint: Assume shear stresses are uniformly distributed over the thickness of the tube, and take their moment about the tube's center.]
CHAPTER 5

BASIC LAWS

5.1 INTRODUCTION

The thermomechanical response of a continuum depends on its material composition. Gases, for example, behave differently from liquids or solids. In the same manner, solids and liquids possess distinct thermomechanical characteristics; Ch. 4. These differences are quantified through the constitutive relations which relate the kinematical and dynamical quantities.

The basic laws, on the other hand, are assumed to apply to all continua, independently of the particular material constitution. In other words, it is assumed that the deformation and motion occur in consonance with certain basic laws, so that, with the aid of these laws, basic equations governing the motion of all continuous bodies can be established. To apply these basic laws to particular cases, the basic equations are then further restricted by the corresponding constitutive relations.

The four basic laws are listed on p. 1-11.

CONSERVATION OF MASS

The law of the conservation of mass requires that the total mass of a continuum remain unchanged during the course of its motion and deformation. The mass-density, on the other hand, may change from particle to particle, and in time, this change being governed by an equation which guarantees the preservation of total mass.
Let at the initial time \( t^0 \) a continuum occupy volume \( V \) and have the mass-density distribution \( \rho_0 = \rho_0(\mathbf{x}) \). At a later time \( t > t^0 \) let the corresponding mass-density be \( \rho = \rho(\mathbf{x}, t) \), and assume that the continuum then occupies volume \( \nu \).

The total mass of the continuum is conserved, so that one has

\[
\int_V \rho_0 \, dV = \int_\nu \rho \, d\nu .
\]

(1.1)

This equation can be reduced to a local form by means of the mean-value theorem.

**MEAN-VALUE THEOREM**

Let \( f(x_1, x_2, x_3) = f(\mathbf{x}) \) be a real-valued continuous\(^1\) function in the closed region \( r + \partial r \) with volume \( \nu \). Then there exists a point \( \mathbf{x}^* \) in \( r \) such that

\[
\nu f(\mathbf{x}^*) = \int_\nu f(\mathbf{\xi}) \, d\nu .
\]

(1.2)

The proof of this theorem can be found in any textbook on differential and integral calculus.\(^2\)

At a typical particle \( P \) consider now a nested sequence of volumes \( \nu_n \) in the initial configuration at time \( t^0 \), each containing \( P \) and each being contained in the preceding one. Denote by \( \nu_n \) the corresponding

---

\(^1\) On the boundary \( \partial r \), one-sided continuity (i.e. as points on \( \partial r \) are approached from within \( r \)) is implied.

sequence of volumes in the current configuration at time \( t \). As in Sec. 1.2, choose the sequence \( V_n \) such that its largest diameter goes to zero as \( n \) becomes large.

Consider a typical volume \( V_n \) which corresponds to \( u_n \), and apply the conservation law (1.1) to obtain

\[
\int_{V_n} \rho_o \, dV = \int_{u_n} \rho \, du .
\]

(1.3)

With the aid of the mean-value theorem this reduces to

\[
V_n \rho_o (x^*) = u_n \rho (x^{**}, t) ,
\]

(1.4)

where \( x^* \) and \( x^{**} \) are suitable points in \( V_n \) and \( u_n \), respectively; the particle at \( x^{**} \) is not necessarily the particle \( x^* \). Rewrite (1.4) as

\[
\rho_o (x^*) = \rho (x^{**}, t) \frac{u_n}{V_n}
\]

(1.5)

and take the limit as \( n \) becomes large, arriving at

\[
\rho_o (x) = \rho (x, t) \frac{du}{dV}
\]

\[
= \rho (x, t) J ,
\]

(1.6)

where \( J \) is the Jacobian; see Eqs. (2-6.4) and (3-4.2). Note that in (1.6), \( x \) is the position at time \( t \) of the particle which initially is at \( X \).

**Balance Law for Linear Momentum**

This law is also called **Euler's first law**.

At a given instant, a considered portion of a continuum is subjected to body forces and surface tractions which represent the effect of the
material points outside, on this portion of the continuum. Denote the surface tractions by \( \mathbf{t} \) which act on the surface \( S \) that bounds the instantaneous volume \( V \), and let the body forces\(^1\) measured per unit mass be denoted by \( f \). The total force \( \mathbf{F} \) instantaneously acting on the considered portion of the continuum then is

\[
\mathbf{F} = \int_S \mathbf{t} \, ds + \int_V \mathbf{f} \rho \, dv .
\]  

(1.7)

The linear momentum of an elementary mass \( dm = \rho \, dv \) at the instant \( t \) is \( dm \mathbf{v} \), where \( \mathbf{v} = \mathbf{v}(x, t) \) is the velocity vector. Hence, the total linear momentum becomes

\[
\mathbf{E} = \int_V \mathbf{v} \rho \, dv .
\]  

(1.8)

Euler's first law states that the instantaneous rate of change of linear momentum, \( \frac{D\mathbf{E}}{Dt} \), is equal to the resultant external force, \( \mathbf{F} \), that acts on the body at the considered instant\(^2\). Hence, one has

\[
\frac{D\mathbf{E}}{Dt} = \mathbf{F}
\]  

(1.9)

From (1.7), (1.8), and (1.9), it follows that

\[
\frac{D}{Dt} \int_V \mathbf{v} \rho \, dv = \int_S \mathbf{t} \, ds + \int_V \mathbf{f} \rho \, dv ,
\]  

(1.10)

---

\(^1\) The body forces may also stem from the interaction of pairs of particles forming the continuum, as for example, in the case of a body under its own gravitation. It is assumed that forces of this kind are accounted for in the specification of \( f \).

\(^2\) Compare with corresponding results for point-mass, pp. 2-5, -13, -14, and -19.
which is the mathematical statement of Euler's first law, i.e. the balance law for linear momentum.

**BALANCE LAW FOR ANGULAR MOMENTUM**

This law is also called Euler's second law. Before it can be stated, one needs the notion of the moment of a vector about a fixed point.

Let \( \mathbf{A} \) be a vector, and consider a fixed point \( O \). The moment \( \mathbf{M} \) of vector \( \mathbf{A} \) about point \( O \) is a vector whose line of action is perpendicular to the plane formed by \( \mathbf{A} \) and \( O \), whose magnitude is equal to the magnitude of \( \mathbf{A} \), i.e. \( A \), times the distance from \( O \) to \( \mathbf{A} \), i.e. the length of the perpendicular from \( O \) to \( \mathbf{A} \), and whose sense is given by the direction of the advancement of a right-handed screw which lies along \( \mathbf{M} \) and turns in the sense defined by \( \mathbf{A} \): Fig. 1.1. If the position vector of the point of application of \( \mathbf{A} \) is \( x \), the moment of \( \mathbf{A} \) about \( O \) becomes

\[
\mathbf{M} = x \times \mathbf{A},
\]

where \( x \) denotes vector-multiplication.

The moment of a force about a point is often called its **torque**.

The instantaneous moment with respect to the origin \( O \) of the applied forces on a portion of the continuum which at that instant occupies volume \( V \) bounded by surface \( S \), is

\[
\mathbf{M} = \int_S x \times \mathbf{t} \, ds + \int_V x \times \mathbf{f} \, dV.
\]

The **angular momentum** about a point \( O \) is the moment of the linear momentum with respect to that point. For the considered continuum this is
\[ \mathbf{G} = \int_{\mathcal{V}} \mathbf{z} \times \mathbf{v} \rho \, dv \]  

(1.13)

Euler's second law states that the \textit{instantaneous rate of change of angular momentum}, \( \frac{d\mathbf{G}}{dt} \), is equal to the \textit{resultant moment}, \( \mathbf{M} \), that acts on the \textit{body at the considered instant}. Hence, one has

\[ \frac{d\mathbf{G}}{dt} = \mathbf{M} \]  

(1.14)

From (1.12), (1.13), and (1.14), it follows that

\[ \frac{d}{dt} \int_{\mathcal{V}} \mathbf{z} \times \mathbf{v} \rho \, dv = \int_{\mathcal{S}} \mathbf{z} \times \mathbf{t} \, ds + \int_{\mathcal{V}} \mathbf{z} \times \mathbf{f} \rho \, dv \]  

(1.15)

---

**Figure 1.1**

The moment \( \mathbf{M} \) of a vector \( \mathbf{A} \) about point 0 is a vector normal to the plane formed by 0 and \( \mathbf{A} \), has a magnitude equal \( A \, d \), where \( d \) is the distance from 0 to \( \mathbf{A} \), and has the sense defined by a right-handed screw as shown.
CONSERVATION OF ENERGY

Energy can exist in various forms. The equivalence of these forms is implied by the law of the conservation of energy. Before it can be stated quantitatively, one needs to examine various forms in which energy is manifested. These are discussed in the sequel.

Potential Energy: When an object with mass \( m \) is raised to height \( h \) near the surface of the earth, it is said that its potential energy is increased by \( mgh \).

This means that, if the object is released, it will move down under gravity, and has the potentiality of doing work equal to \( mgh \). If the object is indeed released with zero initial velocity at height \( h \), it loses the potential energy when it reaches the ground, since this potential energy is converted to kinetic energy. When the object reaches the ground, it has a finite velocity. Neglecting the interaction with air molecules, one has \( mgh = \frac{1}{2} mv^2 \), where \( v \) is the speed of the object.

This is a statement of the conservation of energy in its simplest form. It says that the object gained kinetic energy at the expense of its potential energy.

Heat Energy: When the object hits the ground, its kinetic energy disappears, since it stops upon hitting the ground. This kinetic energy, however, is not "lost". It has simply been converted into the kinetic energy of the particles which are thrown around because of the impact, or are chipped off the object, and into kinetic energy of the random vibration of molecules and atoms of the object and the ground. In fact, if a bullet hits a hard surface almost its entire kinetic energy can be
converted to such (random) molecular kinetic energy; the bullet melts. Hence, another form of energy is the kinetic energy of the random motion of molecules. This is the heat energy.

In gases the heat energy represents mostly the kinetic energy of the random translational motion of the molecules. In solids and liquids, on the other hand, it is mostly the vibrational energy of molecules and atoms, which comprises the heat energy.

**Internal Potential Energy:** When a solid rod is slowly elongated by pulling at both its ends, work is performed on it. Part of this work becomes the heat energy, but a major part of it is stored in the rod in the form of interatomic potential energy; work is done to keep further apart two adjacent atoms, and this work may be recoverable by letting the solid return to its initial equilibrium state. Therefore, when a solid is deformed, energy is stored within it in the form of internal potential energy. The situation can be visualized by considering a coiled spring. If the spring is compressed, potential energy is stored in it, that can be retrieved if the spring is elastic.

**Internal Energy:** From the above remarks it is observed that materials can store energy within themselves in the form of molecular and atomic kinetic energy, and intermolecular and interatomic potential energy. In the continuum theory where one is not directly concerned with the microscopic aspects of matter, the internal energies of the kind mentioned above are represented collectively by a scalar-valued function $e = e(x,t)$ which measures internal energy of the continuum per its unit mass at the particle which at instant $t$ is at point $x$. The $e$ is called the internal energy-density. It is an extensive quantity
in the sense that the total internal energy of a collection of bodies
is equal to the sum of the internal energy of each constituent. Therefore, the total internal energy of a continuum with the instantaneous
volume \( V \) is given by

\[ E = \int_V e \rho \, dV . \]  

(1.16)

**Kinetic Energy:** In addition to this internal energy the body has
kinetic energy. This kinetic energy should not be confused with the
(random) molecular kinetic energy mentioned before, which was called heat
energy. The kinetic energy of a continuum relates to its ordered motion
as a continuous body. If at instant \( t \) the speed of the particles com-
prising the continuum (in its ordered bulk motion) is denoted by \( v \),
then the total kinetic energy of this continuum at that instant is
given by

\[ K = \frac{1}{2} \int_V v^2 \rho \, dV . \]  

(1.17)

**Radiation Energy:** Energy can be transmitted to a body by radia-
tion. The radiation which consists of electromagnetic waves carries
with it a certain amount of energy that can be deposited into matter,
as the radiation is absorbed by the matter. This type of energy usually
serves to increase the heat energy in the matter. In the continuum
theory one can view such radiative heat energy as heat sources that are
distributed continuously over that part of the body that is subjected
to the radiation. Denote by \( h \) the heat deposited per unit time, per
unit mass of the continuum by radiation absorption. Then the rate of
heat absorption is

\[ H_1 = \int h \rho \, d\nu \]  \hspace{1cm} (1.18)

**Heat Flux:** Heat energy can also be transferred to material bodies by heat conduction through their surface boundaries. This represents the exchange of molecular and atomic kinetic energy (of random motion of molecules) across the surface boundary with the adjacent material points. This heat flow is represented by a heat flux vector (or, for short, heat flux) \( \mathbf{q} = q(\mathbf{x}, t) \) defined on the surface boundary \( s \), and which measures the rate of outward (when positive) heat flow per unit area of surface boundary \( s \). Since \( \mathbf{q} \) is not necessarily perpendicular to surface \( s \), the rate of heat flowing into the continuum at an instant \( t \) over an elementary area \( d\mathbf{s} \) is \( -\mathbf{q} \cdot \mathbf{n} \, d\mathbf{s} \), where \( \mathbf{n} \) is the outward unit normal on \( d\mathbf{s} \). Hence, the total heat flowing into the continuum is

\[ H_2 = \int_s -\mathbf{q} \cdot \mathbf{n} \, d\mathbf{s} \]  \hspace{1cm} (1.19)

From Eqs. (1.18) and (1.19) the instantaneous rate at which heat energy is added to the continuum becomes

\[ H = \int h \rho \, d\nu - \int_s \mathbf{q} \cdot \mathbf{n} \, d\mathbf{s} \]  \hspace{1cm} (1.20)

**Mechanical Work:** The body forces and surface tractions acting on the considered portion of a continuum transfer energy to the continuum in the form of mechanical work. The rate of this work is

\[ \frac{D\mathbf{w}}{Dt} = \int h \mathbf{f} \cdot \mathbf{\nu} \, d\nu + \int_t \mathbf{t} \cdot \mathbf{n} \, d\mathbf{s} \]  \hspace{1cm} (1.21)
The first term on the right-hand side represents the rate of work by body forces, while the second term is the rate of work of the surface tractions, \( \mathbf{v} \) being the particle velocity at which the corresponding force is acting; compare with Eqs. (1.6), (2.14), and (3.7) of Ch. 2.

Conservation of Energy: The time rate of change of the kinetic energy (of ordered bulk motion) and the internal energy of any portion of a continuum is equal to the rate at which heat energy is added (by radiation absorption and heat flow through the surface boundary), and the rate at which mechanical work is done (by body forces and surface tractions) on that considered portion of the continuum. From Eqs. (1.16), (1.17), (1.20), and (1.21) it now follows that

\[
\frac{D}{Dt} (H + \mathcal{E}) = \frac{Dw}{Dt} + \mathbf{H}
\]  

(1.22)

which is the mathematical statement of the energy conservation law.

In this chapter the consequences of the four basic laws will be developed and discussed for various cases.
5.2 CONSERVATION OF MASS, RECTILINEAR MOTION

Consider the rectilinear motion of a continuum in a tube of constant cross-sectional area, as discussed in Sec. 2.5; the tube may be regarded as a stream tube. In this motion the mass-density distribution may change from particle to particle, as well as in time. For example, if the continuum represents the gas-continuum discussed in Sec. 1.2, the gas may be compressed or expanded as it flows through the tube. Since it is assumed that particles instantaneously on a common cross-section have equal velocities, accelerations, and mass-densities, one may continue to consider the particles on the center line of the tube as representative. On the other hand, at each cross-section one may multiply the corresponding common mass-density of the particles situated on that section, by the constant cross-sectional area of the tube, and then speak of mass per unit length of the tube. Denoting the mass-density at the particle \( X_1 \) at the initial time \( t^0 \) by \( \rho_o(X_1) \) one writes

\[
\rho^*(X_1) = A \rho_o(X_1),
\]

(2.1)

where \( A \) is the cross-sectional area of the tube, and \( \rho^*(X_1) \) is the mass per unit length of the tube at \( X_1 \) at time \( t^0 \). This mass-density may be given an alternative definition as follows.

At the initial time \( t^0 \) consider on the \( X_1X_1 \)-axis a particle \( X_1 \), and a nested sequence of intervals \( I_n \), each containing particle \( X_1 \) within it, and each being contained within the preceding one. Denote by \( \Delta L_n \) the length of the \( n \)th interval, and by \( \Delta m_n \) the mass of the continuum contained within two cross-sections drawn through the terminals of that interval. Let \( \Delta L_n \) go to zero as \( n \) becomes large. The mass per
unit length at particle $X_1$ at time $t^0$ is now defined by the following limit
\[ \lim_{n \to \infty} \frac{\Delta m_n}{\Delta L_n} = \rho^*(X_1). \] (2.2)

It is clear that definition (2.1) is equivalent to definition (2.2).

Consider now two particles $X_1$ and $X_1 + \Delta X_1$ which are separated from each other at time $t^0$ by the material line segment $\Delta X_1$. The total mass of the continuum contained within the cross-sections passing through these material points is $\Delta m$. As these particles move downstream along the tube, the distance between the cross-sections passing through them (which cross-sections move with the particles), in general, changes.

Since no material is assumed to be added or extracted during this motion, as a basic assumption it is accepted that the total mass contained within the above two cross-sections remains constant throughout the entire motion.

Suppose now that at time $t$ particle $X_1$ is at point $x_1$, and particle $X_1 + \Delta X_1$ is at $x_1 + \Delta x_1$ on the $X_1 x_1$-axis. Denote by $\rho(x_1, t)$ the mass-density at the particle which at time $t$ is at point $x_1$, and observe that $\rho(x_1, t^0) = \rho_0(X_1)$.

Since the cross-sectional area of the tube is constant, the law of conservation of mass, Eq. (1.2), reduces to
\[ \int_{X_1}^{x_1 + \Delta x_1} \rho_0(\xi) \, d\xi = \int_{X_1}^{x_1 + \Delta x_1} \rho(\xi, t) \, d\xi. \] (2.3)

This equation can be reduced to a differential equation with the aid of the mean-value theorem.
MEAN-VALUE THEOREM

Let \( f(x) \) be a real-valued continuous function on the closed interval \( a \leq x \leq b \). Then there exists a point \( x^* \), \( a < x^* < b \), such that

\[
(b - a) f(x^*) = \int_a^b f(x) \, dx.
\]

(2.4)

The proof of this theorem can be found in any textbook on differential and integral calculus. Intuitively, it states that the area under the curve \( y = f(x) \) from \( x = a \) to \( x = b \) in the \( x,y \)-plane, is equal to the area of the rectangle with base equal to \( b - a \) and height equal to \( y^* = f(x^*) \) for a suitable \( x^* \).

SPATIAL CONTINUITY EQUATION

It will be assumed that the mass-density \( \rho(x_1,t) \) is a sufficiently smooth function of its arguments \( x_1 \) and \( t \) in the considered range of variation of \( x_1 \) for all \( t \geq t^0 \), so that it admits derivatives up to any desired order. Then, according to the mean-value theorem, there exist points \( x_1^* \) and \( x_1^{**} \), \( x_1 < x_1^* < x_1 + \Delta x_1 \) and \( x_1 < x_1^{**} < x_1 + \Delta x_1 \), such that Eq. (2.3) is equivalent to

\[
\Delta x_1 \rho_o(x_1^*) = \Delta x_1 \rho(x_1^{**},t)
\]

which can be written as

\[
\rho_o(x_1^*) = \rho(x_1^{**},t) \frac{\Delta x_1}{\Delta x_1}.
\]

(2.5)

---

1 At the end points \( x = a \) and \( x = b \), the right- and the left-hand continuity is implied.

2 See, for example W. Fulks, op. cit., p. 125.
Consider now the limiting case of this equation as the particles \( X_1 \) and \( X_1 + \Delta X_1 \) are chosen closer and closer to each other in the sense of the nested sequence discussed above. At the limit, \( X_1, X_1^*, \) and \( X_1 + \Delta X_1 \), all coincide, and one obtains

\[
\rho_o(X_1) = \rho(x_1, t) \frac{\partial x_1}{\partial x_1} \tag{2.6}
\]

which relates the initial mass-density \( \rho_o \) to the current mass-density \( \rho \).

The left-hand side of (2.6) is not a function of time, since it is fixed as soon as particle \( X_1 \) is fixed. Therefore, its material time derivative is zero, i.e.

\[
\frac{D}{Dt} \left[ \rho(x_1, t) \frac{\partial x_1}{\partial x_1} \right] = 0
\]

which can be written as

\[
\frac{D\rho}{Dt} \frac{\partial x_1}{\partial x_1} + \rho \frac{D}{Dt} \left( \frac{\partial x_1}{\partial x_1} \right) = 0 . \tag{2.7}
\]

But from Example 5.2, p. 2-47, one has

\[
\frac{D}{Dt} \left( \frac{\partial x_1}{\partial x_1} \right) = \frac{\partial x_1}{\partial x_1} \frac{\partial v_1}{\partial x_1}
\]

which reduces (2.7) to

\[
\left[ \frac{D\rho}{Dt} + \rho \frac{\partial v_1}{\partial x_1} \right] \frac{\partial x_1}{\partial x_1} = 0
\]

\[\text{1 Note that since the Jacobian J in a rectilinear motion is given by } J = \frac{\partial x_1}{\partial x_1}, \text{ Eq. (2.6) is the same as Eq. (1.6), as it should be.}\]
which in view of the fact that \( \frac{\partial x_1}{\partial x_1} = \Lambda \) is positive, yields

\[
\frac{D\rho}{Dt} + \rho \frac{\partial v_1}{\partial x_1} = 0. \tag{2.9}
\]

This equation is commonly known as the spatial continuity equation. It is obtained from the conservation of mass, and, therefore, implies that law.

The first term in (2.9) can be written as

\[
\frac{D\rho(x_1,t)}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x_1} v_1.
\]

Hence, (2.9) becomes

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_1} (\rho v_1) = 0. \tag{2.10}
\]

Moreover, since \( \frac{D}{Dt} (\ln \rho) = \left( \frac{D\rho}{Dt} \right) / \rho \), and since \( \rho \) is nonzero (positive), it follows that

\[
\frac{D}{Dt} (\ln \rho) + \frac{\partial v_1}{\partial x_1} = 0. \tag{2.11}
\]

For a steady or stationary motion the kinematical quantities are not explicit functions of time. The mass-density in steady flow does not therefore depend on time explicitly, and hence \( \frac{\partial \rho}{\partial t} = 0 \). From (2.10) it follows that in the steady flow of a continuum through the tube, one has

\[
\rho v_1 = \text{constant}. \tag{2.12}
\]

Multiply (2.12) by the cross-sectional area of the tube, \( A \), and obtain

\[
Q = \rho A v_1 = \text{constant}
\]

which is the mass flux through a cross-section per unit of time.
ALTERNATIVE DERIVATION OF SPATIAL CONTINUITY EQUATION:
CONTROLLED VOLUME

A more intuitive method of deriving the spatial continuity equa-
tion (2.9) is to consider a controlled volume (a fixed volume) in space,
write down the net rate of mass flow into this volume, and equate this
with the time rate of change of mass contained within this volume. Note
that, whereas previously a fixed portion of the continuum having a fixed
mass was involved, here the volume in space is fixed.

Consider the volume of the tube between cross-sections at \( x_1 \) and
\( x_1 + \Delta x_1 \); see Fig. 2.1. The rate of mass flow into this volume at
section \( x_1 \) is \( A \rho(x_1, t) v_1(x_1, t) \), and the rate at which mass flows out
at \( x_1 + \Delta x_1 \) is \( A \rho(x_1 + \Delta x_1, t) v_1(x_1 + \Delta x_1, t) \). Hence the net rate of
mass flow into this controlled volume becomes

\[
\frac{Dm}{Dt} = A \left( \rho(x_1, t) v_1(x_1, t) - \rho(x_1 + \Delta x_1, t) v_1(x_1 + \Delta x_1, t) \right)
\]

\[
= -A \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) v_1(\xi, t) \, d\xi . \tag{2.13}
\]

The time rate of change of the mass within the controlled volume,
on the other hand, can be written as

\[
\frac{Dm}{Dt} = A \int_{x_1}^{x_1 + \Delta x_1} \frac{\partial \rho(\xi, t)}{\partial t} \, d\xi . \tag{2.14}
\]
Since mass must be conserved, it follows that

$$\int_{x_1}^{x_1 + \Delta x_1} \left\{ \frac{\partial p(\xi, t)}{\partial \xi} + \frac{\partial}{\partial \xi} \left[ \rho(\xi, t) \nu_1(\xi, t) \right] \right\} \, d\xi = 0 \quad (2.15)$$

This integral must be zero for all admissible choice of $\Delta x_1$. If it is assumed that the integrand is continuous, then it must vanish identically.

To show this consider the following lemma.

**Lemma**

Let $f(x)$ be a continuous real-valued function of the real variable $x$ for $a < x < b$, and assume that $\int_{x_1}^{x_2} f(x) \, dx = 0$ for all $x_1$ and $x_2$ between $a$ and $b$. Then $f(x) = 0$ for all $a < x < b$.

To prove the lemma assume the converse, and let $f(x)$ be, for example, positive at a point $x = x_0$ in the considered interval. Since $f$ is continuous, there must be an interval $a < x_0 - \delta < x < x_0 + \delta < b$, $\delta > 0$, for which $f$ is positive. Hence $\int_{x_0 - \delta}^{x_0 + \delta} f(x) \, dx > 0$ which is a contradiction. A similar conclusion can be drawn, if one assumes that the function takes on negative value at a point in the interval. Thus $f(x)$ must be identically zero over the entire interval.

Applying this lemma to Eq. (2.15) obtain Eq. (2.10).
Figure 2.1

Controlled Volume: The rate at which the mass inside of the controlled volume between sections at $x_1$ and $x_1 + \Delta x_1$, changes $x_1 + \Delta x_1$ (increases) is $A \int_{x_1}^{x_1 + \Delta x_1} \frac{\partial \rho}{\partial t} \, d\xi$. This must equal the time rate of mass flow into the controlled volume, defined by Eq. (2.13).
5.3 BALANCE OF LINEAR MOMENTUM, RECTILINEAR MOTION

In the one-dimensional motion of the continuum in the tube discussed before, consider at time \( t \) the material contained within cross-sections at points \( x_1 \) and \( x_1 + \Delta x_1 \) on the \( X_1X_1 \)-axis. The corresponding linear momentum is

\[
\mathcal{L}_1 = A \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) \, v_1(\xi, t) \, d\xi ;
\]

(3.1)

this is because \( A \, d\xi \, \rho(\xi, t) \) is the mass of an elementary slice with thickness \( d\xi \) which at time \( t \) has the velocity-component \( v_1(\xi, t) \), the integral representing the sum of the momenta of these elementary slices.

Referring to Ch. 3, observe that one may remove material to the left of the section at \( x_1 \), and that to the right of section \( x_1 + \Delta x_1 \), and replace instead the corresponding tractions on these cross-sections, see Fig. 3.1. If the traction at \( x_1 \) is \( t_1(x_1, t) \), and that at section \( x_1 + \Delta x_1 \) is \( t_1(x_1 + \Delta x_1, t) \), both at the instant \( t \), then one can isolate the material within these two sections and apply Euler's first law (1.5).

In addition to surface tractions, the considered portion of material in the tube may also be acted upon by body forces \( f_1(x_1, t) \), so that the total force in the \( X_1X_1 \)-direction is

\[
\mathcal{F}_1 = A \left\{ -t_1(x_1, t) + t_1(x_1 + \Delta x_1, t) + \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) \, f_1(\xi, t) \, d\xi \right\}.
\]

(3.2)
Material to the left of section at $x_1$, and material to the right of section at $x_1 + \Delta x_1$ are removed and in their place tractions $t^I(x_1, t)$ and $t^I(x_1 + \Delta x_1, t)$ are applied.

Euler's law (1.5) now becomes

$$\frac{D\mathbf{E}_1}{Dt} = \mathbb{F}_1.$$  \hspace{1cm} (3.3)

To obtain the corresponding differential equation, the following limiting case must be considered:

$$\lim_{\Delta x_1 \to 0} \frac{1}{\Delta x_1} \frac{D\mathbf{E}_1}{Dt} = \lim_{\Delta x_1 \to 0} \frac{\mathbb{F}_1}{\Delta x_1}.$$  \hspace{1cm} (3.4)

This limiting process must, of course, be interpreted in the sense of the nested sequence of intervals discussed in the preceding section.

To calculate the right-hand side of (3.4), first apply the mean-value theorem to the integral in the right-hand side of (3.2), and obtain

$$\lim_{\Delta x_1 \to 0} \frac{1}{\Delta x_1} \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) f_1(\xi, t) d\xi = \rho(x_1, t) f_1(x_1, t).$$  \hspace{1cm} (3.5)
Moreover, the second two terms in the right-hand side of (3.2) yield

$$
\lim_{\Delta x_1 \to 0} \frac{t_1(x_1 + \Delta x_1, t) - t_1(x_1, t)}{\Delta x_1} = \frac{\partial t_1(x_1, t)}{\partial x_1} .
$$

(3.6)

To calculate the left-hand side of (3.4), first apply the mean-value theorem to (3.1), obtaining

$$
\Delta \tau = A \rho(x_1^*, t) v_1(x_1^*, t) \Delta x_1 .
$$

(3.7)

From this equation it follows that

$$
\frac{1}{A} \lim_{\Delta x_1 \to 0} \frac{1}{\Delta x_1} \frac{\Delta \tau_1}{\Delta t} = \lim_{\Delta x_1 \to 0} \rho(x_1^*, t) \frac{Dv_1(x_1^*, t)}{Dt} + \lim_{\Delta x_1 \to 0} \frac{1}{\Delta x_1} v_1(x_1^*, t) \frac{\Delta x_1}{\Delta x_1} \rho(x_1^*, t).
$$

(3.8)

It will now be shown that the last term in (3.8) vanishes. To this end consider

$$
\lim_{\Delta x_1 \to 0} \frac{1}{\Delta x_1} \frac{\Delta x_1}{\Delta x_1} \rho_1(x_1^*, t) = \lim_{\Delta x_1 \to 0} \frac{\Delta x_1}{\Delta x_1} \rho_1(x_1^*, t) = \frac{D}{Dx_1} \{ \rho_0(x_1) \} = 0 .
$$
From (3.8) and the above equation, it follows that
\[
\lim_{\Delta x_1 \to 0} \frac{1}{\Delta x_1} \frac{dx_1}{Dt} \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) v_1(\xi, t) d\xi = \rho(x_1, t) \frac{Dv_1(x_1, t)}{Dt} .
\] (3.9)

In view of (3.5), (3.6), and (3.9), Eq. (3.4) now becomes
\[
\frac{\partial t_I}{\partial x_1} + \rho f_1 = \rho \frac{Dv_1}{Dt} .
\] (3.10)

This is the differential equation giving a local description of Euler's first law. It is the local statement for the balance of linear momentum, and is commonly called Cauchy's first law. The right-hand side of (3.10) may also be written as
\[
\rho \frac{Dv_1}{Dt} = \rho \left[ \frac{\partial v_1}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x_1} v_1^2 \right] .
\] (3.11)

When a continuum is in a state of equilibrium, the right-hand side of Eq. (3.10) vanishes. Subject to appropriate boundary conditions, the left-hand side then defines the distribution of internal forces portrayed by $t_I$.

**EXAMPLE 3.1**

Consider a large body of fluid with constant mass-density $\rho$, which is in an equilibrium state under the constant gravitational force characterized by body forces $g$. Assume that a constant pressure $p_0$ (for example, atmospheric pressure) acts on the free surface of the fluid. Take the axis $X_1X_1$ normal to the plane boundary with origin at the free surface. Equation (3.10) becomes
\[ \frac{dt_I}{dx_1} + \rho g = 0, \quad (3.12) \]

where the total derivative with respect to \( x_1 \) is used, since no time-dependency exists. Integrating this equation, obtain

\[ t_I = -\rho g x_1 + C, \quad (3.13) \]

where \( C \) is the integration constant. At the free boundary \( x_1 = 0 \), \( t_I \) must equal the applied pressure, i.e. \( t_I = -p_o \), and hence, \( C = -p_o \).

Equation (3.13) then gives

\[ t_I = -p_o - \rho g x_1, \quad (3.14) \]

At any depth \( x_1 \), the hydrostatic pressure \( p = -t_I \) is

\[ p = p_o + \rho g x_1. \quad (3.15) \]

**BERNOULLI'S EQUATION**

For a steady flow, see Sec. 2.5, the first term in the right-hand side of (3.11) is zero, and Eq. (3.10) then reduces to

\[ \frac{\partial t_I}{\partial x_1} + \rho f_1 = \rho \frac{\partial}{\partial x_1} \left( \frac{1}{2} v_1^2 \right). \quad (3.16) \]

It is customary to set \( t_I = -p \), and when the body force \( f_1 \) admits a potential in the form of Eq. (2.3), Ch. 3, reduce (3.16) to

\[ -\frac{1}{\rho} \frac{\partial p}{\partial x_1} - \frac{\partial \phi}{\partial x_1} = \frac{\partial}{\partial x_1} \left( \frac{1}{2} v_1^2 \right) \quad (3.17) \]

which, upon integration, becomes
\[ \int \frac{dp}{\rho} + \varphi + \frac{1}{2} v_{1}^{2} = \text{constant} \quad (3.18) \]

This is Bernoulli's Equation. It states that, for steady flow, the quantity in the left-hand side must remain constant. Note that, for the flow to be steady, the body forces must be stationary and hence \( \varphi \) cannot depend explicitly on time.

Equation (3.18) ceases to be valid when the flow is not steady, for example, if \( \varphi \) depends explicitly on time. A modified, more general form can be obtained for cases in which the velocity field admits a velocity potential \( \psi \) such that

\[ v_{1} = -\frac{\partial \psi}{\partial x_{1}} \quad (3.19) \]

where \( \psi \) will, of course, be also an explicit function of time. In this more general case, (3.17) modifies to

\[ -\frac{1}{\rho} \frac{\partial p}{\partial x_{1}} - \frac{\partial \varphi}{\partial x_{1}} = \frac{\partial}{\partial x_{1}} \left( \frac{1}{2} v_{1}^{2} - \frac{\partial \psi}{\partial t} \right) \quad (3.20) \]

which, upon integration with respect to \( x_{1} \), yields

\[ \int \frac{dp}{\rho} + \varphi + \frac{1}{2} v_{1}^{2} - \frac{\partial \psi}{\partial t} = f(t) \quad (3.21) \]

This is the Bernoulli equation for one-dimensional non-steady flow. The functions \( \varphi \) and \( \psi \) in this equation may depend explicitly on \( t \) as well as on \( x_{1} \). When the last term in the left-hand side is absent and the potential of the body forces is time-independent, the right-hand side of (3.21) becomes a constant; this equation then reduces to (3.18).

When the continuum is incompressible, the mass-density \( \rho \) is then a constant, and the integral in (3.18) becomes
\[ \int \frac{dp}{\rho} = \frac{p}{\rho}, \]

reducing (3.18) to

\[ \frac{p}{\rho} + \varphi + \frac{1}{2} \nu_1^2 = \text{constant}. \]  

(3.22)

**ALTERNATIVE DERIVATION OF CAUCHY'S FIRST LAW**

A less rigorous but highly intuitive derivation of Eq. (3.10) will now be given. This derivation, however, can be rendered rigorous by means of the mean-value theorem.

Consider a portion of the continuum within a slice of elementary thickness \(dx_1\) between sections at \(x_1\) and \(x_1 + dx_1\). The surface tractions at \(x_1\) are \(t_1(x_1, t)\) and those at \(x_1 + dx_1\) are \(t_1(x_1, t) + \frac{\partial t_1(x_1, t)}{\partial x_1} dx_1\). The corresponding body forces are \(A \rho(x_1, t) f_1(x_1, t) dx_1\).

Let us now apply Newton's second law. Since the mass of the considered portion of the continuum is \(A \rho(x_1, t) dx_1\), this law yields

\[ \frac{\partial t_1}{\partial x_1} + \rho f_1 = \rho a_1, \]  

(3.23)

where \(a_1 = a_1(x_1, t)\) is the acceleration. The left-hand side in (3.23) is proportional to the resultant force consisting of tractions and body forces, which acts on the mass contained within the elementary slice \(dx_1\). The right-hand side, on the other hand, is proportional to the corresponding initial force, i.e. mass times acceleration.

\[ \text{The proportionality factor is the volume of this slice, i.e. } A dx_1. \]
5.4 **Remark on Material Derivative**

In the preceding section the material time derivative of the linear momentum \( \mathbf{L}_1 \) was calculated for the limiting case defined by the left-hand side of Eq. (3.4). This led to Eq. (3.9). Consider now the integral of the right-hand side of (3.9) over the material contained within sections at \( x_1 \) and \( x_1 + \Delta x_1 \),

\[
J = \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) \frac{Dv_1(\xi, t)}{Dt} \, d\xi .
\]  

(4.1)

Because of the conservation of mass, it immediately follows that

\[
A \frac{D}{Dt} \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) v_1(\xi, t) \, d\xi = A \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) \frac{Dv_1(\xi, t)}{Dt} \, d\xi .
\]

(4.2)

To see this, note that \( A \, d\xi \rho(\xi, t) \) represents the elementary mass \( dm \) of the material contained within an elementary slice of thickness \( d\xi \), and rewrite (4.2) as

\[
\frac{D}{Dt} \int_{m} v_1(\xi, t) \, dm = \int_{m} \frac{Dv_1(\xi, t)}{Dt} \, dm ,
\]

(4.3)

where \( m \) is the total mass of the continuum between \( x_1 \) and \( x_1 + \Delta x_1 \). Since the integration is now over a fixed mass of the continuum, the material time differentiation reduces to the partial time derivative which is commutative with the integration over this fixed mass of the continuum, provided that the involved quantities are sufficiently smooth.
The above result has general validity as long as the involved quantities of motion are sufficiently smooth.

Suppose, in general, that \( q(x_1, t) \) represents a quantity of motion defined on the particle that at time \( t \) is at point \( x_1 \), and consider the integral

\[
\mathcal{J} = \int_{x_1}^{x_1 + \Delta x_1} q(\xi, t) \, d\xi ;
\]

the constant \( A \) is not included here. Suppose now that the material time derivative of \( \mathcal{J} \) is needed. To this end, write

\[
\frac{D \mathcal{J}}{Dt} = \frac{D}{Dt} \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) \left[ \frac{q(\xi, t)}{\rho(\xi, t)} \right] d\xi
\]

\[
= \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi, t) \frac{D}{Dt} \left[ \frac{q(\xi, t)}{\rho(\xi, t)} \right] d\xi .
\]

Here the integrand is divided and multiplied by \( \rho(\xi, t) \). Then the quantity in the bracket can be regarded as the quantity of motion defined over a fixed mass of the continuum, in which case the material time differentiation and the integration commute.

In view of (4.2) and the fact that Eq. (3.2) is equivalent to

\[
\mathcal{J} = \int_{x_1}^{x_1 + \Delta x_1} \left\{ \frac{\partial t_1(\xi, t)}{\partial \xi} + \rho(\xi, t) f_1(\xi, t) \right\} d\xi ,
\]

(4.6)
Cauchy's first law (3.10) can be regarded as an immediate consequence of

\[ A \int_{x_1}^{x_1+\Delta x_1} \left\{ \frac{\partial t^I}{\partial s} + \rho f^I - \rho \frac{Dv}{Dt} \right\} \, ds = 0 \]  

(4.7)

and the lemma cited on p. 5-18.
5.5 CONSERVATION OF ENERGY, RECTILINEAR MOTION

Consider the portion of the continuum in the tube, as in the preceding three sections, and let us seek the consequence of the conservation of energy stated by Eq. (1.18).

To calculate the rate of heat flow, observe that the heat flux vectors \( q_1(x_1, t) \) and \( q_1(x_1 + \Delta x_1, t) \) are constant over the cross-sections at \( x_1 \) and \( x_1 + \Delta x_1 \), respectively; see Fig. 5.1. Thus, the net heat flow into the material per unit time through these cross-sections is

\[
A \left[ q_1(x_1, t) - q_1(x_1 + \Delta x_1, t) \right] = -A \int_{x_1}^{x_1 + \Delta x_1} \frac{\partial q_1(\xi, t)}{\partial \xi} \, d\xi. \tag{5.1}
\]

![Figure 5.1](image)

Heat flows in and out through sections at \( x_1 \) and \( x_1 + \Delta x_1 \), respectively. This is represented by the component \( q_1 \) of the heat flux vector \( q \), which points outward when \( q_1 \) is positive. Hence heat flow per unit time into the material between sections \( x_1 \) and \( x_1 + \Delta x_1 \) is equal to \( A \left[ q_1(x_1, t) - q_1(x_1 + \Delta x_1, t) \right] \).
In view of (5.1), Eq. (1.16) becomes

\[ \mathcal{W} = A \int_{x_1}^{x_1 + \Delta x_1} \left[ -\frac{\partial q_1(\xi,t)}{\partial \xi} + \rho(\xi,t) h(\xi,t) \right] d\xi . \]  (5.2)

The rate of the mechanical work, namely the second term in the right-hand side of (1.18), in the present case takes on the form

\[ \frac{Dw}{Dt} = A \int_{x_1}^{x_1 + \Delta x_1} \left[ - t_1(x_1,t) v_1(x_1,t) + t_1(x_1 + \Delta x_1,t) v_1(x_1 + \Delta x_1,t) \right. \]

\[ + \left. \int_{x_1}^{x_1 + \Delta x_1} \rho(\xi,t) f_1(\xi,t) v_1(\xi,t) d\xi \right] \]

\[ = A \int_{x_1}^{x_1 + \Delta x_1} \left[ \frac{\partial}{\partial \xi} (t_1 v_1) + \rho f_1 v_1 \right] d\xi \]

\[ = A \int_{x_1}^{x_1 + \Delta x_1} \left[ \left( \frac{\partial}{\partial \xi} + \rho f_1 \right) v_1 + t_1 \frac{\partial v_1}{\partial \xi} \right] d\xi . \]  (5.3)

The rate of change of the kinetic energy and the internal energy, namely the quantity in the left-hand side of (1.18), for the present case can be expressed as

\[ A \frac{D}{Dt} \int_{x_1}^{x_1 + \Delta x_1} \left[ \frac{1}{2} v_1^2 + e \right] \rho d\xi = A \int_{x_1}^{x_1 + \Delta x_1} \left[ v_1 \frac{Dv_1}{Dt} + \frac{De}{Dt} \right] \rho d\xi , \]  (5.4)

where Eq. (4.6) is used.
In view of (5.2), (5.3), and (5.4), the conservation of energy becomes

\[
\int_{x_1}^{x_1 + \Delta x_1} \left[ \frac{\partial t}{\partial \xi} + \rho f_1 - \rho \frac{Dv_1}{Dt} \right] v_1 \, d\xi + \int_{x_1}^{x_1 + \Delta x_1} \left[ t_1 \, d_1 - \frac{\partial q_1}{\partial \xi} \right] d\xi = 0 ,
\]

(5.5)

where, in addition to some rearrangement, \( \frac{\partial v_1}{\partial \xi} = d_1 (\xi, t) \) which is the stretch rate defined by Eq. (5.27) of Ch. 2, is used. The quantity inside the first integral is identically zero as a consequence of Cauchy's first law, Eq. (3.10). Hence the second integral must be zero for any admissible choice of \( \Delta x_1 \). If it is assumed that the integrand is continuous, then it must be identically zero; see the lemma on p. 5-18.

It thus follows that

\[
\rho \frac{Dv}{Dt} = t_1 d_1 - \frac{\partial q_1}{\partial x_1} + \rho h
\]

(5.6)

which is the local form for the conservation of energy. The first quantity on the right-hand side is called the stress power. This equation states that the local time rate of change of internal energy corresponds to stress power and to the heat energy due to heat conduction and radiation absorption. Note that (5.6) is expressed in terms of the spatial variables \( x_1 \) and \( t \). Thus one has

\[
\frac{De}{Dt} = \frac{\partial e}{\partial t} + \frac{\partial e}{\partial x_1} v_1 .
\]
5.6 BALANCE LAWS ACROSS A DISCONTINUITY FRONT

The results developed so far were based on the assumption that the quantities of motion such as mass-density, displacement, velocity, etc. are continuous and possess continuous derivatives in the domain of their definition. The corresponding equations, for example, Eqs. (2.10), (3.10), and (5.6), are valid within the region where the continuity requirements are met.

There are many interesting problems in which one or several quantities of motion, or their derivatives, may suffer a discontinuity over an isolated surface. For example, in shock waves there is a thin layer of material with the thickness of the order of $10^{-4}$ cm (the shock), across which the particle velocity, the mass-density, and the pressure, for instance, undergo large but finite changes; the displacement, however, remains continuous. In the sequel the example of the motion of a continuum in the tube will be used to develop the conservation equations valid across a shock front. In actuality the quantities of motion vary continuously from one side of the shock to the other side, but with a very steep gradient. Since the shock layer is very thin, one can treat it as a discontinuity surface, and obtain results which have reasonable accuracy. Within the thin layer of shock, even in gases which closely comply with the perfect gas law under normal conditions, there is a great deal of dissipation due to viscous effects which result in the generation of a substantial amount of heat. This particular problem will be discussed further later on in connection with the problem of a shock tube. Here attention will be confined to developing the basic balance equations across a shock which will be treated as a discontinuity plane.
CONSERVATION OF MASS ACROSS DISCONTINUITY PLANE

Figure 6.1 shows the discontinuity (shock) front as plane II which is assumed to be moving toward the right with speed $U^*$ relative to the fixed $X_1X_2$-axis. The mass-density, particle velocity, and the traction, behind (to the left of) the shock, are denoted by superscript (2), and those in front, by (1). To obtain a statement for the conservation of mass one cannot use the procedure outlined in Sec. 5.2, since the mean-value theorem is no longer applicable. Instead one proceeds as follows.

Since the particle velocity (with respect to the $X_1X_2$-axis) behind the shock is $v_2^{(2)} = v_1^{(2)} e_1$, and since the shock velocity is $U^* e_1$, the shock velocity with respect to the particles behind, is $(U^* - v_1^{(2)}) e_1$. Similarly, the velocity of shock relative to the particles in front, is $(U^* - v_1^{(1)}) e_1$. The mass flow into the shock per unit of time at each instant is, therefore, given by

$$\frac{Dm}{Dt} = \left[ U^* - v_1^{(1)} \right] A \rho^{(1)} \quad (6.1)$$

which represents the rate at which plane II sweeps mass as it moves forward with its absolute velocity $U^* e_1$. The rate at which mass flows out of the shock (left behind) is

$$\frac{Dm}{Dt} = \left[ U^* - v_1^{(2)} \right] A \rho^{(2)} \quad . \quad (6.2)$$

The conservation of mass requires that (6.1) should equal (6.2), which then yields

$$\left[ U^* - v_1^{(1)} \right] \rho^{(1)} = \left[ U^* - v_1^{(2)} \right] \rho^{(2)} \quad . \quad (6.3)$$
(a) Across the discontinuity plane I I (shock) the mass-density, particle velocity, tractions, and other quantities of motion undergo finite jumps. Plane I I moves to the right with speed $U^*$. 

(b) The Variation of Mass-Density Along the $X_1X_1$-Axis: A finite jump $<\rho> = \rho^{(2)} - \rho^{(1)}$ occurs across plane I I. Other quantities may have similar finite jumps.
This is the statement of the conservation of mass across the discontinuity front I I. Note carefully the meaning of each term. The quantities \( \rho^{(1)} \) and \( v_1^{(1)} \) are the limiting values of the mass-density and particle velocity, as plane I I is approached from its right along (its normal) the \( X_1X_1' \)-axis. The quantities \( \rho^{(2)} \) and \( v_1^{(2)} \), on the other hand, are the limiting values of the corresponding quantities as plane I I is approached from its left along the \( X_1X_1' \)-axis. \( U^* \) in (6.3), moreover, is the speed with which plane I I moves relative to the \( X_1X_1' \)-axis, to the right (if \( U^* \) is positive).

It is convenient to denote the jump in a typical quantity of motion, say, \( q \), by

\[
< q >= q^{(2)} - q^{(1)}
\]  \hspace{1cm} (6.4)

With this convention (6.3) can be reduced to

\[
< pU > = 0 \ ,
\]  \hspace{1cm} (6.5)

where the notation

\[
U^{(1)} = U^* - v_1^{(1)} \ , \quad U^{(2)} = U^* - v_1^{(2)} \ ,
\]  \hspace{1cm} (6.6)

which define the velocity of plane I I (shock) relative to the particles in its front and its back, respectively, is used.

It may be useful to compare (6.5) with the statement of conservation of mass when no discontinuity exists. If one confines attention to a fixed portion of matter contained within the tube, this latter conservation law can be stated as

\[\]

\[1\] See Eq. (4.6) and note that, in connection with (6.7), the quantity inside of brackets in (11.4) is equal to 1.
where, since $A$ is constant, it is not included. This equation says that the mass instantaneously contained within sections at $x_1$ and at $x_1 + \Delta x_1$, is conserved. Let us now consider at an instant $t$, two cross-sections, one behind at $x_1$, the other ahead at $x_1 + \Delta x_1$, of the discontinuity plane I I. Equation (6.5), therefore, represents the jump condition corresponding to the balance equation (6.7); i.e. the latter replaces the former, across discontinuity planes. In fact, if $\rho$ in Eq. (6.7) is replaced by any other quantity of motion (quantities defined on particles of the continuum), say, $Q$, so that one has

$$\frac{\partial}{\partial t} \int_{x_1}^{x_1 + \Delta x_1} Q(\xi, t) \, d\xi = 0$$

whenever $Q$ is continuous for $x_1 \leq \xi \leq x_1 + \Delta x_1$, then across a discontinuity where $Q$ suffers a finite jump, the balance equation (6.8) implies that

$$< Q U > = 0 . \quad (6.9)$$

**BALANCE OF LINEAR MOMENTUM ACROSS DISCONTINUITY PLANE**

Equation (3.3) states the balance of linear momentum. From (3.1) it is observed that the left-hand side of (3.3) has the form given by the left-hand side of (6.8), and, therefore, according to the preceding discussion, reduces to the left-hand side of (6.9). Hence, if one replaces $Q$ by the integrand in the right-hand side of (3.1), one obtains $A < Q U >$
which represents the left-hand side of (3.3) across the discontinuity plane I I. The right-hand side of (3.3), on the other hand, reduces to the negative of the jump in the traction, \(-A < t \_ I\), if section \(x_1\) is chosen to the left of plane I I, and section \(x_1 + \Delta x_1\) to its right, and then the limit is taken as these two sections are taken closer and closer to plane I I, converging on this plane at the limit. Observe that the integral in the right-hand side of (3.2) goes to zero with \(\Delta x_1\) even though its integrand has a finite jump at the discontinuity plane. To see this, let at the instant \(t\), \(M\) denote the maximum value of \(\rho(\xi,t) f_1(\xi,t)\) over the interval \(x_1 \leq \xi \leq x_1 + \Delta x_1\). Then

\[
\int_{x_1}^{x_1 + \Delta x_1} \rho(\xi,t) f_1(\xi,t) \, d\xi \leq M \Delta x_1 .
\]

Since \(M\) is finite, the right-hand side of this inequality goes to zero with \(\Delta x_1\).

The balance of linear momentum across the discontinuity plane I I, therefore, becomes

\[
< \rho v_1 U > = - < t \_ I >
\]

which can be written as

\[
< \rho v_1 U + t \_ I > = 0 . \tag{6.10}
\]

Written explicitly, this equation is

\[
\left[ \rho \left( v_1 \left( U \right)^{(2)} + t \_ I^{(2)} \right) \right] - \left[ \rho \left( v_1 \left( U \right)^{(1)} + t \_ I^{(1)} \right) \right] = 0 . \tag{6.11}
\]
With the aid of (6.5), Eq. (6.11) may be expressed as

\[ \rho^{(2)} u^{(2)} (v_{1}^{(1)} - v_{1}^{(2)}) = t_{I}^{(2)} - t_{I}^{(1)} \]

But one also has

\[ v_{1}^{(1)} - v_{1}^{(2)} = \left( u^{*} - v_{1}^{(2)} \right) - \left( u^{*} - v_{1}^{(1)} \right) \]

\[ = u^{(2)} - u^{(1)} = < u > \]

where (6.6) is also used. Hence it follows that

\[ \rho^{(2)} u^{(2)} < u > = < t_{I} > \quad (6.12) \]

which may also be rewritten in the following more common form

\[ \rho^{(1)} (u^{(1)})^2 - t_{I}^{(1)} = \rho^{(2)} (u^{(2)})^2 - t_{I}^{(2)} \quad (6.13) \]

In most books on gas-dynamics, where the gas pressure \( p = - t_{I} \) is the only internal force that must be dealt with, Eq. (6.14) is often written as

\[ < \rho u^2 + p > = 0 \quad , \quad p = - t_{I} \quad (6.14) \]

**BALANCE OF ENERGY ACROSS DISCONTINUITY PLANE**

The student by analogy should be able to write down the jump condition for the energy balance equation (1.18). The left-hand side of (5.4) gives \( A < \left( \frac{1}{2} \rho v_{1}^2 + \rho e \right) u > \). The first term in the right-hand side of (1.18), according to the quantity inside the first pair of brackets in (5.3), becomes \( -A < t_{I} v_{1} > \). The last term on the right of (1.18), finally, reduces to \( A < q_{I} > \), provided one notes (5.1), and the fact that the integral corresponding to the heat supply \( h \) approaches zero, as the two
cross-sections (the one behind, the other ahead of plane I I) approach each other\(^1\). Thus one arrives at

\[
\langle \rho u \left( e + \frac{1}{2} v_1^2 \right) + t_1 v_1 - q_1 \rangle = 0
\]

(6.15)

which is the balance equation for energy across a discontinuity plane.

Note that, with the aid of (6.5) and (6.6), this equation may be expressed as

\[
\rho^{(1)} U^{(1)} \langle e + \frac{1}{2} v_1^2 \rangle = \langle q_1 \rangle + \langle t_1 U \rangle - U^* \langle t_1 \rangle .
\]

(6.16)

\(^1\) The situation here is similar to that of the body forces discussed before.
PROBLEMS FOR CHAPTER 5

2.1 For a steady flow through a tube of constant cross-section \( A \), the mass-density is given by

\[
\rho = \frac{2x^2 + 1}{x^2 + 1}.
\]

Find the corresponding velocity when the rate of mass flow is equal to \( Q \).

2.2 For a flow through a tube whose cross-section changes slightly from \( A_1 \) to \( A_2 \), one may assume a uniform velocity over each cross-section.

(a) Let \( A = A(x_1) \) be a given function of \( x_1 \). Modify Eq. (2.15) to obtain the following spatial continuity equation:

\[
\frac{\partial \rho(x_1,t)}{\partial t} A(x_1) + \frac{\partial}{\partial x_1} \left[ \rho(x_1,t) v_1(x_1,t) A(x_1) \right] = 0.
\]

(b) Assume a steady flow of an incompressible material. What is the change in velocity if the cross-sectional area increases by 10%?

2.3 Prob. 3.3, p. 3-36.


3.1 A cylindrical can with one end open is observed to be floating on a liquid of density \( \rho \) with the open end down. The can is of weight \( W \) and is supported by air that is trapped in the can as shown below.
The can floats out of the fluid a distance $h$. If the air is assumed to follow Boyle's law, i.e., pressure times volume is a constant, determine the force $F$ necessary just to submerge the can. The internal cross-sectional area of the can is $A$.

Note: The thickness of the wall is assumed to be zero and the hydrostatic pressure due to the atmosphere may be neglected.

Hint: The distances $x_1$ and $x_2$ may be used as auxiliary quantities, but they should not appear in the final answer.

3.2 A plate of weight 62.4 lbf per unit width is suspended at one end by a hinge at the water level of the reservoir shown below. The bottom end is free to move as in the preceding sketch. Calculate the angle of repose, $\theta$, of the plate, using 62.4 lbf/ft$^3$ as the specific weight of the water.
3.3 A pipe of uniform diameter and of length \( L \) is connected to a large vessel in which the pressure is maintained at a constant value \( p_1 \). A valve at the end of the pipe prevents flow. The length \( L \) is very much greater than the pipe diameter and the vessel is sufficiently large so that the velocity anywhere within the vessel may be neglected. The atmospheric pressure is \( p_a \), and an incompressible, frictionless fluid of density \( \rho \) completely fills the pipe and vessel. Elevation changes are negligible. The valve is suddenly opened and flow commences.

(a) Determine the flow velocity at the end of the pipe as a function of time.

(b) If \( L = 150 \) ft, \( p_1 = 30 \) psia, \( p_a = 14.7 \) psia and the fluid is water, how much time is required to achieve 90 per cent of the ultimate velocity?

Hint: Note that the velocity in the pipe is a function of time only.

3.4 Two very long parallel plates of length \( 2L \) are separated a distance \( b \). The upper plate moves downward at a steady rate \( V \). A nonviscous and incompressible fluid of density \( \rho \) fills the gap between the plates. Fluid is squeezed out between the plates and since the flow is symmetrical the velocity parallel to the plate at the center is zero. Assume that \( b \ll L \), and that the velocity \( u \) parallel to the plate is substantially constant across the gap. Treat the flow as being one dimensional and parallel to the \( x \)-axis.
(a) Show that the velocity at any point \( x \) from the center is \( u = \frac{Vx}{b} \) approximately.

(b) Noting that \( b \) changes with time and assuming that the pressure outside the plates is zero, obtain an expression for the pressure at any point \( x \) along the plate. Neglect gravity.
CHAPTER 6

APPLICATIONS: FLUID

6.1 RECTILINEAR LAMINAR FLOW OF INCOMPRESSIBLE NEWTONIAN FLUIDS

Consider two stationary rigid plates a distance \( h \) apart, as shown in Fig. 1.1a. Assume that the plates are infinitely extended perpendicular to the plane of the figure, and consider the flow of an incompressible (\( \rho = \text{constant} \)) Newtonian fluid between these plates in the \( X_1 X_1 \)-direction.

Since the fluid is viscous, and the velocity varies in the \( X_2 X_2 \)-direction, Eq. (5-3.10) must be modified as to include the shear stresses \( t_{12} \).

To this end, consider a slice of unit thickness perpendicular to the paper in Fig. 1.1, and balance the linear momentum for an elementary parallelepiped ABCD - \( A'B'C'D' \) shown in Fig. 1.1b. The surface tractions, which represent the action of the fluid outside upon that within the parallelepiped, consist of normal stresses \( t_{11} \) and shear stresses \( t_{12} = t_{21} \) acting on face \( AA'D'D \), as shown. On the opposite face, \( BB'C'C \), the normal stress is changed to \( t_{11} + \frac{\partial t_{11}}{\partial x_1} \, dx_1 \), and the shear stress has changed to \( t_{12} + \frac{\partial t_{12}}{\partial x_1} \, dx_1 \), since this face is a distance \( dx_1 \) from face \( AA'D'D \). The shear stress on face \( ABB'A' \) is \( t_{12} \), and that on face \( CC'D'D \) is \( t_{12} + \frac{\partial t_{12}}{\partial x_2} \, dx_2 \). On these faces, normal stresses may also act, but, since the linear momentum is balanced in the \( X_1 X_1 \)-direction only, these forces which must, however, be self-equilibrating, do not enter. Keeping in mind that the parallelepiped is of unit thickness, the total resultant force acting on it in the \( X_1 X_1 \)-direction is
Figure 1.1

(a) Laminar flow between parallel plates.
(b) Surface tractions on an elementary fluid parallelepiped.
(c) Flow down an inclined plane.
(d) Flow in a circular tube.
\[
\left[ \frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{12}}{\partial x_2} + \rho f_1 \right] dx_1 dx_2
\]

where \( f_1 \) is the component of body force per unit mass in the \( x_1 \)-direction. The mass contained within this parallelepiped is \( \rho \, dx_1 dx_2 \), and if the velocity in the \( x_1 \)-direction is denoted by \( v_1 \), the balance of linear momentum requires that the inertial force \( \rho \, dx_1 dx_2 \frac{Dv_1}{Dt} \) equal the above total force acting on the parallelepiped. It thus follows that

\[
\frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{12}}{\partial x_2} + \rho f_1 = \rho \frac{Dv_1}{Dt} .
\] (1.1)

**STEADY FLOW**

Consider now steady flow under a pressure-gradient. Let \( t_{11} = -p_0 \) at \( x_1 = 0 \), and \( t_{11} = -p_1 \) at \( x_1 = L \). Assume that the body forces are zero, and hence reduce (1.1) to

\[
\frac{\partial t_{11}}{\partial x_1} + \frac{\partial t_{12}}{\partial x_2} = 0 .
\] (1.2)

For the Newtonian fluid one has

\[
t_{12} = t_{21} = \mu \frac{\partial v_1}{\partial x_2} ,
\] (1.3)

and, since the flow is steady and the fluid incompressible, \( v_1 = v_1(x_2) \), and hence \( t_{12} = t_{12}(x_2) \). The \( t_{11} \), on the other hand, depends only on \( x_1 \). Consequently, (1.2) and (1.3) can be reduced to

\[
\frac{dt_{11}}{dx_1} = -\mu \frac{d^2 v_1}{dx_2^2} = K_1 ,
\] (1.4)
where, since the first term is only a function of \( x_1 \), and the second, a function of \( x_2 \), their equality implies that both are equal to a constant, \( K_1 \). Upon integration (1.4) yields

\[
t_{11} = K_1 x_1 + K_2 ,
\]

\[
v_1 = -\frac{1}{2} \frac{K_1}{\mu} x_2^2 + K_2' x_2 + K_3 .
\]

To obtain the constants of integration, observe that

\[
t_{11} = -p_0 \text{ at } x_1 = 0 , \quad t_{11} = -p_1 \text{ at } x_1 = L ,
\]

which yield

\[
t_{11} = -p_0 + \frac{p_0 - p_1}{L} x_1 . \tag{1.5}
\]

Moreover, from the no-slip condition it follows that

\[
v_1 = 0 \text{ at } x_2 = 0 \text{ and } x_2 = h ,
\]

leading to

\[
v_1 = \frac{1}{2\mu} \frac{P_0 - p_1}{L} \left( h x_2 - x_2^2 \right) . \tag{1.6}
\]

As is seen from (1.6), the velocity profile is parabolic.

If the upper plate in Fig. 1.1a is not stationary, but moves at a constant velocity \( V^0 \) in the positive \( X_1 \)-direction, the corresponding velocity profile would consist of that of simple shear given by Eq. (4-8.1) superimposed on (1.6) above, so that one has

\[
v_1 = \frac{V^0}{h} x_2 + \frac{1}{2\mu} \frac{P_0 - p_1}{L} \left( h x_2 - x_2^2 \right) . \tag{1.7}
\]
LAMINAR FLOW DOWN AN INCLINED PLANE

When the plates are inclined by an angle $\alpha$, so that body forces of magnitude $g \cos \alpha$ due to the gravitational attraction act on the fluid particles, the same solution is obtained, except for the fact that the pressure-gradient $\frac{P_0 - P_1}{L}$ must be increased by $\rho \ g \cos \alpha$. In this case the constant $K_1$ will have the following value:

$$K_1 = \rho \ g \cos \alpha + \frac{P_0 - P_1}{L}.$$

The solution for the steady flow of a Newtonian fluid down an inclined plane (upper surface free, Fig. 1.1c) is obtained, if in (1.6) $h$ is replaced by $2h$, and $\frac{P_0 - P_1}{L}$ by $\rho \ g \cos \alpha$, arriving at

$$v_1 = \frac{\rho \ g \cos \alpha}{2\mu} \left(2h/2 - x_2^2\right),$$  \hspace{1cm} (1.8)

where $h$ is the depth of the fluid. This is because the flow with the free surface and depth $h$ is equivalent to the flow between two plates $2h$ apart, since in the latter case the velocity profile is symmetric with respect to the plane $X_2 = h$.

LAMINAR FLOW IN CIRCULAR TUBE

The velocity profile for the laminar flow in a circular tube can be obtained in the same manner as outlined above. As one would expect, this profile is parabolic, having zero value at the tube's wall, and a maximum value at the center. If the distance measured from the center of the tube is denoted by $r$, for a cylindrical portion of fluid, having the tube's axis as its axis and a circular cross-section of radius $r$, there are shear stresses $t_{12} = \mu \frac{dv}{dr}$, $v = v(r)$ being the velocity, acting on its cylindrical surface,
and normal tractions (pressure) $p_0$ and $p_1$ acting on the end sections at $X_1 = 0$ and $X_1 = L$, respectively; see Fig. 1.1d. The resultant force due to end pressures, therefore, is $\pi r^2 (p_0 - p_1)$. The total force due to the shear stress on the cylindrical boundary is $2\pi rL\mu \frac{dv}{dr}$. Balancing the forces, one must have

$$\frac{p_0 - p_1}{L} \frac{r}{2} + \mu \frac{dv}{dr} = 0,$$

which, upon integration, yields

$$v = \frac{p_0 - p_1}{L} \frac{1}{4\mu} \left( r_0^2 - r^2 \right), \quad (1.9)$$

where $r_0$ is the radius of the tube.

The maximum velocity occurs at the center, and has the value of

$$v_{max} = \frac{p_0 - p_1}{L} \frac{r_0^2}{4\mu}, \quad (1.10)$$

which is twice the average velocity given by

$$v = \frac{p_0 - p_1}{L} \frac{r_0^2}{8\mu}. \quad (1.11)$$

To obtain the flow rate either multiply the average velocity $V$ by the tube's area $\pi r_0^2$, or calculate it as follows:

$$Q = \int_0^{r_0} v 2\pi r dr = \frac{p_0 - p_1}{L} \frac{\pi}{8\mu} r_0^4. \quad (1.12)$$

This result is often called the Hagen-Poiseuille law after G. Hagen and L. J. M. Poiseuille.
The examples presented above illustrate the laminar flow of Newtonian fluids in which a fluid moves in layers. The flow is very regular, and there is no velocity fluctuation at a given point. This is in contrast to the turbulent flow, which is considered in the sequel.

Note that in the above examples of laminar flows, the pressure-gradient is proportional to the flow velocity.

6.2 TURBULENT FLOW AND REYNOLDS NUMBER

The laminar flow in which fluid layers move in a regular fashion relative to each other, is very special, since many flow regimes of practical importance are turbulent. In turbulent flow there are secondary eddy movements of the fluid superimposed on its bulk motion, resulting in a continuous mixing.

Even if the bulk motion of the fluid is very simple, for example the rectilinear flow in a tube, the secondary motion involves a complex deformation and rotation of material neighborhoods. Material elements are stretched, and material neighborhoods are distorted, as they spin in their secondary eddy movements. Such motion is, however, continuous, and of macroscopic character, although they occur usually on a much smaller scale than that of the bulk motion.

It is clear that, in such secondary complex turbulent movement of the fluid particles, additional retarding forces manifest themselves, which hinder the apparent relative bulk movement of fluid layers, i.e., there is additional viscosity. Roughly speaking, this additional viscosity stems from the exchange of linear momentum between fluid layers, the exchange of angular momentum corresponding to the spin (vorticity)
of fluid elements, and finally, actual viscous effects in the deformation process that is involved in such secondary fluid movements. Various writers have stressed one or all of these factors in attempting to describe turbulence. Osborne Reynolds back in 1894 introduced the concept of additional internal stresses (eddy stresses) associated with the exchange of linear momentum involved in the secondary movement of the fluid particles. It was then L. Prandtl, who in 1925 supplemented Reynolds' idea with the concept of a "mixing length" which, in rough terms, was introduced to represent the average distance that a small fluid mass travels before losing its identity (i.e., its additional linear momentum which it carries with it from the region of higher velocity) by mixing with the fluid in the neighborhood to which it penetrates. The situation may be visualized analogously (but not similarly) to the transport of linear momentum by gas molecules, which was discussed in Sec. 4.8. There are, however, essential differences, in the sense that, (1) the secondary fluid movement in turbulence is macroscopic in character and involves continuous deformation of material neighborhoods, whereas gas molecules are discrete microscopic elements moving randomly as they collide with each other; (2) the transport velocity in the secondary motion is usually smaller than the bulk fluid velocity, while the random translational velocity of the molecules, is, in general, substantially larger than the macroscopic bulk velocity of the gas; (3) it is not easy to give Prandtl's mixing length a well-defined statistical character, while the mean free path involved in molecular collisions can be estimated with a high degree of accuracy.
G. I. Taylor in 1932 considered the same concept of mixing length, but stressed the transport of vorticity (the spin of fluid neighborhoods) in the secondary fluid movement. Theodore Van Karman, on the other hand, in 1937 proposed equations which accounted for the average momentum transfer in turbulent motion, as well as vorticity dissipation and the viscous effect of eddies.

The more modern trend in the study of turbulence has been toward a purely statistical (semi-empirical) approach where, with the aid of experimentation, attempts are made to obtain various statistical parameters which characterize the motion. At the present time a great deal of activity is devoted to this end.

REYNOLDS NUMBER

Referring to Eq. (1.1) and using Newton's viscosity law (1.3), write

\[
\frac{\partial t_{11}}{\partial x_1} + \mu \frac{\partial^2 v_1}{\partial x_2^2} + \rho f_1 = \rho \frac{Dv_1}{Dt}
\]  

which characterizes the rectilinear flow of Newtonian fluids. As is commonly used, replace \( t_{11} \) by \(- p\) which represents the pressure, and obtain

\[
-\frac{\partial p}{\partial x_1} + \mu \frac{\partial^2 v_1}{\partial x_2^2} + \rho f_1 = \rho \frac{Dv_1}{Dt}.
\]  

In the study of laminar flow, the inertial term in the right-hand side of this equation did not occur. For the case of an inviscid fluid, on the other hand, \( \mu = 0 \), and therefore, the second term in (2.2), which

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1 Actually G. I. Taylor considered such an analysis as early as 1915; see his Scientific Papers, Vol. 2, Cambridge University Press, 1960, papers 1 and 24.
represents the viscous effects, disappears. In most flows, however, both effects exist, and the interplay between them establishes the character of the flow. To obtain more insight into the nature of such interplay, it is instructive to render the quantities in (2.2) dimensionless, as follows.

Choose a typical length \( D \) (such as the diameter of the tube, or the distance between the plates in the corresponding flows discussed in the preceding section), a typical velocity \( V \) (for example, the average flow velocity), and introduce the following dimensionless quantities:

\[
\nu = \frac{V}{V}, \quad \tau = \frac{tV}{D}, \quad \xi_1 = \frac{x_1}{D}, \quad \xi_2 = \frac{x_2}{D}.
\]

Equation (2.2) may then be reduced to

\[
-\frac{\partial \tau}{\partial \xi_1} + \frac{1}{R} \frac{\partial^2 \tau}{\partial \xi_2^2} = \frac{\partial \nu}{\partial \tau} + \frac{\partial}{\partial \xi_1} \left( \frac{1}{2} \nu^2 \right),
\]

where body forces are ignored (since they will not affect the following discussion), and where

\[
R = \frac{\rho V^2}{\mu} = \frac{\rho V D}{\mu} = \frac{V D}{\nu},
\]

is called the Reynolds number.

The second term in the left-hand side of (2.3) represents the viscous effect, whereas in the steady (bulk) motion of the fluid, the second term in the right-hand side corresponds to the inertial effect.

If the quantities \( V \) and \( D \) are appropriately selected so that the dimensionless quantities \( \frac{\partial^2 \nu}{\partial \xi_2^2} \) and \( \frac{\partial}{\partial \xi_1} \left( \frac{1}{2} \nu^2 \right) \) are of the same order
of magnitude as will be assumed, then the ratio of the inertial force to the viscous or frictional force, i.e.

\[
\frac{\text{inertial force}}{\text{viscous force}} = \frac{\rho v_1 \frac{\partial v_1}{\partial x_1}}{\frac{\partial^2 v_1}{\partial x_2^2}}
\]

\[= 0 \left( \frac{vD}{v} \right),\]

is defined by the Reynolds number, R. When R is small, viscous effects dominate the flow, whereas the converse occurs for large Reynolds numbers.

It is important to bear in mind that, even for a very large R, Eq. (2.3) has a different characteristic than the corresponding equation for inviscid flow where \(R = \infty\). In this latter case the differential equation is of a lower order, since the second term of (2.3) does not occur. Thus one cannot satisfy, for example, the no-slip condition at the solid boundaries for an inviscid fluid, whereas this condition can be, and is satisfied when \(v\) is not identically zero, albeit it may be very small.

REYNOLDS EXPERIMENT

Experiment shows that, at sufficiently small Reynolds number R, laminar flow can be maintained. As R is increased by increasing the velocity of the flow, the laminar flow regime tends to become less stable, and, for sufficiently large Reynolds numbers, turbulence sets in. In the flow through circular tubes, for example, stable laminar
flow exists for \( R < 2,000 \). For \( R \) between 2,000 and 3,000, this stability depends on the nature of the disturbances, i.e., a sufficiently large disturbance, for example, may render the laminar flow unstable, causing turbulence, whereas small disturbances may be dissipated through viscous damping. By careful experimentation, laminar flow in pipes can be maintained at values of \( R \) much larger than 3,000, i.e. up to \( 10^4 \). Such regimes are, however, highly unstable, and in this sense resemble, for example, supercooled liquids.

Experimental work which demonstrates the transition from laminar to turbulent flow has been first performed by Reynolds (1883). The Reynolds apparatus is schematically shown in Fig. 2.1a. The liquid in the tank is left undisturbed for many hours, and the entrance to the tube \( T \) has a very smooth transition, as sketched. Dye can be injected at point \( D \), and the velocity of the flow through the tube is controlled by value \( V \). At sufficiently low velocities (small Reynolds number) there is no turbulence (no mixing of the fluid), and the colored liquid filaments have a smooth undisturbed shape as they move down the tube, see sketch 2.1a. As the Reynolds number is increased, turbulence sets in, and there is a complete mixing of the dye with the fluid, sketch 2.1b. With further increase in the velocity, the transition point of dye mixing moves upstream and the mixing becomes more violent.

**EFFECTIVE VISCOSITY**

It was apparently Boussinesq who in 1887 proposed the idea of an (apparent) additional viscosity corresponding to eddy motion in turbulent
Reynolds' Experiment:  (a) For small velocities, the flow is laminar. (b) At higher velocity turbulent mixing takes place, and as the Reynolds number is increased by increasing the flow velocity, the point of disruption moves upstream.
flow. In this phenomenological approach one simply modifies the coefficient of viscosity as

\[ \tau_{21} = \mu_e \frac{\partial v_1}{\partial x_2}, \quad \mu_e = \mu + \mu_t, \quad (2.5) \]

where \( \mu \) is the actual molecular viscosity which, for liquids, represents the intermolecular retarding forces, and, in gases, manifests the effect of momentum exchange, as discussed before, whereas \( \mu_t \) is the apparent or turbulent viscosity (also called eddy viscosity) which does not represent a characteristic of the material property, but rather is to account for the nature of the secondary motion, and, therefore, is affected by the Reynolds number and the flow condition. In (2.5) \( \mu_e \) may then be considered as the "effective" viscosity.

The turbulent viscosity \( \mu_t \) can be estimated in a manner similar to that used to estimate the molecular viscosity in gases; see Sec. 4.8. This involves the concept of mixing length together with, either Reynolds' momentum exchange in eddy motion, or Taylor's vorticity transport phenomenon. The two results, in general, do not coincide, although in special cases they give similar estimates of \( \mu_t \). These results will not be presented here, since the arguments are quite similar to those discussed in connection with the kinetics of gas molecules, and also since these arguments cannot be rigorously supported either on physical or mathematical grounds.\(^1\)

For large Reynolds numbers the value of \( \mu_t \) may be several hundred times that of \( \mu \), whereas for sufficiently small Reynolds numbers, \( \mu_t \) may

\(^1\) For a discussion, see Hunter Rouse, Fluid Mechanics for Hydraulic Engineers, Dover, 1961, pp. 175-191.
be negligible compared with $\mu$. For example, for flow through circular pipes, $\frac{\mu L}{\mu}$ is about 500 for Reynolds numbers of the order of $10^5$ to $10^6$.

6.3 **BOUNDARY-LAYER**

The no-slip condition requires that the fluid velocity reduce to zero on stationary solid boundaries. At high Reynolds' numbers the flow velocity rapidly increases away from the solid boundaries, attaining (almost) its full value a short distance from the boundary. Next to the solid boundary, therefore, there is a thin layer of fluid with very high velocity-gradients. This layer is called the boundary-layer in which the viscous effects are very important. Outside this layer, on the other hand, the fluid behaves more or less as if inviscid. The concept of boundary-layer was introduced by Prandtl in 1904, and has proved to be significant in the treatment of complicated fluid motion. The boundary-layer theory constitutes one of the important areas of fluid dynamics where a great deal of practically significant work has been done.\(^1\) In this section this concept will be illustrated by considering the flow over a plate.

In Fig. 3.1a, a Newtonian fluid flows over plate $P$ of length $L$. The flow velocity is $V$, and is uniform upstream from the plate. The boundary-layer thickness is zero at the leading edge, and attains the value $\delta$ at the trailing edge of the plate. The velocity profile at several points along the plate is sketched in the figure. Since the change in velocity in the $X_2 X_2$-direction occurs gradually, reaching

---

(a) Next to the plate P there is a thin layer of fluid with high velocity-gradient, called the boundary-layer, in which the viscous effects are very significant. Outside this layer, the flow is more or less inviscid.

(b) At low flow velocities the boundary-layer spans the entire pipe. For sufficiently large velocities, on the other hand, turbulence sets in, and the boundary-layer reduces to a thin laminar sublayer and a transition layer, leading to the turbulent core with a very flat velocity profile.
asymptotically the value \( V \), it is customary to define the thickness of the boundary-layer as the distance to the point where 99% of the full velocity is reached.

To obtain an estimate for \( \delta \) one may argue in the following qualitative manner. In the boundary-layer the viscous effects are important, and, for steady flow, they balance the corresponding inertial forces. The viscous term \( \mu \frac{\partial}{\partial x_2} \frac{\partial v_1}{\partial x_2} \) is of the order of \( \mu \frac{V}{\delta^2} \) close to the plate. The inertial term \( \rho v_1 \frac{\partial v_1}{\partial x_1} \), on the other hand, is of the order of \( \rho \frac{V^2}{L} \).

Hence, within the boundary-layer one has

\[
\alpha^2 \frac{\mu}{\delta^2} = \rho \frac{V^2}{L},
\]

(3.1)

where \( \alpha^2 \) is the numerical factor of proportionality to be calculated.

Solving for \( \delta \) obtain

\[
\delta = \alpha \sqrt{\frac{UL}{\rho V}} = \alpha \sqrt{\frac{UL}{V}} = \frac{\alpha L}{\sqrt{R_L}},
\]

(3.2)

where \( R_L \) is the Reynolds number, i.e., \( R_L = \frac{UL}{V} \). Accurate calculations\(^1\) show that \( \alpha = 5 \) approximately. To give an estimate of the thickness involved, consider air with mass-density \( \rho = 1.23 \text{ N sec}^2/\text{m}^4 \), and viscosity \( \mu = 1.77 \times 10^{-5} \text{ N sec/m}^2 \), at 1 atm and 20°C, and let \( L = 1 \text{ m} \), and \( V = 15 \text{ m/sec} \). The Reynolds number then is about \( 10^6 \), and (3.2) yields \( \delta \approx 0.005 \text{ m} = 5 \text{ mm} \).

Equation (3.2) shows that the boundary-layer thickness increases with the length of the plate. In the flow between two stationary plates, or in a pipe, therefore, the boundary-layer thickness increases

\(^1\) Ibid.
as one moves downstream from the entrance, and eventually spans the entire distance between two plates or the diameter of the pipe, leading to laminar flow. This will, however, be the case, if the flow velocity is sufficiently small, so that laminar flow is attainable. For sufficiently large flow velocity, on the other hand, the Reynolds number \( R_\delta = \frac{\frac{\nu}{\delta}}{\nu} \) increases with \( \delta \), reaching a value at which turbulence in the boundary-layer sets in. Hence, next to the solid boundary there is then a very thin laminar sublayer which is followed by a (turbulent) transition region, leading to the major portion of the fluid which constitutes a turbulent core, see Fig. 3.1b. The velocity profile is nearly uniform in the turbulent core, for large Reynolds numbers. And the layer next to the boundary where large velocity-gradients occur, is very thin. Therefore, except for the frictional forces, the flow calculation based on the assumption of uniform velocity distribution leads to reasonable results at high Reynolds numbers.

With the aid of Eq. (3.2) the value of the shear stress at the plate can be estimated. Denoting this value by \( \tau_0 \), observe that \( \frac{\partial v}{\partial x_2} \) is of the order of \( \frac{V}{\delta} \) and, therefore,

\[
\tau_0 = \beta \mu \frac{V}{\delta} = \beta \sqrt{\frac{\mu \nu^2}{L}} ,
\]

where \( \beta \) is the numerical factor to be calculated, and where (3.2) is used. Exact calculation\(^1\) shows that the numerical factor \( \beta \) is 1.328.

\(^1\) *Ibid.*
The drag coefficient $C_D$ for the plate is defined by $C_D = \frac{\tau_0}{\rho V^2}$.

From (3.3) one obtains

$$C_D = 1.328 \sqrt{\frac{\mu}{\rho L}} = \frac{1.328}{\sqrt{R_L}} \quad .$$

(3.4)

For the example cited before $C_D = 0.0013$.

6.4 RECTILINEAR FLOW OF COMPRESSIBLE PERFECT FLUID

The perfect fluid is inviscid by definition, and, therefore, does not adhere to solid boundaries. While no real fluid is perfect, the model provides good results at sufficiently high Reynolds numbers.

Consider the flow of a compressible fluid in a tube of constant cross-sectional area $A$. In steady flow the Bernouilli equation (5-3.18) must be satisfied, and if no body forces are applied, one must have

$$\int \frac{dp}{\rho} + \frac{1}{2} v^2 = \text{constant} \quad ,$$

(4.1)

where, for simplicity, the subscript 1 is eliminated, and the flow velocity is denoted by $v$. For the adiabatic flow, where no heat is added or subtracted, the pressure and mass-density are related by

$$p = k_o \rho^\gamma ;$$

see Eq. (4-2.7), p. 4-4. In this case the first term in the right-hand side of (4.1) becomes

$$\int \frac{dp}{\rho} = k_o \gamma \int \rho^{\gamma-2} d\rho = \frac{k_o \gamma \rho^{\gamma-1}}{\gamma-1} = \frac{\gamma - 1}{\gamma - 1} \frac{p}{\rho} \quad .$$

(4.2)
The quantity $\gamma p/\rho = \frac{dp}{d\rho}$ represents the square of the speed of sound in the gas; see Eq. (4-2.22), p. 4-16. It is commonly denoted by $c^2$. Hence

$$\int \frac{dp}{\rho} = \frac{c^2}{\gamma - 1}. \quad (4.3)$$

Note that, in general, the speed of sound $c$ is not constant. The Bernoulli equation (4.1) now becomes

$$\frac{c^2}{\gamma - 1} + \frac{1}{2} v^2 = \text{constant}. \quad (4.4)$$

The point at which the velocity $v$ is zero is called the stagnation point. Denote by $c_o$ the sound velocity at a stagnation point. Then Eq. (4.4) can be written as

$$\frac{1}{2} v^2 = \frac{1}{\gamma - 1} (c_o^2 - c^2). \quad (4.5)$$

The maximum velocity, $v_{\text{max}}$, occurs when $c = 0$, and is defined by

$$v_{\text{max}} = c_o \sqrt{\frac{2}{\gamma - 1}}$$

which, for air ($\gamma = 1.4$), becomes $v_{\text{max}} = c_o \sqrt{5}$.

The flow is called sonic when $v = c$. It is called supersonic when $v > c$, and subsonic when $v < c$. If the sonic velocity is denoted by $c_*$, then one obtains

$$\frac{1}{2} c_*^2 = \frac{1}{\gamma - 1} (c_o^2 - c_*^2) \quad (4.7)$$

which yields

$$c_* = c_o \sqrt{\frac{2}{\gamma - 1}}. \quad (4.8)$$

1 See also next section.
Observe that $v_{\text{max}}$ can be written as

$$v_{\text{max}} = c_\star \sqrt{\frac{\gamma+1}{\gamma-1}}.$$  \hspace{1cm} (4.9)

For air, therefore, $v_{\text{max}} = c_\star \sqrt{6}$, and $c_\star = c_o \sqrt{\frac{5}{6}}$.

With the aid of $c^2 = \gamma \frac{p}{\rho}$ and (4.8), the Bernouilli equation becomes

$$\frac{1}{2} v^2 + \frac{\gamma}{\gamma-1} \frac{p}{\rho} = \frac{1}{2} \frac{\gamma+1}{\gamma-1} c_\star^2 \hspace{1cm} (4.10)$$

which, together with (4.6), gives

$$c^2 - v^2 = \frac{\gamma+1}{2} (c_\star^2 - v^2). \hspace{1cm} (4.11)$$

This equation shows that if $v \gtrless c_\star$, then $v \gtrless c_\star$.

It is convenient to introduce the dimensionless parameter

$$M = \frac{v}{c} \hspace{1cm} (4.12)$$

which is called the Mach number, and observe that $M \gtrless 1$ according to whether $v \gtrless c$, and $M = 1$ for sonic flow. The Bernouilli equation now yields

$$\frac{\gamma-1}{2} M^2 = \frac{c_o^2}{c^2} - 1 \hspace{1cm} (4.13)$$

which may also be reduced to

$$\frac{\gamma-1}{2} M^2 = \left( \frac{\rho_o}{\rho} \right)^{\frac{1}{\gamma-1}} - 1, \hspace{1cm} (4.14)$$

where $\rho_o$ is the density at a stagnation point.
6.5 ACOUSTIC WAVES IN PERFECT FLUID

The propagation of small pressure disturbances in compressible materials is commonly referred to as acoustic waves. This will be illustrated by considering such waves in perfect fluids.

Eliminate for simplicity the subscripts, and write \( x \) for \( x_1 \) and \( v \) for \( v_1 \), to obtain the equations defining the balance of mass and linear momentum, as

\[
\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial x} + v \frac{\partial \rho}{\partial x} = 0 ,
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0 ,
\]

(5.1)

where \( \rho \) is the mass-density, \( v \) the velocity, and \( p \) the pressure. Since

\[
\frac{\partial p}{\partial x} = \frac{dp}{\partial p} \frac{\partial p}{\partial x} = c^2 \frac{\partial \rho}{\partial x} , \quad \frac{dp}{\partial p} = c^2 ,
\]

(5.2)

the second equation in (5.1) becomes

\[
\frac{c^2}{\rho} \frac{\partial \rho}{\partial x} + v \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = 0
\]

(5.3)

which, together with (5.1), defines \( \rho \) and \( v \) as functions of \( x \) and \( t \).

These equations are not easily integrable. For small disturbances, however, they can be linearized, as follows, which then yield explicitly solvable equations.

Let the perfect fluid be initially undisturbed, with uniform mass-density \( \rho_0 \), pressure \( p_0 \), and constant temperature \( \theta_0 \). Consider small disturbances with velocity \( v \) and mass-density \( \rho = (1 + s)\rho_0 \), where the dimensionless \( s = \frac{\rho - \rho_0}{\rho_0} \) is commonly referred to as the condensation.
Substitution into (5.1) and (5.3) now yields

\[
\frac{\partial s}{\partial t} + \frac{\partial v}{\partial x} + \left( s \frac{\partial v}{\partial x} + v \frac{\partial s}{\partial x} \right) = 0 ,
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{c^2}{1 + s} \frac{\partial s}{\partial x} = 0 .
\]  

(5.4)

To linearize these equations, first observe that

\[
c^2 = \frac{dp}{d\rho} = \left( \frac{dp}{d\rho} \right)_0 + \left( \frac{\partial^2 p}{\partial \rho^2} \right)_0 (\rho - \rho_0) + \ldots
\]

\[
= c_0^2 + \rho_0 s \left( \frac{\partial^2 p}{\partial \rho^2} \right)_0 + \ldots
\]

which can now be substituted into the last term of the right-hand side of (5.4). Neglect all the second and higher order terms, and reduce Eqs. (5.4) to

\[
\frac{\partial s}{\partial t} + \frac{\partial v}{\partial x} = 0 ,
\]

\[
\frac{\partial v}{\partial t} + c_0^2 \frac{\partial s}{\partial x} = 0 ,
\]  

(5.5)

which can be combined to yield

\[
\frac{\partial^2 s}{\partial t^2} - c_0^2 \frac{\partial^2 s}{\partial x^2} = 0 , \quad \frac{\partial^2 v}{\partial t^2} - c_0^2 \frac{\partial^2 v}{\partial x^2} = 0 ,
\]

(5.6)

These are the linear wave equations which characterize the propagation of small disturbances in the fluid.

---

1 Note that this requires that the velocity \(v\) and its gradient together with other quantities of motion be small. For further discussion see p. 6-29.
SIMPLE WAVE SOLUTIONS

The general solution of (5.6) is given by

\[ s = F(x - c_0 t) + G(x + c_0 t), \quad v = f(x - c_0 t) + g(x + c_0 t) \] (5.7)

as can be verified by direct substitution. The functions \( F \) and \( G \) are arbitrary, and are to be fixed by the initial form of the disturbances, i.e., the prescribed values of \( s \) and \( \frac{\partial s}{\partial t} \) at \( t = 0 \). The functions \( f \) and \( g \) are defined in terms of \( F \) and \( G \) by

\[ f = c_0 F, \quad g = -c_0 G, \] (5.8)

as is evident from (5.5).

Suppose that \( F(x) \) is given at \( t = 0 \). Then \( F(x - c_0 t) \) represents a disturbance propagating to the right along the \( X_1X_1 \)-axis with speed \( c_0 \). This pulse according to the present solution retains its initial form. This is exemplified in Fig. 5.1a. At \( t = 0 \), \( F(x) \) is shown to be nonzero for \( a < x < b \). At a later time \( t > 0 \), this pulse has moved to the right, having nonzero values for \( a + c_0 t < x < b + c_0 t \). In this range the argument of the function \( F \) still takes on values between \( a \) and \( b \). The pulse, therefore, is simply shifted to the right by distance \( c_0 t \).

\( F(x - c_0 t) \) is called the simple wave function. Its argument \( x - c_0 t \) is called the phase of the wave function.

Similar remarks apply to the simple wave function \( G(x + c_0 t) \), except for the fact that this function represents a wave traveling to the left along the \( X_1X_1 \)-axis.
(a) At time $t = 0$ the simple wave $F(x)$ is defined for $a < x < b$. At a later time $t > 0$, this pulse has moved to the right having nonzero values for $a + c_0 t < x < b + c_0 t$, so that $F(x_0) = F(x - c_0 t)$.

(b) In the $x, t$-plane, the characteristics $\xi = x - c_0 t$ (left) and $\eta = x + c_1 t$ (right) are shown. The pulse $F(x - c_0 t)$ moves along the characteristics $\xi = \text{constant}$, and the pulse $G(x + c_0 t)$ moves along the characteristic $\eta = \text{constant}$, in the $x, t$-plane.
CHARACTERISTICS

Setting $\xi = x - c_o t$ and $\eta = x + c_o t$, plot in the $x,t$-plane the lines of constant $\xi$ and constant $\eta$, as shown in Fig. 5.1b. These lines which have the slope $\frac{dx}{dt} = \pm c_o$, are called the characteristics. The characteristics $\xi =$ constant have a common positive slope, $+ c_o$, whereas those corresponding to $\eta =$ constant have a negative slope, $- c_o$. In the $x,t$-plane the disturbances travel along the characteristics, as shown in Fig. 5.1b.

Suppose now that at time $t = 0$ the initial values of $s$ and $\frac{\partial s}{\partial t}$ are prescribed, i.e.,

$$s = H(x), \quad \frac{\partial s}{\partial t} = K(x), \quad \text{at } t = 0.$$  \hspace{1cm} (5.9)

From (5.7), it follows that

$$F(x) + G(x) = H(x), \quad -F'(x) + G'(x) = \frac{1}{c_o} K(x),$$

where primes denote differentiation with respect to the corresponding argument. Differentiating the first equation with respect to $x$, and using the second equation, obtain

$$F'(x) = \frac{1}{2} H'(x) - \frac{1}{2 c_o} K(x)$$

which, upon integration, yields

$$F(x) = \frac{1}{2} H(x) - \int_0^x K(\theta) d\theta + C,$$ \hspace{1cm} (5.10)

where $C$ is the integration constant. In a similar manner obtain

$$G(x) = \frac{1}{2} H(x) + \int_0^x K(\theta) d\theta - C.$$ \hspace{1cm} (5.11)
Hence, the final solution becomes

\[ s(x,t) = \frac{1}{2} [H(x + c_o t) + H(x - c_o t)] + \frac{1}{2c_o} \int_{x - c_o t}^{x + c_o t} K(\theta) d\theta, \quad (5.12) \]

where the integration constant cancels out.

The solution for \( v \) is obtained with the aid of Eq. (5.8).

The small changes in pressure corresponding to the above general solutions can be characterized, as follows. For small \( s \) write

\[ p = k_o \rho^\gamma = k_o \rho_o^\gamma (1 + s)^\gamma \]

\[ = k_o \rho_o^\gamma (1 + \gamma s + \ldots) \approx p_o + p_o \gamma s, \]

so that, to the first order of approximation in \( s \), one has

\[ \frac{p - p_o}{p_o} = \frac{\Delta p}{p_o} \approx \gamma s \quad (5.13) \]

which defines the pressure change in terms of the change in mass-density, i.e., the condensation \( s \).

Note that, from (5.7) and (5.8) it follows that

\[ \frac{v}{c_o s} = \frac{F - G}{F + G}. \quad (5.14) \]

It is important to note that one of the essential characteristics of the wave motion discussed above is that, while the disturbances travel through the medium carrying with them mechanical energy, the medium as a whole does not acquire a permanent displacement; i.e., as the pulse passes through the material points, these points move, but then settle at their original position as the pulse moves on.
TIME-HARMONIC WAVES

As special solutions of wave Eqs. (5.6) consider

\[ a \cos \frac{2\pi}{\lambda} (x - c_0 t) \quad \text{and} \quad a \sin \frac{2\pi}{\lambda} (x - c_0 t) \]  \hspace{1cm} (5.15)

which represent sinusoidal waves moving to the right along the \( X_1X_1 \)-axis. Waves of this kind are called harmonic. At a fixed time, the spatial variation of, say, mass-density in this kind of waves, is harmonic. Moreover, at a given point \( x \), the wave has a harmonic variation in time. Here \( \lambda \) is the wavelength which characterizes the spatial periodicity of the wave at fixed time. The wave period \( \tau \) is defined by

\[ \tau = \frac{\lambda}{c_0} \]  \hspace{1cm} (5.16)

which represents the time periodicity of the wave at a fixed point.

The circular frequency \( \omega \) is defined by

\[ \omega = \frac{2\pi}{\tau} \]  \hspace{1cm} (5.17)

and the wave number \( k \) by

\[ k = \frac{2\pi}{\lambda} \]  \hspace{1cm} (5.18)

Observe, therefore, that

\[ \lambda = \frac{2\pi c_0}{\omega} = \frac{2\pi}{k} = c_0 \tau \]  \hspace{1cm} (5.19)

The above sine and cosine solutions of the wave equation can be collectively represented in terms of the exponential function as

\[ e^{ik(x-c_0t)} , \quad i = \sqrt{-1} \]
whose real part gives the cosine solution, and whose imaginary part, the sine solution, since

\[ \frac{ik(x-c_0 t)}{e} = \cos k(x-c_0 t) + i \sin k(x-c_0 t). \]  

(5.20)

Similarly, a harmonic wave travelling with speed \( c_0 \) to the left along the \( X_1X_1 \)-axis can be represented by

\[ \frac{ik(x+c_0 t)}{e}. \]

Since the wave equations (5.6) are linear, their solution can be constructed by the superposition of various elementary solutions. In particular, one may consider solutions consisting of various time-harmonic waves with different wave numbers \( k \), which are superimposed on each other. In this manner one can, by means of a Fourier series, develop the solution corresponding to a given smooth pulse (wave function) moving in a given direction.

ON THE NATURE OF LINEARIZATION

In arriving at linearized wave equations (5.6), it was assumed that the velocity and its various gradients are so small that all the second order terms are negligible. To obtain an insight to the exact nature of such linearization, one must proceed systematically and state explicitly each assumption, as is demonstrated in the sequel.

Denoting the initial particle positions by \( X \), write the displacement field as

\[ U(X,t) = \Xi(X,t) - X \]

\[ u(x,t) = x - \xi(x,t) \]

\[ = x - X, \]

\[ = x - X, \]  

(5.21)
which are the same equations as (2-5.22) and (2-5.25), respectively, except for the subscript 1 which is eliminated here for the sake of simplicity. Let the pressure \( p \) expressed in terms of material variables \( X \) and \( t \) be designated by \( P \), so that \( P(X,t) = p [x(X,t),t] \). Then Eq. (5.1)_2 can be expressed as

\[
- \frac{\partial P}{\partial x} = - \frac{\partial P}{\partial x} \frac{\partial X}{\partial x} = \frac{\partial X}{\partial x} \left[ \frac{\partial X}{\partial x} \rho \right] \frac{\partial^2 U}{\partial t^2},
\]

or

\[
- \frac{\partial P}{\partial x} = \rho_o \frac{\partial^2 U}{\partial t^2},
\]

where Eq. (5-2.6) and the chain rule of differentiation are used, and where it is implied that, after carrying out the indicated differentiation, the resulting quantity is expressed in terms of the material variables \( X \) and \( t \).

Observe that (5.22) represents exactly (5.1)_2.

Denote by \( S(X,t) \) the condensation \( s(x,t) \) expressed in material coordinates. Then from (5-2.6) and the definition \( \rho = (1 + s) \rho_o \) obtain

\[
S(X,t) = - \frac{\partial U}{\partial x} \left( 1 + \frac{\partial U}{\partial x} \right)^{-1}
\]

which is also exact.

As the first approximation, assume that

\[
\left| \frac{\partial U}{\partial x} \right| \ll 1,
\]

and seek its consequences.
Under this assumption, (5.23) yields

\[ s = -\frac{\partial u}{\partial x} \]  \hspace{1cm} (5.25)

Furthermore, the constitutive relation \( p = k_o \rho^\gamma \) can immediately be reduced to

\[ p = p_o - p_o \gamma \frac{\partial u}{\partial x} \]  \hspace{1cm} (5.26)

which then can be employed to reduce (5.22) to

\[ \frac{\partial^2 u}{\partial t^2} - c_o^2 \frac{\partial^2 u}{\partial x^2} = 0 \]  \hspace{1cm} (5.27)

which is the linear wave equation, but for the displacement field \( U \), and in terms of the material variables \( X \) and \( t \).

Observe that, even though (5.27) is linear, and therefore has all the properties of the linear wave equation (for example, superposition is valid), the equations governing fields \( u(x,t) \), \( s(x,t) \), and \( v(x,t) \), are not, in general, linear as is shown in the sequel.

The general solution of (5.27) is given by

\[ U = F_*(X - c_o t) + G_*(X + c_o t) \]  \hspace{1cm} (5.28)

and one may consider also time-harmonic solutions.

Let us now seek to develop the equation governing the variation of \( u(x,t) \). To this end note Eq. (2-5.35), p. 2-44, and obtain

\[ c_o^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} (1 - \frac{\partial u}{\partial x})^2 + 2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t \partial x} (1 - \frac{\partial u}{\partial x}) + \left( \frac{\partial u}{\partial t} \right)^2 \frac{\partial^2 u}{\partial x^2} \]  \hspace{1cm} (5.29)
which is an exact counterpart of (5.27) in terms of the spatial variables \( x \) and \( t \), and which obviously is nonlinear.

It is remarkable that, notwithstanding its nonlinearity, (5.29) still admits a solution of the form \( f_*(x - c_o t) \), a solution of the form \( g_*(x + c_o t) \), but not a solution in the form of the sum of these two solutions. The fact that \( f_*(x - c_o t) \), for example, is a solution of (5.29), can be verified by direct substitution, and the fact that \( f_*(x - c_o t) + g_*(x + c_o t) \) is not a solution, can similarly be checked. Hence, (5.29) does not admit superposition, i.e., the sum of two solutions is not, in general, itself a solution.

Now observe that if \( \frac{\partial u}{\partial x} \) is small, so is \( \frac{\partial u}{\partial t} \) which, according to Eq. (2-5.34), leads to

\[
v(x,t) \approx \frac{\partial u}{\partial t} \quad (5.30)
\]

Moreover, Eq. (2-5.35), p. 2-44, becomes

\[
a(x,t) = \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial t \partial x} + \left( \frac{\partial u}{\partial t} \right)^2 \frac{\partial^2 u}{\partial x^2} \quad (5.31)
\]

which cannot be reduced further, unless approximating assumptions in addition to (5.24) are used. These assumptions are that the velocity (5.30) is small together with its gradient, and that \( \frac{\partial^2 u}{\partial x^2} \) is, at most, of the order of 1. Under these conditions one immediately arrives at the linear wave equations (5.6). For a special class of solutions, however, no such additional assumptions are necessary. These solutions are in the form

\[
u = f(x - c_o t) \quad (5.32)
\]
from which it follows that

\[ \left| \frac{du}{dx} \right| = |f'| \ll 1, \quad \text{and} \quad \left| \frac{1}{c_0} \frac{du}{dt} \right| = |f'| \ll 1. \quad (5.33) \]

Thus the last two terms in the right-hand side of (5.31) can be neglected, leading to the linear wave equation.
6.6 **STEADY FLOW OF INCOMPRESSIBLE FLUID IN PIPES**

In practice, fluids are usually conveyed through pipes at subsonic velocities. In the subsonic velocity range, even gases can be regarded incompressible without noticeable error. The compressibility becomes important when one deals with acoustic waves, as discussed in the preceding section, or when one considers flow at supersonic velocities, as will be discussed later on in connection with shock waves. In addition to this, in usual practical situations, most flows occur at a sufficiently large Reynolds number, so that it is turbulent. As discussed in Section 5.3, the velocity profile is almost uniform for sufficiently large Reynolds numbers, and, except for the frictional losses associated with a thin laminar layer close to the pipe's walls, which must be accounted for separately, the flow can be regarded (almost) rectilinear for most practical purposes.

The Bernoulli equation (5-3.18) for rectilinear steady flow of incompressible fluids has been obtained from the balance of linear momentum Eq. (5-3.3). In order to account for the energy losses which occur because of skin friction, and because of turbulent mixing that takes place as fluid moves through various valves and other control devices which are commonly used in practice, it is simpler and more enlightening to seek to deduce the corresponding Bernoulli equation from the conservation of energy.

To this end, consider the steady flow of an incompressible fluid of constant mass-density $\rho$ through pipes from Station 1 to Station 2, as
sketched in Fig. 6.1. For simplicity, let the body forces consist of only gravitational force of constant intensity $g$ per unit mass. Since the flow is steady, it follows that

$$ (vA)_1 = (vA)_2 , $$

where $v$ is the average velocity, $A$ is the normal cross-sectional area, and the subscripts 1 and 2 indicate that these quantities correspond to Stations 1 and 2, respectively. Choosing an arbitrary datum to measure elevations, let the elevation of the center line of the pipe at Station 1 be denoted by $h_1$, and that at Station 2 by $h_2$. The change per unit time in the potential energy as the flow moves from Station 1 to Station 2, therefore, is

$$ \varphi_1 - \varphi_2 = (\rho vAg h)_1 - (\rho vAg h)_2 $$

$$ = \rho vAg (h_1 - h_2) , $$

where (6.1) is used. This is because $\rho vA$ is the mass flux transmitted from elevation $h_1$ to elevation $h_2$.

The flux of kinetic energy at Station 1 is $\left( \rho v^2 A_1 \right)$, and that at Station 2 is given by the same expression, except with the subscript 2.

The work done per unit time by the pressure $p$ at Station 1 is $-(pAv)_1$, since $pA$ evaluated at Station 1 is the total compressive force transmitted across the cross-section at this station, and $v$ evaluated at Station 1 is the corresponding velocity.

The conservation of energy may now be stated, as follows:
The Steady Flow of Incompressible Fluid from Station 1 to Station 2: At Station 1, the pressure head is $\frac{p_1}{\rho g}$, the velocity head is $\frac{v_1^2}{2g}$, and the potential head is $h_1$, whereas at Station 2 these quantities have changed to $\frac{p_2}{\rho g}$, $\frac{v_2^2}{2g}$, and $h_2$, respectively. The total head loss between these Stations is $h_2$. 

Figure 6.1
\begin{align*}
\left(\rho v A \frac{1}{2} v^2\right)_1 - \left(\rho v A \frac{1}{2} v^2\right)_2 + \rho v A g (h_1 - h_2) = \\
- (p A v)_1 + (p A v)_2 + \dot{e}_k ,
\end{align*}

(6.3)

where \( \dot{e}_k \) is the rate at which energy is lost, because of frictional and other effects between Stations 1 and 2. This equation states that the change in the kinetic energy plus the change in the potential energy, both measured per unit of time between Stations 2 and 1 in a steady flow, equals the rate at which the traction forces at the boundaries do work on the considered portion of the system, plus the rate at which energy is lost by dissipative mechanisms.

Since \( \rho \) is constant, Eq. (6.3) can be reduced to

\begin{align*}
\left(\frac{p}{\rho} + gh + \frac{1}{2} v^2\right)_1 - \left(\frac{p}{\rho} + gh + \frac{1}{2} v^2\right)_2 = \frac{\dot{e}_k}{\rho v A} ,
\end{align*}

(6.4)

where (6.1) is also used. It is customary to express the above equation in the following form:

\begin{align*}
\left(\frac{p}{\rho g} + h + \frac{1}{2g} v^2\right)_1 - \left(\frac{p}{\rho g} + h + \frac{1}{2g} v^2\right)_2 = \dot{h}_l ,
\end{align*}

(6.5)

where

\begin{align*}
\dot{h}_l = \frac{\dot{e}_k}{\rho v A g}
\end{align*}

(6.6)

is called the head loss. In (6.5) all quantities are expressed in terms of an equivalent potential head. The quantity \( \frac{p}{\rho g} \) is called the pressure head, \( \frac{1}{2g} v^2 \) is called the velocity head, and \( h \) is the elevation or the potential head. All these quantities have the physical dimension of length, and therefore can be expressed in, say, feet, or inches.